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Ideally r-Constrained Graded Lie Subalgebras of Maximal Class Algebras

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Abstract. Let $E \supseteq F$ be a field extension and M a graded Lie algebra of maximal class over E. We investigate the F-subalgebras L of M, generated by elements of degree 1. We provide conditions for L being either ideally r-constrained or not just infinite. We show by an example that those conditions are tight. Furthermore, we determine the structure of L when the field extension $E \supseteq F$ is finite. A class of ideally r-constrained Lie algebras which are not $(r-1)$-constrained is explicitly constructed, for every $r \geq 1$.

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1. Introduction

Narrowness conditions for $\mathbb{N}$-graded Lie algebras in zero characteristic have been introduced in [7]. Shalev and Zelmanov built on the remarkable example of the variety of *algébres filiformes*, whose study was started by Vergne in [8, 9], showing that the most important algebras are narrow in some sense. *Thin algebras* arise as the class of positively graded, infinite dimensional Lie algebras, generated by two elements of degree 1 and satisfying the following narrowness condition: every nonzero graded ideal is trapped between two consecutive Lie powers of the Lie algebra. As an immediate consequence of the definition, every homogeneous component of a thin algebra has dimension 1 or 2. Thin algebras have been introduced in [2] and are currently widely studied. Although classification results have been provided for several families of these algebras, the special class of thin algebras, whose homogeneous components except for the second one have dimension 2, is less understood. In [6] it is proved that all metabelian thin algebras belong to this class and they are in one-to-one correspondence with the quadratic extensions of the underlying field F. These results have been extended to the non-metabelian case in [1]. The main idea in those papers is to consider a quadratic field extension $E \supseteq F$ and a Lie algebra M of *maximal class* over E, that is M is a thin algebra over E whose homogeneous components, but the first one, have dimension 1 (see

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Section 2 for details). In the cited papers, the authors consider the Lie algebras generated over F by two elements of M of degree 1 and they characterise the thin ones, showing that their homogeneous components except for the second one have dimension 2.

Ideally r-constrained Lie algebras arise naturally as a generalisation of thin Lie algebras and they are the main topic of this paper. According to [4], we say that a Lie algebra $L := \bigoplus_{i\geq 1} L_i$ is *ideally r-constrained* (or r-constrained for short) if, for every non-zero graded ideal I of L, there exists an integer i such that $L^i \supseteq I \supseteq L^{i+r}$, where $L^k = \bigoplus_{j\geq k} L_j$ is the k-th Lie power of L. Here, r is a positive integer and when $r = 1$ we will simply say that L is *ideally constrained*. Thus, a thin algebra is an ideally constrained Lie algebra whose first homogeneous component has dimension 2. Note also that a finitely generated r-constrained Lie algebra is *just infinite*, meaning that every non-zero graded ideal of the algebra has finite codimension. Periodic just infinite Lie algebras have been studied in [5], where it is proved that those algebras are necessarily r-constrained.

In the spirit of [1], we consider an arbitrary field extension $E \supseteq F$ and a Lie algebra M of maximal class over E. We provide conditions for an F-subalgebra L of M, generated in degree 1, to satisfy a dichotomy, that is to be either ideally r-constrained or not just infinite. Precisely, we associate to L a sequence of intermediate fields F_i's, depending on the intersection of the first homogeneous component of L with the 2-*step centralisers* C_i's of M (see Section 2). Differently from [1], where only the dimensions of the fields F_i's matter, in this paper the field $\mathcal{K}$ generated by the F_i's plays a role in the above mentioned dichotomy. In particular, we can determine the structure of L when the field extension $E \supseteq F$ is finite or when E is algebraic over F and M has only finitely many C_i's.
The paper is structured as follows. In Section 2, we recall the standard definitions and properties of Lie algebras of maximal class and introduce the techniques needed to define the fields F_i's. Section 3 is devoted to collecting the properties of the F_i's and to proving the main result stated in Theorem 3.8. Section 4 concludes the paper by providing examples and open problems. In particular, for every $r \geq 1$ we exhibit an r-constrained Lie algebra which is not $(r-1)$-constrained. We also show that the assumptions of Theorem 3.8 are tight by considering the case where E is transcendental over F.

2. Preliminaries

If X is a subset of a (left) module V over a ring R, we denote by RX the R-submodule generated by X. We will simply write Rx when $X = \{x\}$. By a Lie algebra we mean a graded Lie algebra $L := \bigoplus_{i\geq 1} L_i$ over some field, generated by its first homogeneous component L_1. If not otherwise stated, we shall assume that $L_i \neq 0$ for every i: in particular L is infinite dimensional. Given a Lie algebra L and subsets X and Y of L, we denote by $[X, Y]$ the additive subgroup of L generated by the elements $[x, y]$ as x ranges in X and y ranges in Y. We will simply write $[X, y]$ when $Y = \{y\}$. In particular, $L_i = [L_{i-1}, L_1]$ for every $i \geq 2$, as we are assuming that L is generated by L_1. A Lie algebra $M := \bigoplus_{i\geq 1} M_i$ over a field E is said to be of *maximal class* if $\dim_E M_1 = 2$ and $\dim_E M_i = 1$ for every $i \geq 2$. Borrowing some terminology from the theory of (pro) p-groups of maximal class, the i-th 2-*step centraliser* for $i \geq 2$ is defined as $C_i = \mathrm{C}_{M_1}(M_i) = \{a \in M_1 : [a, M_i] = 0\}$:

this is a subspace of dimension 1. Note that for every non-zero $l_i \in M_i$ we have $C_i = \mathrm{C}_{M_1}(l_i)$. The following lemma, which we quote for easy reference, is implicitly stated in [3, Proposition 4.1].

Lemma 2.1. *Let M be a Lie algebra of maximal class and C be a 2-step centraliser. If t is the smallest integer such that $C_t = C$, then in every interval of integers of length t there is at least one j such that $C_j = C$. In particular, there are infinitely many occurrences of C.*

We say that a Lie algebra $L := \bigoplus_{i\geq 1} L_i$ is *ideally r-constrained* if, for every non-zero graded ideal I, there exists an integer i such that $L^i \supseteq I \supseteq L^{i+r}$, where $L^k = \bigoplus_{j\geq k} L_j$ is the k-th Lie power of L. Equivalently, L is ideally r-constrained if, for every positive integer i and every non-zero homogeneous element z of degree i, we have that $[z, {}_r L_1] = L_{i+r}$, where $[z, {}_k L_1]$ is defined recursively by setting $[z, {}_1 L_1] := [z, L_1]$ and $[z, {}_k L_1] := [[z, {}_{k-1} L_1].L_1]$ for $k \geq 2$. When $r = 1$ we will simply say that L is *ideally constrained.* A Lie algebra of maximal class is ideally constrained. A *just infinite* (dimensional) Lie algebra is an algebra whose non-zero graded ideals have finite codimension. A finitely generated ideally r-constrained Lie algebra is just infinite.

Definition 2.2. Let V be a vector space over a field E and U a subgroup of the additive group of V. We denote by E_U the subset of E of the elements α such that $\alpha U \subseteq U$.

Lemma 2.3. *With the previous notation, the subset E_U of E is a subring of E. The subset U is an R-module with respect to a given subring R of E if and only if $R \subseteq E_U$: in particular, U is an E_U-module. If there exists a subfield F of E_U such that $\dim_F U$ is finite, then E_U is a subfield of E.*

Proof. We prove the last statement since the rest is trivial. If $U = 0$ then $E_U = E$; if $U \neq 0$, take $u \in U \setminus \{0\}$ and note that the map from E_U to U sending k in ku is an injective F-linear map so $\dim_F E_U$ is finite, and therefore E_U is a field. ∎

Remark 2.4. Let L be a Lie algebra over some field E. If U is an additive subgroup of L and X is a subset of L, then $E_{[U,X]} \supseteq E_U$.

Lemma 2.5. *Let X and Y be subsets of a field E such that $X = Yu$ for some non-zero u in E and $1 \in X \cap Y$. Then the subfield generated by X and the subfield generated by Y coincide.*

Proof. Since $1 \in Y$, we have that $u = 1u \in X$. Thus, given y in Y, we get that $y = yu \cdot u^{-1}$ belongs to the subfield generated by X. For the other inclusion, just note that $Y = Xu^{-1}$. ∎

Definition 2.6. Let $E \supseteq F$ be fields, V an E-vector space, C an E-subspace of V of codimension 1, and W an F-subspace of V, such that $W \nsubseteq C$. Let $\phi\colon V \to E$ be an E-linear map such that $\ker \phi = C$ and $1 \in \phi(W)$ (such a map clearly exists). We denote by $\mathcal{F}(W/C)$ the subfield of E generated by $\phi(W)$.

The notation just introduced is unambiguous since the following result holds.

Lemma 2.7. *The subfield $\mathcal{F}(W/C)$ contains F and does not depend on the choice of ϕ. Furthermore, $\mathcal{F}(W/C) = F$ if and only if $\dim_F(W/W \cap C) = 1$. Finally, $|\mathcal{F}(W/C) : F| \geq \dim_F(W/W \cap C)$, and, under the additional assumption that $\dim_F(W/W \cap C)$ is finite, equality holds if and only if $\phi(W)$ is a field.*

Proof. If ψ is another map with the required properties, then there exists a non-zero u in E such that $\psi = u\phi$: by Lemma 2.5, $\psi(W)$ and $\phi(W)$ generate the same subfield. Since $\phi(W)$ is an F-subspace of E containing 1, it contains F and so does $\mathcal{F}(W/C)$. Moreover, $|\mathcal{F}(W/C) : F| \geq \dim_F \phi(W) = \dim_F(W/W \cap C)$ and, under the additional requirement that $\dim_F(W/W \cap C)$ is finite, equality holds if and only if $\phi(W) = \mathcal{F}(W/C)$, that is $\phi(W)$ is a field; in particular, $\mathcal{F}(W/C) = F$ if and only if $\dim_F(W/W \cap C) = 1$. ∎

3. Finitely generated F-subalgebras of M

In the rest of this paper $E \supseteq F$ will be fields and M will be a Lie E-algebra of maximal class. We are interested in the description of the structure of the Lie F-algebra generated by an F-subspace L_1 of M_1. First note that if $\dim_E EL_1 \leq 1$ then the elements of L_1 commute and the Lie F-algebra generated by L_1 is L_1 itself equipped with the trivial Lie product. Thus we will assume that $EL_1 = M_1$: in particular, $L_1 \nsubseteq C_i = \mathrm{C}_{M_1}(M_i)$ for every $i \geq 2$. We associate to L_1 a sequence of subfields $\{F_i\}_{i\geq 2}$ of E by setting $F_i := \mathcal{F}(L_1/C_i)$ as in Definition 2.6. If K is a subfield of E containing F and T_1 denotes the K-subspace KL_1, then $ET_1 = M_1$, so we may similarly associate to T_1 a sequence of subfields $K_i := \mathcal{F}(T_1/C_i)$. It easily turns out that $K_i = K(F_i)$ for every $i \geq 2$.

Remark 3.1. When $\dim_F L_1 = 2$, a sequence of integers $\{d_i\}_{i\geq 2}$ associated to L_1 is defined in [1] by setting $d_i := \dim_F(L_1 \cap C_i)$: since $L_1 \nsubseteq C_i$, the possible values for d_i are just 0 and 1 and $d_i = 1$ if and only if $F_i = F$ so the sequence of the F_i's (possibly) carries more information than the sequence of the d_i's.

The following result, whose proof is immediate, will be used repeatedly hereafter.

Lemma 3.2. *If x is an element of $M_1 \setminus C_i$ for some $i \geq 2$, then the adjoint map $l \mapsto [l, x]$ is an E-isomorphism between M_i and M_{i+1}.*

Lemma 3.3. *Let U be an additive subgroup of M_i for some $i \geq 2$ and x an element in $M_1 \setminus C_i$. Then $E_U = E_{[U,x]}$.*

Proof. By Remark 2.4, the inclusion $E_U \subseteq E_{[U,x]}$ holds. To prove the reverse inclusion, we must show that $\alpha u \in U$ for every $\alpha \in E_{[U,x]}$ and $u \in U$. Since U is an additive group, we have $[U, x] = \{[u, x] \mid u \in U\}$: thus $[\alpha u, x] = \alpha[u, x] = [u', x]$ for some $u' \in U$. By Lemma 3.2, $\alpha u = u' \in U$. ∎

Lemma 3.4. *Let U be a finite-dimensional F-subspace of M_i for some $i \geq 2$. Then $\dim_F[U, L_1] \geq \dim_F U$ and equality holds if and only if E_U contains F_i, in which case $E_U = E_{[U,L_1]}$.*

Proof. Take $x \in L_1 \setminus C_i$: by Lemma 3.2, $\dim_F[U, L_1] \geq \dim_F[U, x] = \dim_F U$. Equality holds if and only $[U, x] = [U, L_1]$, that is $[U, y] \subseteq [U, x]$ for every $y \in L_1$.

Let $\phi\colon M_1 \to E$ be the E-linear map such that $\ker\phi = C_i$ and $\phi(x) = 1$. Then $y - \phi(y)x \in C_i$, so that $[U, y] = \phi(y)[U, x]$. Thus, $[U, y] \subseteq [U, x]$ for every $y \in L_1$ if and only if $\phi(L_1) \subseteq E_{[U,x]}$. By Lemma 3.3, $E_{[U,x]} = E_U$, and this is a field by Lemma 2.3. Thus $\phi(L_1) \subseteq E_{[U,x]}$ if and only if $E_U \supseteq F_i$. Finally, if equality holds, then $[U, x] = [U, L_1]$ and so $E_{[U,L_1]} = E_{[U,x]} = E_U$. ∎

Proposition 3.5. *Let K be a subfield of E containing F_i for every $i \geq 2$. If $T := \bigoplus_{i\geq 1} T_i$ is the K-algebra generated by L_1 then $\dim_K T_i = \dim_K T_1 - 1$ for every $i \geq 2$.*

Proof. Let $i \geq 2$. Since $EL_1 = M_1$, it follows that $L_1 \nsubseteq C_i$ and, a fortiori, $T_1 \nsubseteq C_i$. We choose $x_i \in L_1 \setminus C_i$ and the E-linear map $\phi_i\colon M_1 \to E$ such that $\ker\phi_i = C_i$ and $\phi_i(x_i) = 1$: thus F_i is generated by $\phi_i(L_1)$ and K_i is generated by $\phi_i(T_1)$. Since $T_1 = KL_1$, we have $K_i = K(F_i) = K$: Lemma 2.7 implies that $\dim_K(T_1/T_1 \cap C_i) = 1$ so that $T_1 = Kx_i + (T_1 \cap C_i)$. Thus $T_{i+1} = [T_i, T_1] = [T_i, x_i]$ and $T_2 = [T_1, T_1] = [T_1 \cap C_2, x_2]$ (note that the elements in C_2 commute each other). By Lemma 3.2, $\dim_K T_{i+1} = \dim_K T_i$ for $i \geq 2$ and $\dim_K T_2 = \dim_K(T_1 \cap C_2)$, therefore $\dim_K T_2 = \dim_K T_1 - \dim_K(T_1/T_1 \cap C_2) = \dim_K T_1 - 1$. ∎

Definition 3.6. The 2-*step field* of the Lie algebra L is the field $\mathcal{K}$ generated by $\{F_i\}_{i\geq 2}$.

Proposition 3.7. *Suppose that $t := |\mathcal{K} : F|$ is finite. Let X_0 be a finite-dimensional F-subspace of M_i for some $i \geq 2$. Define recursively $X_j := [X_{j-1}, L_1]$ for $j \geq 1$. Then $\dim_{\mathcal{K}} \mathcal{K}X_j = \dim_{\mathcal{K}} \mathcal{K}X_0$ for every $j \geq 0$ and there exists an integer l, independent of X_0 and i, such that X_{l+k} is a $\mathcal{K}$-vector space for every $k \geq 0$.*

Proof. Since $\mathcal{K}X_j = [\mathcal{K}X_{j-1}, L_1]$ for every $j \geq 1$, Lemma 3.4 yields the first claim.

In order to prove the second claim, we first consider the case $\dim_F X_0 = 1$, so that $\dim_F \mathcal{K}X_j = \dim_F \mathcal{K}X_0 = |\mathcal{K} : F| = t$ for every $j \geq 0$.

Since t is finite, there exists an integer r such that $\mathcal{K}$ is generated by the F_u's with $u \leq r$. Let $l := (t-1)r$. We need to prove that $E_{X_{l+k}} \supseteq \mathcal{K}$ for every $k \geq 0$. We proceed by contradiction assuming, in virtue of Remark 2.4, that $E_{X_j} \not\supseteq F_u$ for some $u \leq r$ and every $0 \leq j \leq l$. By Lemma 2.1, we may choose $t-1$ indices $0 \leq c_1 < c_2 < \cdots < c_{t-1} < l$ such that $F_{i+c_1} = \cdots = F_{i+c_{t-1}} = F_u$. Thus, Lemma 3.4 yields

$$\begin{aligned} 1 = \dim_F X_0 \leq \dim_F X_{c_1} < \dim_F X_{c_1+1} \leq \dim_F X_{c_2} < \dim_F X_{c_2+1} \leq \dots \\ \cdots \leq \dim_F X_{c_{t-1}} < \dim_F X_{c_{t-1}+1} \leq \dim_F X_l. \end{aligned}$$

Hence $\dim_F X_l \geq t = \dim_F \mathcal{K}X_l$ and therefore $X_l = \mathcal{K}X_l$, a contradiction. When $\dim_F X_0 > 1$, decomposing X_0 as sum of one-dimensional subspaces completes the proof. ∎

We are now in position to state a dichotomy for the F-subalgebras of a Lie E-algebra of maximal class, showing that they are either ideally r-constrained for some r or they are not just infinite.

Theorem 3.8. *Let $E \supseteq F$ be fields, M a Lie E-algebra of maximal class and $L = \bigoplus_{i\geq 1} L_i$ the Lie F-algebra generated by a finite-dimensional F-subspace L_1 of M_1 such that $EL_1 = M_1$. Assume that the 2-step field $\mathcal{K}$ of L is a finite extension of F. One of the following holds*

(1) $\dim_{\mathcal{K}} \mathcal{K}L_1 = 2$ *and L is ideally r-constrained for some r;*

(2) $\dim_{\mathcal{K}} \mathcal{K}L_1 > 2$ *and L is not just infinite.*

Proof. Let $T = \bigoplus_{i\geq 1} T_i$ be the $\mathcal{K}$-algebra generated by L_1, so $T_1 = \mathcal{K}L_1$. By Proposition 3.5, $\dim_{\mathcal{K}} T_i = \dim_{\mathcal{K}} T_1 - 1$ for every $i \geq 2$.

Assume first that $\dim_{\mathcal{K}} T_1 = 2$. Let I be a non-zero graded ideal of L: set $I_j := I \cap L_j$ for every $j \geq 1$. Let i be the smallest integer such that $I_i \neq \{0\}$. Suppose that $i \geq 2$: by Proposition 3.7, there exists l independent of i such that I_{i+l} contains a non-zero $\mathcal{K}$-subspace of T_{i+l}. Since $\dim_{\mathcal{K}} T_{i+l} = 1$ this implies that $I_{i+l} = T_{i+l}$ and, a fortiori, $I_{i+l} = L_{i+l}$. If $i = 1$, then $I_2 \neq \{0\}$ and the same argument yields $I_{2+l} = L_{2+l}$. In any case $L^i \supseteq I \supseteq L^{i+l+1}$, that is L is ideally $(l+1)$-constrained.
Assume now that $\dim_{\mathcal{K}} T_1 > 2$ so that $\dim_{\mathcal{K}} T_i > 1$ for every $i \geq 2$. By Proposition 3.7, L_i is a $\mathcal{K}$-vector space for i large enough. Since $\mathcal{K}L_i = T_i$, this means that $L_i = T_i$ for i large enough: in particular, $\dim_F L_i \geq \dim_{\mathcal{K}} T_i > 1$. Choose such an i and let X be a non-zero proper $\mathcal{K}$-subspace of L_i. Denote by I and J the ideals generated by X respectively in L and T: clearly $I \subseteq J$. Lemma 3.4 implies that $\dim_{\mathcal{K}}(T_j \cap J) = \dim_{\mathcal{K}} X$ for every $j \geq i$ so that $T_j \cap J$ is a proper subspace of T_j: as an obvious consequence $L_j \cap I$ is a proper subspace of L_j (remind that $L_j = T_j$): therefore I has infinite codimension in L, which is not just infinite. ■

Remark 3.9. There are assumptions that imply the finiteness of $|\mathcal{K} : F|$. The simplest one is that $|E : F|$ is finite. Another possibility is that E is an algebraic extension of F and M has finitely many distinct 2-step centralisers.

Corollary 3.10. *In the same hypotheses of Theorem 3.8, if L is ideally constrained then $\dim_F L_i = |\mathcal{K} : F|$ for every $i \geq 3$ and $F_i = \mathcal{K}$ for every $i \geq 2$.*

Proof. Let z be a (non-zero) homogeneous element of L of degree i with $i \geq 2$: since $[z, L_1] = L_{i+1}$ we have $\dim_F L_{i+1} = \dim_F(L_1/L_1 \cap C_i)$. By Lemma 2.1, there are infinitely many values of i such that $\dim_F L_{i+1} = \dim_F L_3$. By Lemma 3.4, $\{\dim_F L_i\}_{i\geq 2}$ is a weakly increasing sequence, so it is constant for $i \geq 3$. By the proof of Theorem 3.8, L_i is a $\mathcal{K}$-vector space of dimension 1 for i large enough: thus $\dim_F L_i = |\mathcal{K} : F|$ for $i \geq 3$. By Lemma 2.7, we have

$$|\mathcal{K} : F| \geq |F_i : F| \geq \dim_F(L_1/L_1 \cap C_i) = \dim_F L_{i+1}$$

for $i \geq 2$, whence the last claim. ■

Remark 3.11. If $\dim_F L_1 = 2$ then, since $\dim_F L_1 \geq \dim_{\mathcal{K}} \mathcal{K}L_1 \geq \dim_E EL_1 = 2$, the algebra generated by L_1 is ideally r-constrained. However this can happen even if $\dim_F L_1 > 2$: see Example 4.2.

4. Examples and open problems

We now show that both cases of Theorem 3.8 actually occur. We remind that all the 2-step centralisers of the metabelian Lie algebra of maximal class coincide.

Example 4.1. Let $E \supsetneq F$ be fields with $|E : F|$ finite and let M be the metabelian Lie E-algebra of maximal class. Let L_1 be the F-subspace of M_1 generated by an element x in $M_1 \setminus C_2$ and an F-subspace U of C_2 of dimension $d \geq 2$. Take the E-linear map $\phi\colon M_1 \to E$ such that $\phi(x) = 1$ and $\ker\phi = C_2$; thus $\mathcal{K} = F_2 = F$ and $\dim_{\mathcal{K}} \mathcal{K}L_1 = \dim_F L_1 \geq 3$ so that the Lie F-algebra generated by L_1 is not just infinite.

Example 4.2. Let α be an algebraic element of degree $d \geq 2$ over a field F and let $E := F(\alpha)$. Let M be a Lie algebra of maximal class over E with at most two distinct 2-step centralisers. Denote by C a 2-step centraliser and choose a non-zero element y in C. Choose another non-zero element x of M_1 as follows: if M has two distinct 2-step centralisers, x belongs to the 2-step centraliser different from C; otherwise x is just an element not in C. Let L_1 be the F-subspace of M_1 generated by x, αx and y. Given an index i such that $C_i = C$, consider the E-linear map $\phi\colon M_1 \to E$ such that $\phi(x) = 1$ and $\ker\phi = C$; thus $F_i = F(\alpha) = E = \mathcal{K}$ and $\dim_{\mathcal{K}} \mathcal{K}L_1 = 2$ so that the Lie F-algebra L generated by L_1 is ideally r-constrained for some r.

We now compute r: some notation is needed. Given a non-zero homogeneous element z of L of degree i, we denote by $\mathrm{cd}(z)$ the smallest k such that $[z, {}_k L_1] = L_{i+k}$. If r is the maximum of $\mathrm{cd}(z)$ as z ranges over the set of homogeneous elements of L, then L is ideally r-constrained but not ideally $(r-1)$-constrained.

Given a non-zero homogeneous element z of degree $i \geq 2$ of M and a nonnegative integer t, we denote by $U(z,t)$ the set of the elements of the form $p(\alpha)z$ as $p(\alpha)$ ranges over the set of the polynomials in α with coefficients in F and degree at most t. The set $U(z,t)$ is clearly an F-subspace of M_i and its dimension is $t+1$ for $t < d$ and d for $t \geq d-1$. If $C_i = C$ then $[U(z,t), L_1] = U([z,x], t+1)$; this is the only possibility if M is metabelian. If M is not metabelian there are indices i such that $C_i \neq C$: for those indices, we have $[U(z,t), L_1] = U([z,y], t)$. In particular, $\dim_F U(z,t) < \dim_F [U(z,t), L_1]$ if and only if $C_i = C$ and $t < d-1$.

If w is a non-zero homogeneous element of M of the same degree $i \geq 2$ as z and t and s are nonnegative integers such that $U(z,t)$ is strictly contained in $U(w,s)$ but $[U(z,t), L_1] = [U(w,s), L_1]$, then $C_i = C$ and $t = d-2$.

Given an integer $i \geq 2$ and a nonnegative integer k, we denote by $m_{i,k}$ the number of integers j such that $i \leq j < i+k$ and $C_j = C$; we denote by m_i the smallest k such that $m_{i,k} = d-1$. Since $m_{i,k} \leq k$, we have that $m_i \geq d-1$, and equality holds for every i when M is metabelian. An easy induction shows that $[U(z,t), {}_k L_1] = U(w, t + m_{i,k})$ for some homogeneous element w of degree $i+k$. In particular, since $L_2 = U([y,x], 1)$, we have that for every $j \geq 2$, $L_j = U(w, 1 + m_{2,j-2})$ for some homogeneous element w of degree j.

If z is an element of L of degree $i \geq 2$, then the F-subspace generated by z, that is $U(z,0)$, is strictly contained in $L_i = U(w,t)$ for some w of degree i and some positive integer t. By the previous discussion, $[U(z,0), {}_k L_1] = [L_i, {}_k L_1]$ if and only if $k \geq m_i$, so that $\mathrm{cd}(z) = m_i$. It remains to consider non-zero elements in L_1: if z is such an element then $[z, L_1]$ is a non-zero subspace of L_2 and therefore $\mathrm{cd}(z) \leq m_2 + 1$; if we choose $z := x$ then $[x, L_1]$ is the F-vector space generated by $[x,y]$ and therefore $\mathrm{cd}(z) = m_2 + 1$. Summarising, the maximum of $\mathrm{cd}(z)$, that is the integer r such that L is ideally r-constrained but not $(r-1)$-constrained, is m_2+1 if $m_2=\max\{m_i\}$ and is $\max\{m_i\}$ otherwise. In particular, if M is the metabelian Lie algebra of maximal class, then $r=d$ thus showing that there exist examples of ideally r-constrained Lie algebras for every positive integer r.

We close the paper with some problems left open to the reader.

Problem 4.3. In Theorem 3.8 we have assumed that $\mathcal{K}\supseteq F$ is a finite extension. What can be said when the degree of the extension is infinite? For instance, let $E=F(\alpha)$, where α is transcendental and M be the metabelian Lie E-algebra of maximal class. Consider L to be the F-algebra generated by x and $\alpha x+y$, where $x\in M_1\setminus C_2$ and $0\neq y\in C_2$. Then $E=\mathcal{K}$ and an argument similar to Example 4.2 shows that L is the free 2-generated metabelian algebra. Note also that L coincides with the algebra constructed in [6, Lemma 1].

Problem 4.4. Is it possible to compute the smallest r in the first case of Theorem 3.8?

Problem 4.5. What can be said if we take subalgebras of a thin algebra or even more generally of an ideally r-constrained algebra over E?

References

[1] M. Avitabile, A. Caranti, N. Gavioli, V. Monti, M. F. Newman, E. A. O'Brien: *Thin subalgebras of Lie algebras of maximal class*, Israel J. Math. **253** (2023) 101–112.

[2] A. Caranti, S. Mattarei, M. F. Newman, C. M. Scoppola: *Thin groups of prime-power order and thin Lie algebras*, Quart. J. Math. Oxford Ser. (2) 47/187 (1996) 279–296.

[3] A. Caranti, M. F. Newman: *Graded Lie algebras of maximal class II*, J. Algebra 229 (2000) 750–784.

[4] N. Gavioli, V. Monti: *Ideally constrained Lie algebras*, J. Algebra 253 (2002) 31–49.

[5] N. Gavioli, V. Monti, C. M. Scoppola: *Just infinite periodic Lie algebras*, in: *Finite Groups 2003*, Proc. Conf. Gainesville Finite Groups 2003, C. Y. Ho et al. (eds.), De Gruyter, Berlin (2004) 73–85.

[6] N. Gavioli, V. Monti, D. S. Young: *Metabelian thin Lie algebras*, J. Algebra 241 (2001) 102–117.

[7] A. Shalev, E. Zelmanov: *Narrow Lie algebras: a coclass theory and a characterization of the Witt algebra*, J. Algebra 189 (1997) 294–331.

[8] M. Vergne: *Réductibilité de la variété des algèbres de Lie nilpotentes*, C. R. Acad. Sci. Paris Sér. A-B 263 (1966) A4–A6.

[9] M. Vergne: *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes*, C. R. Acad. Sci. Paris Sér. A-B 267 (1968) A867–A870.

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