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Quasi-Hamiltonian Model Spaces

Kay Paulus and Bart Van Steirteghem

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Abstract. Let K be a simple and simply connected compact Lie group. We call a (twisted) quasi-Hamiltonian K-manifold M a quasi-Hamiltonian model space if it is multiplicity free and its momentum map is surjective. We explicitly identify the subgroups of the Lie algebra of a maximal torus of K, which, by F. Knop's classification of multiplicity free quasi-Hamiltonian manifolds, are in one-to-one correspondence with the isomorphism classes of quasi-Hamiltonian model K-spaces.

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1. Introduction

A quasi-affine variety equipped with an action of a complex connected reductive group G is called a *model variety* for G if its coordinate ring contains every irreducible representation of G exactly once. The study of such 'representation models' started in [4] and has been quite fruitful, see for example [1, 7, 9, 10, 19].

In this paper, we classify analogous *model spaces* in the setting of the quasi-Hamiltonian manifolds introduced by A. Alekseev, A. Malkin and E. Meinrenken in [2]. Roughly speaking, a quasi-Hamiltonian K-manifold is a smooth manifold M equipped with an action of a compact connected Lie group K, a 2-form ω and a smooth K-equivariant map $m : M \to K$, called the *(group valued) momentum map*, fulfilling certain compatibility conditions (see Definition 2.1).

In fact, this notion can be generalized by allowing a twist of the conjugation action of K on itself. Indeed, given a smooth automorphism τ of K one can require that the momentum map $m : M \to K$ be equivariant with respect to the twisted conjugation action

$$k \cdot_\tau g = kg\tau(k)^{-1}$$

of K on itself. In this case, we use $K\tau$ for K equipped with this τ-twisted action and denote the momentum map by $m : M \to K\tau$. After having been considered by P. Boalch and D. Yamakawa in the context of twisted wild character varieties in [5], such quasi-Hamiltonian $K\tau$-manifolds were first defined by Meinrenken in [22] and later independently by F. Knop in [14].

From now on, *we assume that K is simply connected.* As is known and will be recalled in Theorem 2.4, there is a natural homeomorphism $c : \mathcal{A} \to K\tau/K$ between the set $K\tau/K$ of τ-twisted conjugacy classes in K and the fundamental alcove $\mathcal{A}$

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of a certain affine root system. In [2, Theorem 7.2] and [22, Theorem 4.4] it was shown (for $\tau = id_K$ and general τ, respectively) that when M is a compact and connected quasi-Hamiltonian $K\tau$-manifold, then the image of the map

$$m_+ := c^{-1} \circ \pi \circ m : M \to \mathcal{A}, \tag{1}$$

where $\pi : K\tau \to K\tau/K$ is the quotient map, is a convex polytope $\mathcal{P}_M$ in $\mathcal{A}$, which is called the *momentum polytope* of M.

Alekseev, Malkin and Meinrenken also ported the classical notion of symplectic reduction of Hamiltonian manifolds to the setting of quasi-Hamiltonian manifolds [2, Section 5]. In analogy to the multiplicity free Hamiltonian manifolds of Guillemin and Sternberg [11], Knop then made the following definition in [14]: a compact connected quasi-Hamiltonian manifold is called *multiplicity free* if all its symplectic reductions are points, see also [15, Def. 2.4.1 and Prop. 2.4.2].

In [15, Corollary 2.6.2], Knop showed that compact connected multiplicity free quasi-Hamiltonian $K\tau$-manifolds M are uniquely determined by the pair $(\mathcal{P}_M, \Lambda_M)$, where Λ_M is a certain lattice which encodes the principal isotropy group of the K-action on M. In addition, he characterized which pairs $(\mathcal{P}, \Lambda)$ consisting of a polytope and a lattice can occur this way.

Knop also studied certain series of examples of multiplicity free quasi-Hamiltonian manifolds in [15, Section 2.7], and in [25] the first author obtained a classification of those for which $\dim \mathcal{P}_M = 1$, see also [16]. In [15, Proposition 2.7.3], Knop identified some multiplicity free quasi-Hamiltonian manifolds which are 'as big as possible.' Their explicit combinatorial classification is the purpose of this paper.

Definition 1.1. A compact connected quasi-Hamiltonian $K\tau$-manifold is called a *(quasi-Hamiltonian) model $K\tau$-space* if it is multiplicity free and its momentum map is surjective.

The main result of this paper is Theorem 2.12 in which we combinatorially classify model $K\tau$-spaces for K *simple* and simply connected. The necessary prerequisites for stating this theorem are reviewed in Section 2. It is an application of Theorem 2.10, which is a specialization of Knop's aforementioned classification theorem. Since the momentum polytope of a model $K\tau$- space is always the full alcove $\mathcal{A}$, our classification is in terms of the possible lattices Λ_M.

The analogous problem in the (untwisted) Hamiltonian setting, where $\mathcal{A}$ is replaced by a dominant Weyl chamber, was essentially solved in [26, §3]. Indeed, Knop's Hamiltonian version [13, Theorem 11.2] of his classification theorem implies that multiplicity free Hamiltonian manifolds with surjective momentum map are also determined by an associated lattice. The subgroups of the weight lattice of K that can be realized this way and the manifolds realizing them were classified in [26, §3], see also Lemma 3.5, Proposition 3.6 and Remark 3.7(e) below. Note that in this case the manifolds are not compact.

Knop's characterization in [15] of the pairs $(\mathcal{P}_M, \Lambda_M)$ realized by multiplicity free quasi-Hamiltonian manifolds M is in terms of *weight monoids* of smooth affine *spherical varieties*. This weight monoid is a basic representation-theoretic invariant of such varieties (see Definition 2.6). As explained in [15, Remark 2.5.4(f,g)] these varieties also yield local descriptions of the manifold M, see Remark 3.7(e) below as well. It would be interesting to have global descriptions of the model spaces we classify, but this lies beyond the scope of this paper.

In Section 3 we present the tools from the combinatorial theory of spherical varieties that we will use in Section 4 to prove Theorem 2.12. The main tool is Proposition 3.10. It is a special case, adapted to our setting, of the combinatorial characterization of the weight monoids of smooth affine spherical varieties in [27] and may be of some independent interest.

Notation. Unless stated otherwise, K will be a simple and simply connected compact Lie group with Lie algebra $\mathfrak{k}$. Furthermore K will be equipped with a (possibly trivial) smooth automorphism τ, also called 'twist', and its Lie algebra $\mathfrak{k}$ with a scalar product $\langle\cdot\,|\,\cdot\rangle$ which is invariant for K and τ. When A is a subset of a free abelian group $\mathcal{X}$, we will use $\langle A\rangle_{\mathbb{Z}}$ for the smallest subgroup of $\mathcal{X}$ containing A and when $A=\{a_1,a_2,\ldots,a_n\}$ is a finite set, we will also use $\langle a_1,a_2,\ldots,a_n\rangle_{\mathbb{Z}}$ for this group. When D is a subset of a real vector space V, we will use $\operatorname{cone}(D)$ for the closed convex cone generated by D in V.

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2. Prerequisites and main result

In this section we briefly recall, mostly following [15], the necessary notions to state both Theorem 2.10, which is the special case of Knop's classification theorem [15, Corollary 2.6.2] that we will use, and Theorem 2.12, which is our main result.

Although it will not play a direct role in what follows, we begin by giving, for completeness, the definition of a quasi-Hamiltonian $K\tau$-manifold, following [15, Definition 2.1.2].

Definition 2.1. A *quasi-Hamiltonian $K\tau$-manifold* is a smooth K-manifold M equipped with a K-invariant 2-form ω and a K-equivariant smooth map $m: M\to K\tau$, called the (group valued) *momentum map*, such that

(1) $d\omega=-m^*\chi$,

(2) $\omega(\xi x,\eta)=\langle\xi\,|\,m^*\theta_\tau(\eta)\rangle$ for all $\xi\in\mathfrak{k}, x\in M$ and $\eta\in T_xM$, and

(3) $\ker\omega_x=\{\xi x\in T_xM:\xi\in\mathfrak{k}$ with $\operatorname{Ad} m(x)({}^\tau\xi)+\xi=0\}$,

where $\theta_\tau:=\frac{1}{2}({}^{\tau^{-1}}\theta^L+\theta^R)$ with θ^L,θ^R being the left- and right-invariant Maurer-Cartan-forms on K and

$$\chi:=\frac{1}{12}\left\langle\theta^L\,\middle|\,[\theta^L,\theta^L]\right\rangle=\frac{1}{12}\left\langle\theta^R\,\middle|\,[\theta^R,\theta^R]\right\rangle$$

is the canonical biinvariant closed 3-form on K with respect to the chosen scalar product $\langle\cdot\,|\,\cdot\rangle$ on $\mathfrak{k}$.

We move on to affine root systems, extracting from [15, Section 1.1], which is based on [20] and [21], what we will need. Let $\bar{\mathfrak{a}}$ be a Euclidean vector space with inner product $\langle\cdot\,|\,\cdot\rangle$ and associated affine space $\mathfrak{a}$. We denote by $L(\mathfrak{a})$ the set of affine linear functions on $\mathfrak{a}$. The gradient of $\alpha \in L(\mathfrak{a})$ is denoted by $\overline{\alpha} \in \bar{\mathfrak{a}}$ and is characterized by the property

$$\alpha(x+t) = \alpha(x) + \langle \overline{\alpha}\,|\,t\rangle\,, \quad \text{for all } x \in \mathfrak{a}, t \in \bar{\mathfrak{a}}. \tag{2}$$

If $\alpha \in L(\mathfrak{a})$ is a non-constant affine linear function, we define the *reflection*

$$s_\alpha : \mathfrak{a} \to \mathfrak{a} \ \text{ by } \ s_\alpha(x) := x - \alpha(x)\overline{\alpha}^\vee$$

where

$$\overline{\alpha}^\vee := \frac{2}{\langle\overline{\alpha}\,|\,\overline{\alpha}\rangle}\,\overline{\alpha} \in \bar{\mathfrak{a}}. \tag{3}$$

Its induced action on an affine linear function $\beta \in L(\mathfrak{a})$ is:

$$s_\alpha(\beta) = \beta - \left\langle \overline{\beta}\,\middle|\,\overline{\alpha}^\vee \right\rangle \alpha.$$

Definition 2.2. A *(reduced) affine root system* on $\mathfrak{a}$ is a set $\Phi \subset L(\mathfrak{a}) \setminus \mathbb{R}$ of non-constant affine linear functions such that:

(a) $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,
(b) $\left\langle \overline{\beta}\,\middle|\,\overline{\alpha}^\vee \right\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$,
(c) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$, and
(d) $\overline{\Phi} := \{\overline{\alpha} \in \bar{\mathfrak{a}} : \alpha \in \Phi\}$ is finite.

Every $\alpha \in L(\mathfrak{a}) \setminus \mathbb{R}$ defines the affine hyperplane

$$H_\alpha := \{x \in \mathfrak{a} : \alpha(x) = 0\}.$$

An *alcove* of Φ is the closure of a connected component of $\mathfrak{a} \setminus \cup_{\alpha\in\Phi} H_\alpha$. The Weyl group W_Φ of Φ, which is the subgroup generated by $\{s_\alpha : \alpha \in \Phi\}$ in the group of isometries of $\mathfrak{a}$, acts simply transitively on the set of alcoves of Φ. Each such alcove $\mathcal{A}$ is a fundamental domain for the action of W_Φ on $\mathfrak{a}$.

Put $\overline{\Phi} := \{\overline{\alpha} : \alpha \in \Phi\}$ and $\overline{\Phi}^\vee := \{\overline{\alpha}^\vee : \alpha \in \Phi\}$. These are possibly non-reduced finite root systems on $\bar{\mathfrak{a}}$.

Definition 2.3. An *integral root system* on $\mathfrak{a}$ is a pair (Φ, Ξ) where $\Phi \subset L(\mathfrak{a})$ is an affine root system and $\Xi \subseteq \bar{\mathfrak{a}}$ is a lattice with $\overline{\Phi} \subseteq \Xi$ and $\langle \Xi\,|\,\overline{\Phi}^\vee\rangle \subseteq \mathbb{Z}$.

Recall that K is assumed to be simply connected. It is known that the twisted conjugacy classes in K are in bijection with an alcove $\mathcal{A}$ of an affine root system that is determined by K and τ, cf. [29, 23, 22]. We give the description of the twisted conjugacy classes from [15, Section 2.2]. To do so, we first recall that if $T \subset K$ is a maximal torus, then its character group $\Xi(T)$ can (and will) be identified with a lattice in $\mathfrak{t} = \mathrm{Lie}(T)$ via the map

$$\Xi(T) \to \mathfrak{t} : \chi \mapsto a_\chi,$$

where a_χ is the unique element of $\mathfrak{t}$ such that

$$\chi(\exp\xi) = e^{2\pi i\langle a_\chi\,|\,\xi\rangle} \quad \text{for all } \xi \in \mathfrak{t}$$

and $\langle\cdot\,|\,\cdot\rangle$ is the chosen scalar product on $\mathfrak{k}$.

Consequently we also view the (finite) root system $\overline{\Phi}(\mathfrak{k},\mathfrak{t})$ of K as a subset of $\mathfrak{t}$. In what follows, we will slightly abuse notation and no longer distinguish between $\mathfrak{a}$ and $\overline{\mathfrak{a}}$.

Theorem 2.4 ([15, Theorem 2.2.1 and Remark 2.2.2])**.** *Let τ be an automorphism of the simply connected compact Lie group K. Then there exists a τ-stable maximal torus T in K and an integral root system (Φ_τ,Λ_τ) on the τ-fixed part $\mathfrak{a}:=\mathfrak{t}^\tau$ of $\mathfrak{t}$, with the following properties:*

(a) *If $\mathrm{pr}_{\mathfrak{a}}:\mathfrak{t}\to\mathfrak{a}$ is the orthogonal projection, then $\overline{\Phi}_\tau=\mathrm{pr}_{\mathfrak{a}}\overline{\Phi}(\mathfrak{k},\mathfrak{t})$ and $\Lambda_\tau=\mathrm{pr}_{\mathfrak{a}}\Xi(T)$.*

(b) *The lattice Λ_τ is the weight lattice of $\overline{\Phi}_\tau$, that is*
$$\Lambda_\tau=\{\lambda\in\mathfrak{a}:\langle\lambda\,|\,\overline{\alpha}^\vee\rangle\in\mathbb{Z}\ \textit{for all}\ \alpha\in\Phi_\tau\}.$$

(c) *If $\mathcal{A}\subset\mathfrak{a}$ is an alcove of Φ_τ, then the composition $c:\mathcal{A}\subset\mathfrak{a}\xrightarrow{\exp}K\to K\tau/K$ is a homeomorphism.*

(d) *If $\mathcal{A}\subset\mathfrak{a}$ is an alcove of Φ_τ and for every $a\in\mathcal{A}$ we set*
$$K_{a\tau}:=\{k\in K:k\cdot_\tau\exp(a)=\exp(a)\}$$
and
$$\Phi_\tau(a):=\{\alpha\in\Phi_\tau:\alpha(a)=0\}.$$
then $K_{a\tau}$ is a closed connected subgroup of K with maximal torus $\exp(\mathfrak{a})$ *and integral root system $\left(\overline{\Phi_\tau(a)},\Lambda_\tau\right)$.*

Moreover, the quadruple $(T,\mathfrak{a},\Phi_\tau,\Lambda_\tau)$ is uniquely determined by K and τ up to conjugation by the subgroup K^τ of τ-fixed points in K.

Remark 2.5 (see [15, Remark 2.2.4])**.** Using standard arguments (like those in the proof of [15, Theorem 2.2.1], for example) one shows that

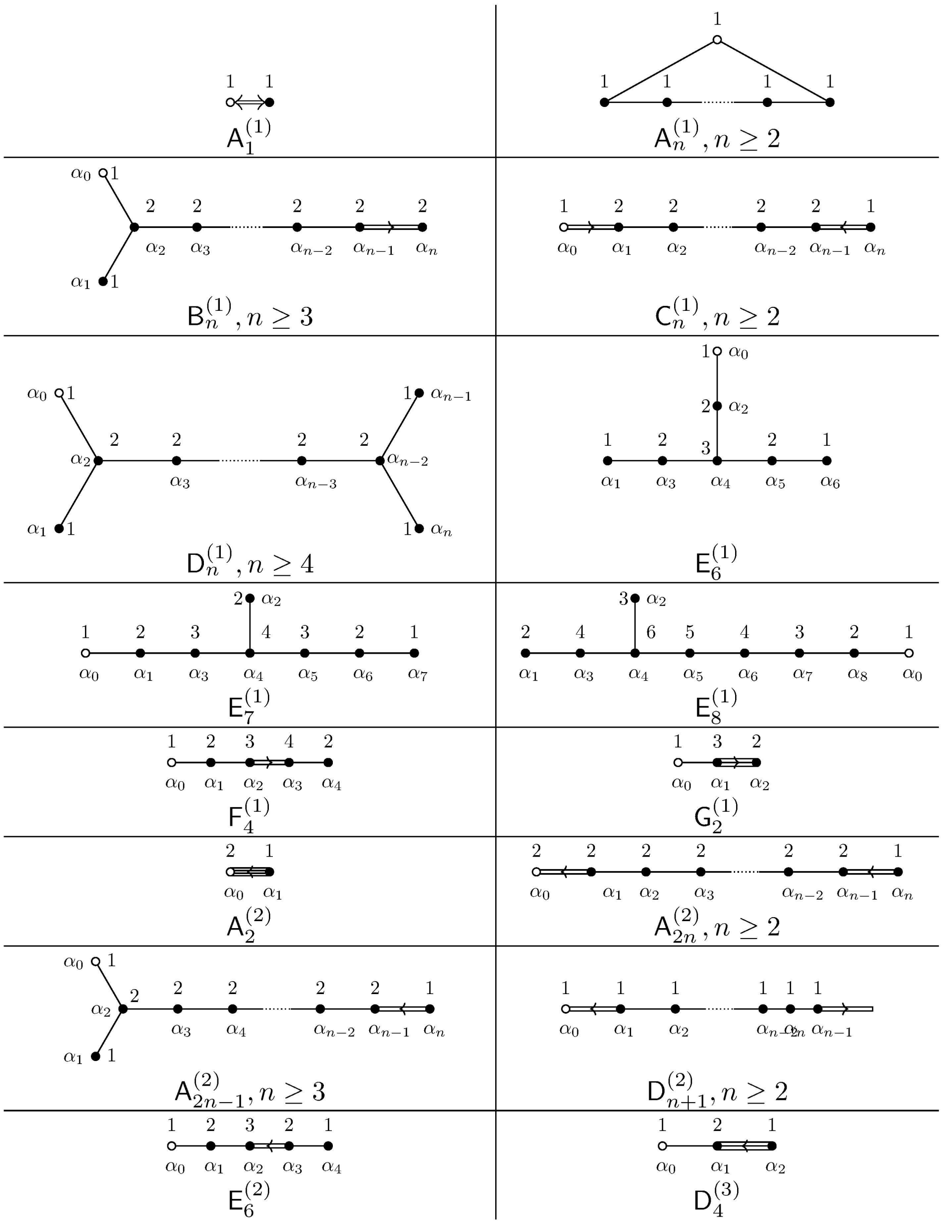
(a) The type of the root system Φ_τ in Theorem 2.4 only depends on the image $\overline{\tau}$ of τ in the group of outer automorphisms of K;

(b) If K is simple then Φ_τ is the irreducible affine root system of type $\mathsf{X}_n^{(r)}$ in Table 1, where X_n is the Dynkin type of K and $r\in\{1,2,3\}$ is the order of $\overline{\tau}$.

Suppose now, that M is a compact connected quasi-Hamiltonian $K\tau$-manifold, with momentum map m. Fixing an alcove $\mathcal{A}$ and a homeomorphism c as in Theorem 2.4, one defines the so-called *invariant momentum map* $m_+:M\to\mathcal{A}$ as in Equation (1). Recall from the introduction that its image
$$\mathcal{P}_M:=m_+(M)\subset\mathcal{A}$$
is a convex polytope, called the *momentum polytope* of M.

The second invariant used in Knop's classifcation theorem of compact connected *multiplicity free* quasi-Hamiltonian $K\tau$-manifolds is a subgroup of $\mathfrak{a}$ which encodes the principal isotropy group of the K-action on M. We introduce it following [15, Section 2.3]. By Theorem 2.4, the isotropy group $K_{a\tau}$ is the same subgroup of K for all a in the relative interior $\mathcal{P}_M^0$ of $\mathcal{P}_M$. Let's call this group L_M. Then the

Table 1. The Dynkin diagrams of the reduced and irreducible affine root systems, with Dynkin labels as given in [12, Theorem 4.8]. The Dynkin labels will play a role in Section 4.



quotient of L_M by the kernel L'_M of its action on $m_+^{-1}(\mathcal{P}_M^0)$ is a torus which we call A_M. Furthermore L'_M is the principal isotropy group of the K-action on M and it is encoded by the character group Λ_M of A_M, which we call the *lattice of* M. Because $\exp(\mathfrak{a})$ is a maximal torus of L_M, the quotient map $L_M \to A_M$ restricts to a surjective homomorphism $\exp(\mathfrak{a}) \to A_M$. Consequently, the lattice Λ_M can and will be viewed as subgroup of Λ_τ, which is itself a subgroup of $\mathfrak{a}$.

An immediate consequence of Definition 1.1 is that the momentum polytope of a quasi-Hamiltonian model space M is the alcove $\mathcal{A}$, so that the only relevant invariant is Λ_M. In order to characterize the lattices of quasi-Hamiltonian model spaces we need to make a few more recollections. Let G be a complex connected reductive group and let $B \subset G$ be a Borel subgroup. Write P_+ for the subset of

$\mathrm{Hom}(B, \mathbb{C}^\times)$ of dominant weights of G (with respect to B). Recall that highest weight theory gives us a one-to-one correspondence $\lambda \to V(\lambda)$ between P_+ and the set of isomorphism classes of irreducible representations of G. If G acts on a variety X, then G acts linearly on the ring $\mathbb{C}[X]$ of regular functions $X \to \mathbb{C}$.

Definition 2.6. A normal G-variety X is called *spherical* if it contains a dense orbit of the Borel subgroup B of G. The *weight monoid* $\Gamma(X)$ of an irreducible affine G-variety is the set of B-weights of B-eigenvectors in $\mathbb{C}[X]$, that is,

$$\Gamma(X) := \{\lambda \in P_+ : \mathrm{Hom}_G(V(\lambda), \mathbb{C}[X]) \neq 0.\}.$$

Remark 2.7. A well-known result due to Vinberg and Kimel'fel'd [28] says that an irreducible affine G-variety X contains a dense B-orbit if and only if every irreducible representation of G occurs at most once in $\mathbb{C}[X]$. In particular, the weight monoid $\Gamma(X)$ of an affine spherical G-variety X describes $\mathbb{C}[X]$ as a representation of G:

$$\mathbb{C}[X] \cong \oplus_{\lambda \in \Gamma(X)} V(\lambda).$$

Next we define the subgroups of $\mathfrak{a}$ that can occur as lattices of quasi-Hamiltonian model $K\tau$-spaces. First recall that for every $a \in \mathcal{A}$, the subgroup $K_{a\tau}$ has $\exp(\mathfrak{a})$ as a maximal torus, whose character group is $\Lambda_\tau \subset \mathfrak{a}$. The weight lattice of the complexification $K_{a\tau}^{\mathbb{C}}$ of $K_{a\tau}$ can naturally be identified with the weight lattice Λ_τ of $K_{a\tau}$. Furthermore $\mathrm{cone}(\mathcal{A} - a) \subset \mathfrak{a}$ is a Weyl chamber for $K_{a\tau}$ and thus determines a Borel subgroup of $K_{a\tau}^{\mathbb{C}}$ with respect to which $\mathrm{cone}(\mathcal{A} - a) \cap \Lambda_\tau$ is the set of dominant weights. When X is a smooth affine spherical $K_{a\tau}^{\mathbb{C}}$-variety, we define its weight monoid $\Gamma(X) \subset \mathrm{cone}(\mathcal{A} - a) \cap \Lambda_\tau$ with respect to this Borel subgroup and view it as a subset of $\mathfrak{a}$.

Definition 2.8. Let $\mathfrak{a} = \mathfrak{t}^\tau$ and $\mathcal{A} \subset \mathfrak{a}$ be as in Theorem 2.4. Let Λ be a subgroup of $\mathfrak{a}$. We will say that Λ is *$K\tau$-admissible* if for every vertex a of $\mathcal{A}$ there exists a smooth affine spherical $K_{a\tau}^{\mathbb{C}}$-variety whose weight monoid Γ_a satisfies $\mathrm{cone}(\Gamma_a) = \mathrm{cone}(\mathcal{A} - a)$ and $\langle \Gamma_a \rangle_{\mathbb{Z}} = \Lambda$.

Remark 2.9. (a) A subgroup Λ of $\mathfrak{a}$ is $K\tau$-admissible if and only if, in the parlance of [15, Definition 2.5.1], $(\mathcal{A}, \Lambda)$ is a *spherical pair.*

(b) If Λ is $K\tau$-admissible, then the weight monoids Γ_a as in Definition 2.8 are uniquely determined by Λ; see Remark 3.2 below. Indeed, $\Gamma_a = \mathrm{cone}(\mathcal{A} - a) \cap \Lambda$.

(c) Note that it follows from Definition 2.8 that if Λ is a $K\tau$-admissible subgroup of $\mathfrak{a}$, then Λ is a *lattice (of full rank)* in the vector space $\mathfrak{a}$: it is a finitely generated free abelian subgroup of $(\mathfrak{a}, +)$ and $\mathrm{rank}(\Lambda) = \dim_{\mathbb{R}}(\mathfrak{a})$.

Here is the announced specialization of [15, Corollary 2.6.2].

Theorem 2.10 (Knop)**.** *Let τ be an automorphism of the compact and simply connected Lie group K and let $\mathfrak{a} = \mathfrak{t}^\tau$. The map $M \mapsto \Lambda_M$ yields a bijection between the set of isomorphism classes of quasi-Hamiltonian $K\tau$-model spaces and the set of $K\tau$-admissible subgroups of $\mathfrak{a}$.*

Remark 2.11. It follows from Definition 2.4.1 and Proposition 2.4.2 in [15] that the dimension of a quasi-Hamiltonian model $K\tau$-space is equal to $\dim_{\mathbb{R}} K + \dim_{\mathbb{R}} \mathfrak{a}$.

We need a bit more notation before we can state our main result. Let S_τ be the set of simple roots of Φ_τ corresponding to the choice of alcove $\mathcal{A}$, that is, $\alpha \in \Phi_\tau$ belongs to S_τ if and only if the affine hyperplane H_α is a wall of $\mathcal{A}$ and $\alpha(a) \geq 0$ for all $a \in \mathcal{A}$. For K simple, we number the simple roots in $S_\tau = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ as in the Dynkin diagram $\mathsf{X}_n^{(r)}$ in Table 1 corresponding to Φ_τ.

Here is the main result of this paper. The proof will be given in Section 4.

Theorem 2.12. *Let K be a simple and simply connected compact Lie group and τ a smooth automorphism of K. Let $T, \mathfrak{a}, \Lambda_\tau, \Phi_\tau$ and $\mathcal{A}$ be as in Theorem 2.4 and number the simple roots S_τ of Φ_τ as in Table 1. Finally, let $\operatorname{ord}(\overline{\tau})$ be the order of the image of τ in the group of outer automorphisms of K.*

Then the map $M \mapsto \Lambda_M$ yields a bijection between the set of isomorphism classes of quasi-Hamiltonian model $K\tau$-spaces and the subgroups Λ of $\mathfrak{a}$ in the following table:

	K	$\operatorname{ord}(\overline{\tau})$	*$K\tau$-admissible subgroups Λ of $\mathfrak{a}$.*
1	*any (K,τ), except $(\mathrm{SU}(2n+1),\sigma)$ with $n \geq 1$ and $\operatorname{ord}(\overline{\sigma}) = 2$*		*any subgroup Λ of $\mathfrak{a}$ with $\{2\overline{\alpha}_1, 2\overline{\alpha}_2, \ldots, 2\overline{\alpha}_n\} \subset \Lambda \subset 2\Lambda_\tau$*
2	$\mathrm{SU}(n+1), n \geq 2$ *even*	*1*	*any subgroup Λ of $\mathfrak{a}$ with $\{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n\} \subset \Lambda \subset \Lambda_\tau$*
3	$\mathrm{SU}(n+1), n \geq 1$ *odd*	*1*	$\langle \overline{\alpha}_2 + \overline{\alpha}_3, \overline{\alpha}_3 + \overline{\alpha}_4, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}$ *with* $r, e \in \mathbb{Z}_{\geq 0}$, $e \mid \frac{n+1}{2}$, $0 \leq r \leq e-1$, *where* ω_{n-1}, $\omega_n \in \mathfrak{a}$ *are defined by* $\langle \omega_k \mid \overline{\alpha}_j^\vee \rangle = \delta_{kj}$ *for all* $k \in \{n-1, n\}, j \in \{1, 2, \ldots, n\}$.
4	$\mathrm{Sp}(2n), n \geq 2$	*1*	Λ_τ
5	$\mathrm{SU}(5)$	*2*	$\langle \overline{\alpha}_1, \overline{\alpha}_2 \rangle_{\mathbb{Z}}$
6	$\mathrm{SU}(2n+1), n \geq 1$	*2*	Λ_τ
7	$\mathrm{SU}(2n+1), n \geq 1$	*2*	$2\Lambda_\tau$
8	$\mathrm{Spin}(2n+2), n \geq 2$	*2*	$\langle \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n \rangle_{\mathbb{Z}}$
9	$\mathrm{Spin}(2n+2)$, $n \geq 3$ *odd*	*2*	$\langle \overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_2 + \overline{\alpha}_3, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, 2\overline{\alpha}_n \rangle_{\mathbb{Z}}$

Remark 2.13. We keep the notations from Theorem 2.12 in this remark.

(a) When $\operatorname{ord}(\overline{\tau}) = 1$, Λ_τ is simply the weight lattice $\Xi(T)$ of K and $\{\overline{\alpha}_1, \overline{\alpha}_2, ..., \overline{\alpha}_n\}$ is a set of simple roots of K. This claim about Λ_τ holds because $\overline{\alpha}_0^\vee$ is an integral linear combination of $\overline{\alpha}_1^\vee, \overline{\alpha}_2^\vee, \ldots, \overline{\alpha}_n^\vee$, which holds, for example, because $-\overline{\alpha}_0$ is the highest root in the root system $\overline{\Phi}(\mathfrak{k}, \mathfrak{t})$ of K and the coroots of the simple roots form a basis of the dual root system.

(b) More generally one can check for each irreducible affine root system in Table 1 that Λ_τ is the weight lattice of the root system $\overline{\Phi_\tau(\mathsf{v}_0)}$ of $K_{\mathsf{v}_0\tau}$; see Remark 3.4.

(c) The lattices Λ in cases (1) and (2) are in natural bijective correspondence with the subgroups of the (finite) quotient $\Lambda_\tau / \langle \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n \rangle_{\mathbb{Z}}$. For each irreducible finite root system, this quotient group is given in [6, Planches I-IX].

(d) The following cases in Theorem 2.12 had already been found in [14, Theorem 11.4], see also [15, Proposition 2.7.3]: (4), (6), (3) with $e = 1, r = 0$ and (2) with $\Lambda = \Lambda_\tau = \Xi(T)$.

(e) As is well known, and can be read in [6, Planche I], the weights ω_{n-1} and ω_n used to describe the lattices Λ in case (3) of Theorem 2.12 can be expressed as (rational) linear combinations of the simple roots $\overline{\alpha}_1, \overline{\alpha}_2, \dots, \overline{\alpha}_n$. We have not found an elegant basis of Λ in terms of these simple roots. Note that for $e = 1$, we have $\Lambda = \Lambda_\tau = \Xi(T)$. In particular $\Xi(T) = \langle \omega_1 \rangle_{\mathbb{Z}}$ is the only $K\tau$-admissible subgroup of $\mathfrak{a}$ in case (3) of Theorem 2.12 with $n = 1$.

We illustrate Definition 2.8, Theorem 2.10 and Theorem 2.12 with two low-dimensional examples.

Example 2.14. (a) The spinning 4-sphere of [3, Appendix A]: It is shown in *loc.cit.* that the unit 4-sphere $M = S^4 \subset \mathbb{R}^5$, equipped with the action of $K = \mathrm{SU}(2)$ obtained by restricting the linear K-action on $\mathbb{R}^5 \cong \mathbb{C}^2 \oplus \mathbb{R}$, where K acts on the first factor $\mathbb{C}^2$ via the standard representation and trivially on the second factor $\mathbb{R}$, can be given the structure of a multiplicity free quasi-Hamiltonian $K\tau$-manifold with $\tau = \mathrm{id}_K$. In fact, its momentum map is surjective and therefore it is a model $K\tau$-space. As pointed out in [15, page 515], the corresponding $K\tau$-admissible subgroup Λ_M of $\mathfrak{a} = \mathfrak{t}$ is the weight lattice $P \cong \mathbb{Z}$ of $K = \mathrm{SU}(2)$. That P is indeed $K\tau$-admissible can be seen as follows. In this case, the root system Φ_τ provided by Theorem 2.4 is of type $\mathsf{A}_1^{(1)}$ with simple roots α_0, α_1 and $\mathcal{A}$ is an interval. Let v_0 be the endpoint of $\mathcal{A}$ with $\alpha_1(\mathsf{v}_0) = 0$ and v_1 the endpoint with $\alpha_0(\mathsf{v}_1) = 0$. Defining $\omega \in \mathfrak{a}$ by $\langle \overline{\alpha}_1^\vee \,|\, \omega \rangle = 1$, we have $P = \mathbb{Z}\omega$, $\mathrm{cone}(\mathcal{A} - \mathsf{v}_0) = \mathbb{R}_{\geq 0}\omega$ and $\mathrm{cone}(\mathcal{A} - \mathsf{v}_1) = \mathbb{R}_{\geq 0}(-\omega)$. Moreover $K^{\mathbb{C}}_{\mathsf{v}_0\tau} = K^{\mathbb{C}}_{\mathsf{v}_1\tau} = K^{\mathbb{C}} \cong \mathrm{SL}(2, \mathbb{C})$ and the Borel subgroup of $K^{\mathbb{C}}_{\mathsf{v}_0\tau}$, which corresponds to the Weyl chamber $\mathrm{cone}(\mathcal{A} - \mathsf{v}_0) \subset \mathfrak{a}$, is opposite to the Borel subgroup of $K^{\mathbb{C}}_{\mathsf{v}_1\tau}$, which corresponds to the Weyl chamber $\mathrm{cone}(\mathcal{A} - \mathsf{v}_1) \subset \mathfrak{a}$. Equipped with the standard linear action of $\mathrm{SL}(2, \mathbb{C})$, the smooth affine variety $\mathbb{C}^2$ is spherical. As a $K^{\mathbb{C}}_{\mathsf{v}_0\tau}$-variety its weight monoid Γ_{v_0} is $\mathbb{N}\omega$, whereas $\Gamma_{\mathsf{v}_1} = \mathbb{N}(-\omega)$, and we have checked that P is $K\tau$-admissible. This model space corresponds to case (3) in Theorem 2.12 with $n = 1, e = 1, r = 0$.

(b) $K = \mathrm{Spin}(8)$ with the triality automorphism τ: Here Φ_τ is a root system of type $\mathsf{D}_4^{(3)}$ with simple roots $\alpha_0, \alpha_1, \alpha_2$ and $\mathcal{A}$ is a triangle. For each $j \in \{0, 1, 2\}$, we let v_j be the vertex of $\mathcal{A}$ where $\alpha_k(\mathsf{v}_j) = 0$ for all $k \in \{0, 1, 2\} \setminus \{j\}$. Then $K^{\mathbb{C}}_{\mathsf{v}_0\tau}$ is the complex connected group of type G_2 and Λ_τ is its weight lattice. Furthermore $K^{\mathbb{C}}_{\mathsf{v}_1\tau} \cong (\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}))/\Delta Z$, where $\Delta Z = \left\{e, \left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right)\right\}$, and $K^{\mathbb{C}}_{\mathsf{v}_2\tau} \cong \mathrm{PGL}(3)$. As $\Lambda_\tau = \langle \overline{S}(\mathsf{v}_0) \rangle_{\mathbb{Z}}$ for this root system, it follows from Corollary 4.3 below, that $2\Lambda_\tau$ is the only $K\tau$-admissible subgroup of $\mathfrak{a}$. This model space is part of case (1) in Theorem 2.12. We sketch a direct argument that $2\Lambda_\tau$ is indeed $K\tau$-admissible. First, let Γ_{v_0} be the weight monoid of the $K^{\mathbb{C}}_{\mathsf{v}_0\tau}$-variety $\mathsf{G}_2/(\mathsf{A}_1 \times \mathsf{A}_1)$. To define Γ_{v_1} we first recall that if H is a maximal torus of $\mathrm{SL}(2, \mathbb{C})$, then its normalizer $N(H)$ in $\mathrm{SL}(2, \mathbb{C})$ has a unique nontrivial character $c : N(H) \to \mathbb{C}^\times$. This follows from the fact that any character of $N(H)$ is trivial on H, which holds because each $t \in H$ is conjugate to t^{-1} in $N(H)$, and from the fact that $|N(H)/H| = 2$. We let $K \subset N(H) \times N(H)$ be the kernel of cc', where c' is the nontrivial character of the second factor of $N(H) \times N(H)$, and Γ_{v_1} the weight monoid of the $K^{\mathbb{C}}_{\mathsf{v}_1\tau}$-variety

$(\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C}))/K$. Finally, we let Γ_{v_2} be the weight monoid of the $K^{\mathbb{C}}_{\mathsf{v}_2\tau}$-variety $\mathrm{PGL}(3,\mathbb{C})/L$, where L is the image of $\mathrm{SO}(3,\mathbb{C})$ under the quotient map $\mathrm{SL}(3,\mathbb{C})\twoheadrightarrow\mathrm{PGL}(3,\mathbb{C})$. It now follows from [26, Tables 2 and 3] and [8, Lemme 4.3] that $\langle\Gamma_{\mathsf{v}_i}\rangle_{\mathbb{Z}} = 2\Lambda_\tau$ for all $i\in\{0,1,2\}$, which proves the claim that $2\Lambda_\tau$ is $K\tau$-admissible.

3. *G*-adapted lattices

Let K be simple and simply connected. If Λ is a subgroup of $\mathfrak{a}$ that is $K\tau$-admissible and v is a vertex of $\mathcal{A}$, then there exists a smooth affine spherical $K^{\mathbb{C}}_{\mathsf{v}\tau}$-variety whose weight monoid Γ_a satisfies $\langle\Gamma_a\rangle_{\mathbb{Z}}=\Lambda$ and $\mathrm{cone}(\Gamma_a)=\mathrm{cone}(\mathcal{A}-\mathsf{v})$. The first ingredient in the proof in Section 4 of our classification of quasi-Hamiltonian model spaces is Proposition 3.6, which was obtained in [26, Section 3] and provides, up to replacing $K^{\mathbb{C}}_{\mathsf{v}_0\tau}$ by its simply connected covering group $G(\mathsf{v}_0)$, all the lattices that satisfy the condition for being $K\tau$-admissible at the vertex v_0 of $\mathcal{A}$ corresponding to the node α_0 in the Dynkin diagram of Φ_τ (the vertex v_0 is defined in Equation (6)). It will then remain to check which of these lattices verify the condition at every vertex of $\mathcal{A}$. When we do this in Section 4, we will make use of Lemma 3.3 and Proposition 3.10. The latter is a special case of the combinatorial characterization of the weight monoids of smooth affine spherical varieties due to G. Pezzini and the second author, see [27]. This section also contains the necessary preliminaries to state Proposition 3.10.

For the remainder of this section, G is a complex connected reductive group, B a chosen Borel subgroup, H a chosen maximal torus in B and

$$P := \mathrm{Hom}(B,\mathbb{C}^\times)\equiv\mathrm{Hom}(H,\mathbb{C}^\times)$$

the weight lattice of G. Furthermore $\overline{S}$ is the set of simple roots of G and P_+ the subset of dominant weights in P with respect to B and H. Whenever necessary, we number the simple roots $\overline{\alpha}_1,\overline{\alpha}_2,\ldots\in\overline{S}$ and the fundamental weights $\omega_1,\omega_2,\ldots\in P$ as in [6, Planches I–IX].

Definition 3.1. Let Ξ be a subgroup of P. We say that Ξ is G-*adapted* if there exists a smooth affine spherical G-variety whose weight monoid Γ satisfies

$$\langle\Gamma\rangle_{\mathbb{Z}}=\Xi,\ \text{and} \tag{4}$$

$$\mathrm{cone}(\Gamma)=\mathrm{cone}(P_+)\ \text{in}\ P\otimes_{\mathbb{Z}}\mathbb{R} \tag{5}$$

Remark 3.2. (a) Because a smooth affine spherical variety is normal, its weight monoid Γ satisfies the equality $\Gamma=\mathrm{cone}(\Gamma)\cap\langle\Gamma\rangle_{\mathbb{Z}}$ in $P\otimes_{\mathbb{Z}}\mathbb{R}$. This means in particular that if Ξ is G-adapted then there is only one monoid Γ for which Equations (4) and (5) hold, namely $\Gamma=\Xi\cap\mathrm{cone}(P_+)$.

(b) Furthermore, thanks to a theorem of I. Losev's [18, Theorem 1.3], a smooth affine spherical G-variety X is uniquely determined by its weight monoid (up to G-equivariant isomorphism).

Part (c) of the following lemma allows us to "ignore" the lattice Λ_τ when determining whether a subgroup of $\mathfrak{a}$ is $K\tau$-admissible, making it a purely local problem at every vertex of $\mathcal{A}$.

Lemma 3.3. *We make the same assumptions as in Theorem* 2.12*. For each vertex* v *of* $\mathcal{A}$ *we let* $P_{\mathsf{v}} \subset \mathfrak{a}$ *be the weight lattice of the root system* $\overline{\Phi_\tau(\mathsf{v})}$*. Then the following hold:*

(a) $\bigcap_{\mathsf{v}} P_{\mathsf{v}} = \Lambda_\tau$*, where the intersection is over all vertices* v *of* $\mathcal{A}$*;*

(b) $(\overline{\Phi_\tau(\mathsf{v})}, P_{\mathsf{v}})$ *is the integral root system of the simply connected covering group of* $K^{\mathbb{C}}_{\mathsf{v}\tau}$*, which we will denote by* $G(\mathsf{v})$*;*

(c) *A subgroup* Ξ *of* $\mathfrak{a}$ *is* $K\tau$*-admissible if and only if* Ξ *is* $G(\mathsf{v})$*-adapted for every vertex* v *of* $\mathcal{A}$*.*

Proof. Assertion (a) is essentially a restatement of part (b) of Theorem 2.4. Assertion (b) is a standard fact of Lie theory. We come to assertion (c). The "only if" statement holds because if X is a smooth affine spherical $K^{\mathbb{C}}_{\mathsf{v}\tau}$-variety, then the action lifts to $G(\mathsf{v})$. The "if" statement is true because if Ξ is $G(\mathsf{v})$-adapted at every vertex v of $\mathcal{A}$, then Ξ lies in P_{v} for every v. By (a) it then follows that $\Xi \subset \Lambda_\tau$. This implies that at each vertex v, the $G(\mathsf{v})$-action on the smooth affine spherical $G(\mathsf{v})$-variety X_{v} associated to v factors through $K^{\mathbb{C}}_{\mathsf{v}\tau}$, as follows by applying highest weight theory to $\mathbb{C}[X_{\mathsf{v}}]$. ■

Remark 3.4. We keep the notations of Lemma 3.3. Using the numbering of the simple roots $S_\tau = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ of Φ_τ as in Table 1, we define the vertex v_k of $\mathcal{A}$ by

$$\{\mathsf{v}_k\} := \{a \in A : \alpha(a) = 0 \text{ for all } \alpha \in S_\tau \setminus \{\alpha_k\}\} \tag{6}$$

for each $k \in \{0, 1, \ldots, n\}$. We will then show in Lemma 4.1(a) that $P_{\mathsf{v}_0} \subset P_{\mathsf{v}_k}$ for all $k \in \{0, 1, ..., n\}$. Because every vertex of $\mathcal{A}$ is of the form v_k for some $k \in \{0, 1, ..., n\}$, Lemma 3.3(a) then yields the following helpful formula for computing Λ_τ:

$$\Lambda_\tau = P_{\mathsf{v}_0}. \tag{7}$$

Part (d) of the next lemma gives a different description of G-adapted lattices. We will say that a subgroup Ξ of P has *full rank* if $\operatorname{rank}(\Xi) = \operatorname{rank}(P)$. Furthermore, we recall that a submonoid Γ of P_+ is called G*-saturated* if $\langle \Gamma \rangle_{\mathbb{Z}} \cap P_+ = \Gamma$.

Lemma 3.5. (a) *If* Ξ *is a subgroup of* P*, then* $\Xi \cap P_+ = \Xi \cap \operatorname{cone}(P_+)$ *(as subsets of* $P \otimes_{\mathbb{Z}} \mathbb{R}$*).*

(b) *If* Ξ *is a subgroup of* P *of full rank, then* $\operatorname{cone}(\Xi \cap P_+) = \operatorname{cone}(P_+)$*.*

(c) *The map* $\Xi \mapsto \Xi \cap P_+$ *is a bijection from the set*

$$\{\Xi : \Xi \textit{ is a subgroup of full rank of } P\}$$

to the set $\{\Gamma : \Gamma$ *is a* G*-saturated submonoid of* P_+ *with* $\langle \Gamma \rangle_{\mathbb{Z}}$ *of full rank*$\}$ *with inverse map* $\Gamma \mapsto \langle \Gamma \rangle_{\mathbb{Z}}$*.*

(d) *A subgroup* Ξ *of* P *is* G*-adapted if and only if* Ξ *is of full rank and* $\Xi \cap P_+$ *is the weight monoid of a smooth affine spherical* G*-variety.*

Proof. Assertion (a) follows from the well-known fact that $\operatorname{cone}(P_+) \cap P = P_+$. Assertion (b) holds because every extremal ray of the convex polyhedral cone $\operatorname{cone}(P_+)$ contains an element of P_+ and, since Ξ has finite index in P, also an element of Ξ. Part (c) is a consequence of the fact that when Ξ is a subgroup of full rank of P, then

$$\langle \Xi \cap P_+ \rangle_{\mathbb{Z}} = \Xi. \tag{8}$$

Equation (8) in turn can be shown with essentially the proof of [24, Prop. 1.1(iii)]. We turn to assertion (d) and begin with the "only if" statement. Suppose that Ξ is G-adapted. Since P_+ spans $P \otimes_{\mathbb{Z}} \mathbb{R}$ as a vector space, it follows from Equation (5) that Ξ has full rank. Furthermore, it follows from Remark 3.2(a) that $\Xi \cap P_+$ is the weight monoid of a smooth affine spherical G-variety. The "if" statement holds by Equation (8) and assertion (b). ■

Proposition 3.6 below summarizes Propositions 3.7 and 3.16 of [26]. Note that these two propositions in *loc.cit.* are stated in terms of G-saturated submonoids of P_+ of full rank and that parts (c) and (d) of Lemma 3.5 show that Proposition 3.6 just uses different terminology for the same information. We'll make use of the following notation:

$$2\overline{S} := \{2\overline{\alpha} : \overline{\alpha} \in \overline{S}\} \quad \text{and} \quad \overline{S}^+ := \{\overline{\alpha} + \overline{\beta} : \overline{\alpha}, \overline{\beta} \in \overline{S}, \overline{\alpha} \neq \overline{\beta}, \overline{\alpha} \not\perp \overline{\beta}\}. \tag{9}$$

Proposition 3.6. *Suppose G is simply connected and simple and let P be its weight lattice. A subgroup Ξ of P of full rank is G-adapted if and only if one of the following holds*

(AL1) $2\overline{S} \subset \Xi \subset 2P$;

(AL2) *G is of type A_n with $n \geq 1$, n even and $\overline{S}^+ \subset \Xi$;*

(AL3) *G is of type A_n with $n \geq 1$, n odd, $\overline{S}^+ \subset \Xi$ and the odd coroots $\overline{\alpha}_1^\vee|_\Xi, \overline{\alpha}_3^\vee|_\Xi, \ldots, \overline{\alpha}_n^\vee|_\Xi$ are part of a basis of the dual lattice $\Xi^* := \mathrm{Hom}_{\mathbb{Z}}(\Xi, \mathbb{Z})$;*

(AL4) *G is of type B_n with $n \geq 2$ and $\Xi = \langle S^+ \cup \{2\overline{\alpha}_n\}\rangle_{\mathbb{Z}}$;*

(AL5) *G is of type B_n with $n \geq 2$ and $\Xi = \langle \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n\rangle_{\mathbb{Z}}$;*

(AL6) *G is of type C_n with $n \geq 2$ and $\Xi = P$.*

Remark 3.7. (a) For the cases (AL4) and (AL5), we chose a description that differs from [26, Prop. 3.16]. It is straightforward to check that the lattices are the same.

(b) If G is of type $\mathsf{B}_2 \cong \mathsf{C}_2$, then there are five G-adapted subgroups of P:

$$P, \quad 2\langle \overline{S}\rangle_{\mathbb{Z}}, \quad 2P, \quad \langle \overline{\alpha}_1 + \overline{\alpha}_2, 2\overline{\alpha}_2\rangle_{\mathbb{Z}}, \quad \langle \overline{\alpha}_1, \overline{\alpha}_2\rangle_{\mathbb{Z}}, \tag{10}$$

where $\overline{\alpha}_1$ is the long simple root and $\overline{\alpha}_2$ the short one.

(c) The lattices Ξ as in (AL1) are in natural correspondence with the subgroups of the (finite) quotient $2P/2\langle \overline{S}\rangle_{\mathbb{Z}} \cong P/\langle \overline{S}\rangle_{\mathbb{Z}}$. For each simple and simply connected G, the group $P/\langle \overline{S}\rangle_{\mathbb{Z}}$ is given in [6, Planches I-IX].

(d) For concreteness and as we will make use of it in what follows, we recall from [26, Lemma 3.10] the explicit description of the lattices in (AL2) and (AL3). Let G be of type A_n with $n \geq 1$ and Ξ a subgroup of P.

- Suppose n is even. Then Ξ satisfies (AL2) if and only if $\Xi = \langle \overline{S}^+\rangle_{\mathbb{Z}} \oplus \mathbb{Z}(k\omega_{n-1})$ for some $k \in \mathbb{N} \setminus \{0\}$.
- Suppose n is odd. Then Ξ satsifies (AL3) if and only if

$$\Xi = \langle \overline{\alpha}_2 + \overline{\alpha}_3, \overline{\alpha}_3 + \overline{\alpha}_4, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n\rangle_{\mathbb{Z}},$$

for some $e, r \in \mathbb{N}$ with $e|\frac{n+1}{2}$ and $0 \leq r \leq e-1$.

(e) For each lattice Ξ in Proposition 3.6, Tables 2 and 3 in [26] contain an explicit description of the smooth affine spherical G-variety X such that $\langle\Gamma(X)\rangle_{\mathbb{Z}} = \Xi$. These provide "local models" of quasi-Hamiltonian model spaces, in the following sense. Suppose Λ is $K\tau$-admissible, let (M, m) be the $K\tau$-model space determined by Λ and let a be a vertex of $\mathcal{A}$. If X is the smooth affine spherical $K_{a\tau}^{\mathbb{C}}$-variety whose weight monoid is $\mathrm{cone}(\mathcal{A} - a) \cap \Lambda$, then Remark 2.5.4(f) of [15] explains how X describes a neighborhood of $m_+^{-1}(a)$ in M.

(f) In Section 4 we will use an expression like "Ξ satisfies (AL1) at [the vertex] v [of $\mathcal{A}$]" to say that Ξ is a $G(\mathsf{v})$-adapted subgroup of P_{v} satisfying (AL1) for $G = G(\mathsf{v})$.

Proposition 3.6 was proved in [26] by using the combinatorial characterization of the weight monoids of smooth affine spherical varieties from [27]. We now present, in Proposition 3.10, a special case of Theorem 1.12 of *loc.cit.*, which we will use in Section 4 when we verify whether a lattice is $G(\mathsf{v})$-adapted for a group $G(\mathsf{v})$ which is not simple. We first need to introduce two objects.

Definition 3.8. Let Ξ be a subgroup of P of full rank. We define the set $\Sigma^N(\Xi)$ of *N-spherical roots of* Ξ as follows:

$$\Sigma^N(\Xi) = (\overline{S}^+ \cap \Xi) \cup \{2\overline{\alpha} \in 2\overline{S} \cap \Xi : \langle \overline{\alpha}^\vee \,|\, \gamma\rangle \in 2\mathbb{Z} \text{ for all } \gamma \in \Xi\},$$

where $\overline{S}^+$ and $2\overline{S}$ are the sets defined in Equation (9).

Proposition 3.9 (see [27, Prop 1.7])**.** *Let Ξ be a subgroup of P of full rank. Among all the subsets $F \subseteq \overline{S}$ such that the relative interior of the cone spanned by $\{\overline{\alpha}^\vee : \overline{\alpha} \in F\}$ in $\mathrm{Hom}_{\mathbb{Z}}(\Xi, \mathbb{Q}) = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Q})$ contains a point x with $\langle \sigma \,|\, x\rangle \leq 0$ for all $\sigma \in \Sigma^N(\Xi)$ there is a unique one, denoted $\overline{S}_\Xi$, which contains all the others.*

Here is the announced specialization of [27, Theorem 1.12]. In order to save space, we freely use notions from [27, §1] and [26, §§2 and 3] in its proof. For the convenience of the reader, we point out that everything we need from [27, §1] is also contained in [26, §2].

Proposition 3.10. *Let Ξ be a subgroup of P of full rank. Then Ξ is G-adapted if and only if*

(1) *$\{\overline{\alpha}^\vee|_\Xi : \overline{\alpha} \in \overline{S}_\Xi\}$ is a subset of a basis of the dual lattice Ξ^*,*
(2) *if $\overline{\alpha}, \overline{\beta}$ in $\overline{S}_\Xi$ and $\overline{\alpha} \neq \overline{\beta}$, then $\overline{\alpha} \perp \overline{\beta}$, and*
(3) *if $\overline{\alpha} \in \overline{S}_\Xi$, then $2\overline{\alpha} \notin \Sigma^N(\Xi)$.*

Proof. By Lemma 3.5(d) it suffices to show that Ξ satisfies the conditions (1), (2), (3) of the proposition if and only if the G-saturated monoid $\Gamma := \Xi \cap P_+$ satisfies the conditions (a), (b), (c) of [27, Theorem 1.12].

We first show that the set $\Sigma^N(\Xi)$ of Definition 3.8 is the same set as $\Sigma^N(\Gamma)$ in [27, Thm. 1.12] and in [26, §3]. By [26, Lemma 3.2(b)], $\Sigma^N(\Gamma) \subset \overline{S}^+ \cup 2\overline{S}$. Furthermore, by [26, Lemma 3.2(a)], the set $\overline{S}^p(\Gamma) := \{\overline{\alpha} \in \overline{S} : \langle \overline{\alpha}^\vee \,|\, \lambda\rangle = 0 \text{ for all } \lambda \in \Gamma\}$ is empty. It is now immediate to check, using [27, Prop. 1.7], that $\Sigma^N(\Gamma) = \Sigma^N(\Xi)$.

The equality $\Sigma^N(\Gamma) = \Sigma^N(\Xi)$ immediately gives us that the set $\overline{S}_\Xi$ in Proposition 3.9 is the same as the set $\overline{S}_\Gamma$ in [27, Prop. 1.7]. Because $\langle\Gamma\rangle_{\mathbb{Z}} = \Xi$ by Equation (8), it follows that condition (1) of the current proposition is the same as condition (a) in [27, Theorem 1.12].

Suppose now that Γ fulfills conditions (a), (b) and (c) of [27, Theorem 1.12]. Then, Ξ satisfies (2) and (3) of the current proposition by [26, Lemma 3.4.].

Conversely, suppose that Ξ fulfills (1), (2), (3) of the current proposition. Because $\langle\Gamma\rangle_{\mathbb{Z}}$ is of full rank, there are no simple roots $\overline{\alpha}, \overline{\beta} \in \overline{S}$ such that $\overline{\alpha} \neq \overline{\beta}$ and $\overline{\alpha}^\vee|_{\langle\Gamma\rangle_{\mathbb{Z}}} = \overline{\beta}^\vee|_{\langle\Gamma\rangle_{\mathbb{Z}}}$, and so condition (b) of [27, Theorem 1.12] is trivially met. It follows from (2) and (3) that $\langle\overline{S}_\Gamma\rangle_{\mathbb{Z}} \cap \Sigma^N(\Gamma) = \langle\overline{S}_\Xi\rangle_{\mathbb{Z}} \cap \Sigma^N(\Xi) = \emptyset$. Together with (2) this implies that the triple $(\overline{S}_\Gamma, \overline{S}^p(\Gamma), \langle\overline{S}_\Gamma\rangle_{\mathbb{Z}} \cap \Sigma^N(\Gamma)) = (\overline{S}_\Gamma, \emptyset, \emptyset)$ is a (possibly empty) "disjoint union" of copies of the triple $(\mathsf{A}_1, \emptyset, \emptyset)$ in [27, List 1.10]. In particular, the triple satisfies condition (c) of [27, Theorem 1.12], and we have shown that Γ satisfies all three conditions in *loc.cit.* ■

We conclude this section with a generalization of (AL1), which has the same proof as [26, Prop. 3.7]

Proposition 3.11. *If Ξ is a subgroup of full rank of P satisfying $2\overline{S} \subset \Xi \subset 2P$, then Ξ is G-adapted.*

Proof. Because $2\overline{S} \subset \Xi \subset 2P$, we have $\Sigma^N(\Xi) = 2\overline{S}$. One then computes, that $\overline{S}_\Xi = \emptyset$. Consequenlty, the conditions in Proposition 3.10 are trivially satisfied. ■

4. Proof of Theorem 2.12

In this section we will prove Theorem 2.12. *For the remainder of this paper, K is a simply connected and compact Lie group, τ an automorphism of K, and we fix $T, \mathfrak{a}, \Lambda_\tau, \Phi_\tau$ and $\mathcal{A}$ as in Theorem 2.4.* As before we will use S_τ for the set of simple roots of Φ_τ determined by the choice of alcove $\mathcal{A}$ and we will number the elements of $S_\tau = \{\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ as in Table 1. We will also use the notations P_{v} and $G(\mathsf{v})$ from Lemma 3.3 and we set

$$\overline{S}_\tau := \{\overline{\alpha} : \alpha \in S_\tau\}.$$

If v is a vertex of $\mathcal{A}$, we set $\overline{S}(\mathsf{v}) := \{\overline{\alpha} \colon \alpha \in S_\tau, \alpha(\mathsf{v}) = 0\}$.

Then $\overline{S}(\mathsf{v})$ is the set of simple roots of $K_{\mathsf{v}\tau}$, $K^{\mathbb{C}}_{\mathsf{v}\tau}$ and $G(\mathsf{v})$ corresponding to the choice of cone$(\mathcal{A} - \mathsf{v})$ as the positive Weyl chamber. Let $\ell \in \{0, 1, 2, \ldots, n\}$. We recall the definition of the vertex v_ℓ of $\mathcal{A}$ in Equation (6). Then the Dynkin diagram of $\overline{S}(\mathsf{v}_\ell)$ is obtained by removing from the Dynkin diagram $\mathsf{X}_n^{(r)}$ of S_τ the simple root α_ℓ and all the edges adjacent to it. The following notation will also be useful:

$$\overline{S}(\mathsf{v}_\ell)^+ := \{\overline{\alpha} + \overline{\beta} : \overline{\alpha}, \overline{\beta} \in \overline{S}(\mathsf{v}_\ell), \overline{\alpha} \neq \overline{\beta}, \overline{\alpha} \not\perp \overline{\beta}\}.$$

Furthermore, we will use $\omega_1, \omega_2, \ldots, \omega_n$ for the fundamental weights of the root system $\overline{\Phi_\tau(\mathsf{v}_0)}$, that is, $(\omega_i)_{i=1}^n$ are those elements of $\mathfrak{a}$ such that $\langle \omega_i \,|\, \overline{\alpha}_j^\vee \rangle = \delta_{ij}$ for all $i, j \in \{1, 2, \ldots, n\}$. We will make frequent use of the expression "satisfies (AL1) at v" introduced in Remark 3.7(f).

We begin by explaining the Dynkin labels $k(\alpha)$ attached to the simple roots α in each diagram in Table 1. They are the unique coprime positive integers such that $\delta := \sum_{\alpha \in S_\tau} k(\alpha)\alpha$ is a constant function, see [12, Theorem 4.8(c)].

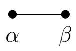
Taking gradients we obtain the equation

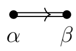
$$\sum_{\alpha\in S_\tau} k(\alpha)\overline{\alpha} = 0 \tag{11}$$

which will be important in what follows. One immediate consequence, using the definition (3) of $\overline{\alpha}^\vee$, is

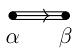
$$\sum_{\alpha\in S_\tau} k(\alpha)\|\overline{\alpha}\|^2\overline{\alpha}^\vee = 0. \tag{12}$$

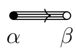
Since it will play a role, we recall that the number of edges between two simple roots in a Dynkin diagram gives information about their relative lengths:



means $\dfrac{\|\overline{\alpha}\|^2}{\|\overline{\beta}\|^2} = 1$;


means $\dfrac{\|\overline{\alpha}\|^2}{\|\overline{\beta}\|^2} = 2$;



means $\dfrac{\|\overline{\alpha}\|^2}{\|\overline{\beta}\|^2} = 3$;


means $\dfrac{\|\overline{\alpha}\|^2}{\|\overline{\beta}\|^2} = 4$.

The following summarizes some immediate consequences of Equations (11) and (12).

Lemma 4.1. (a) $P_{\mathsf{v}_0} \subset P_{\mathsf{v}_\ell}$ *for all* $\ell \in \{0, 1, \ldots, n\}$.

(b) *If* $\ell \in \{0, 1, , \ldots, n\}$ *with* $k(\alpha_\ell) = 1$ *then* $\langle\overline{S}(\mathsf{v}_\ell)\rangle_{\mathbb{Z}} = \langle\overline{S}_\tau\rangle_{\mathbb{Z}}$.

(c) *If* $\ell \in \{0, 1, \ldots, n\}$ *with* $k(\alpha_\ell) > 1$ *then* $\langle\overline{S}(\mathsf{v}_\ell)\rangle_{\mathbb{Z}} \subsetneq \langle\overline{S}_\tau\rangle_{\mathbb{Z}}$.

Proof. To show (a) one uses Equation (12) to check for each affine root system in Table 1 that, with the chosen numbering of the simple roots, $\overline{\alpha}_0^\vee$ is an integral linear combination of $\overline{\alpha}_1^\vee, \overline{\alpha}_2^\vee, \ldots, \overline{\alpha}_n^\vee$ (for the untwisted diagrams $\mathsf{X}_n^{(1)}$ one can also argue as in part 2.13 of Remark 2.13.) To show (b) we first observe that $\overline{S}(\mathsf{v}_\ell) = \overline{S}_\tau \setminus \{\overline{\alpha}_\ell\}$. The claim now follows because Equation (11) implies that $\overline{\alpha}_\ell \in \langle\overline{S}(\mathsf{v}_\ell)\rangle_{\mathbb{Z}}$ when $k(\alpha_\ell) = 1$. Part (c) follows from the fact that the Dynkin labels are coprime, which together with the linear independence of the simple roots in $\overline{S}(\mathsf{v}_\ell)$ implies that $\overline{\alpha}_\ell \notin \langle\overline{S}(\mathsf{v}_\ell)\rangle_{\mathbb{Z}}$ when $k(\alpha_\ell) > 1$. ■

We now start the actual *proof of Theorem 2.12.* For each irreducible affine root system in Table 1 we will check which of the $G(\mathsf{v}_0)$-adapted subgroups Ξ of P_{v_0} are $K\tau$-admissible. It is Proposition 3.6 which provides those $G(\mathsf{v}_0)$-adapted subgroups. This next proposition will determine all the $K\tau$-admissible subgroups of $\mathfrak{a}$ for many of the root systems Φ_τ and justifies entry (1) in Theorem 2.12.

Lemma 4.2. *If* Φ_τ *is not of type* $\mathsf{A}_{2n}^{(2)}$, *with* $n \geq 1$ *and* Ξ *is a subgroup of* P_{v_0} *with* $2\overline{S}(\mathsf{v}_0) \subset \Xi \subset 2P_{\mathsf{v}_0}$, *then* Ξ *is* $K\tau$*-admissible.*

Proof. One checks in Table 1 that $k(\alpha_0) = 1$, and so it follows from Lemma 4.1 that

$$2\overline{S}(\mathsf{v}_\ell) \subset 2\langle\overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}} \subset \Xi \subset 2P_{\mathsf{v}_0} \subset 2P_{\mathsf{v}_\ell}$$

for all $\ell \in \{1, 2, \ldots, n\}$. Proposition 3.11 tells us that Ξ is $G(\mathsf{v}_\ell)$-adapted for all ℓ. By Lemma 3.3(c) we obtain that Ξ is $K\tau$-admissible. ■

Case: Φ_τ has type $\mathsf{D}_n^{(1)}$ with $n \geq 4$, $\mathsf{E}_6^{(1)}$, $\mathsf{E}_7^{(1)}$, $\mathsf{E}_8^{(1)}$, $\mathsf{F}_4^{(1)}$, $\mathsf{G}_2^{(1)}$, $\mathsf{E}_6^{(2)}$, or $\mathsf{D}_4^{(3)}$.

By Proposition 3.6 the only $G(\mathsf{v}_0)$-adapted subgroups of P_{v_0} for these affine root systems are those satisfying (AL1) at v_0. Therefore, Lemma 4.2 yields the following.

Corollary 4.3. *Suppose Φ_τ has one of the following Dynkin types:*

$$\mathsf{D}_n^{(1)} \text{ with } n \geq 4,\ \mathsf{E}_6^{(1)}, \mathsf{E}_7^{(1)},\ \mathsf{E}_8^{(1)},\ \mathsf{F}_4^{(1)},\ \ \mathsf{G}_2^{(1)},\ \mathsf{E}_6^{(2)},\ \mathsf{D}_4^{(3)}.$$

Then the $K\tau$-admissible subgroups Ξ of $\mathfrak{a}$ are the subgroups Ξ of $2\Lambda_\tau$ containing $2\overline{S}(\mathsf{v}_0)$.

This shows that Theorem 2.12 contains all the $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of one of the types listed in Corollary 4.3.

Case: Φ_τ is of type $\mathsf{A}_1^{(1)}$.

Here we have $G(\mathsf{v}_0) \cong \mathrm{SL}(2)$ and the only $G(\mathsf{v}_0)$-adapted lattice not satisfying (AL1) is $P_{\mathsf{v}_0} = \Lambda_\tau$. Because $k(\overline{\alpha}_0) = k(\overline{\alpha}_1) = 1$ and $\|\overline{\alpha}_0\|^2 = \|\overline{\alpha}_1\|^2$ it follows from Equation (12) that $P_{\mathsf{v}_0} = P_{\mathsf{v}_1}$, which is $G(\mathsf{v}_1)$-admissible, because $G(\mathsf{v}_1) \cong \mathrm{SL}(2)$. Together with Lemma 4.2 we have shown

Lemma 4.4. *If Φ_τ is of type $\mathsf{A}_1^{(1)}$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are $\Lambda_\tau, 2\Lambda_\tau$ and $4\Lambda_\tau$.*

We have justified entry (3) for $n = 1$ in Theorem 2.12 and shown that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_1^{(1)}$.

Remark 4.5. The $K\tau$-admissible lattices for $K \cong \mathrm{SU}(2)$ (and the corresponding manifolds) are already contained in [14, §11, Example 2], see also [15, page 515].

Case: Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 2$ even.

Lemma 4.6. *Suppose Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 2$ even and let Ξ be a $G(\mathsf{v}_0)$-adapted subgroup of P_{v_0} that does not satisfy (AL1) at v_0. Then the following are equivalent*

(a) *Ξ is $K\tau$-admissible;*

(b) *$\overline{\alpha}_0 + \overline{\alpha}_1, \overline{\alpha}_n + \overline{\alpha}_0 \in \Xi$;*

(c) *$\overline{S}(\mathsf{v}_0) \subset \Xi$;*

(d) *$\overline{S}_\tau \subset \Xi$.*

Proof. It follows from the assumptions on Ξ and from Proposition 3.6, that Ξ satisfies (AL2) at v_0, i.e. $\overline{S}(\mathsf{v}_0)^+ \subset \Xi$. We first show that (a) and (b) are equivalent. Let $\ell \in \{0, 1, \ldots, n\}$. The Dynkin type of $\overline{S}(\mathsf{v}_\ell)$ is A_n and $\overline{S}(\mathsf{v}_0)^+ \not\subset 2P_{\mathsf{v}_\ell}$, because $\langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle = \langle \overline{\alpha}_2^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle = 1$, which is odd. This implies that $\Xi \not\subset 2P_{\mathsf{v}_\ell}$ and it follows, again from Proposition 3.6, that Ξ is $G(\mathsf{v}_\ell)$-adapted if and only if $\overline{S}(\mathsf{v}_\ell)^+ \subset \Xi$. The equivalence of (a) and (b) now follows, with Lemma 3.3(c), from the fact that $\bigcup_{\ell=0}^n \overline{S}(\mathsf{v}_\ell)^+ \setminus \overline{S}(\mathsf{v}_0)^+ = \{\overline{\alpha}_0 + \overline{\alpha}_1, \overline{\alpha}_n + \overline{\alpha}_0\}$.

We now show that (c) implies (d). For this root system, Equation (11) becomes

$$\overline{\alpha}_0 + \overline{\alpha}_1 + \cdots + \overline{\alpha}_n = 0, \tag{13}$$

which implies, since Ξ is a subgroup of P_{v_0}, that if Ξ contains $\overline{S}(\mathsf{v}_0) = \{\overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n\}$, then it also contains $\overline{S}_\tau = \overline{S}(\mathsf{v}_0) \cup \{\overline{\alpha}_0\}$. That (b) follows from (d) is clear.

Finally, we prove that (b) implies (c).

Observe that, since n is even, Equation (13) can be rewritten as

$$-\overline{\alpha}_n = \sum_{k=1}^{n/2}(\overline{\alpha}_{2(k-1)} + \overline{\alpha}_{2k-1}) = (\overline{\alpha}_0 + \overline{\alpha}_1) + (\overline{\alpha}_2 + \overline{\alpha}_3) + \cdots + (\overline{\alpha}_{n-2} + \overline{\alpha}_{n-1}).$$

If we now assume that (b) holds, and in particular that $\overline{\alpha}_0 + \overline{\alpha}_1 \in \Xi$, then this equation implies that $\overline{\alpha}_n \in \Xi$, since $\overline{S}(\mathsf{v}_0)^+ \subset \Xi$. Again using that $\overline{S}(\mathsf{v}_0)^+ \subset \Xi$ one then (recursively) deduces (c). ■

Remark 4.7. It follows from straightforward computations like in the proof of Lemma 4.6 that, under the assumptions of the lemma, assertion (b) of the lemma holds if and only if $\overline{\alpha}_0 + \overline{\alpha}_1 \in \Xi$ if and only if $\overline{\alpha}_n + \overline{\alpha}_0 \in \Xi$.

Together with Lemma 4.2 we have proven

Lemma 4.8. *Suppose Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 2$ even. The subgroups Ξ of $\mathfrak{a}$ that are $K\tau$-admissible are those that satisfy $2\overline{S}(\mathsf{v}_0) \subset \Xi \subset 2\Lambda_\tau$ and those satisfying $\overline{S}(\mathsf{v}_0) \subset \Xi \subset \Lambda_\tau$.*

We have justified entry (2) in Theorem 2.12 and shown that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 2$ even.

Case: Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 3$ odd.

Lemma 4.9. *Suppose Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 3$ odd and let Ξ be a $G(\mathsf{v}_0)$-adapted subgroup of P_{v_0} that does not satisfy (AL1) at v_0. Then Ξ is $K\tau$-admissible if and only if $\overline{\alpha}_0 + \overline{\alpha}_1, \overline{\alpha}_n + \overline{\alpha}_0 \in \Xi$ and the even coroots $\overline{\alpha}_0^\vee, \overline{\alpha}_2^\vee, \ldots, \overline{\alpha}_{n-1}^\vee$ are part of a $\mathbb{Z}$-basis of Ξ^*.*

Proof. As Ξ is $G(\mathsf{v}_0)$-adapted and does not satisfy (AL1), Proposition 3.6 implies that Ξ satisfies (AL3) at v_0. In particular, it contains $\overline{\alpha}_1+\overline{\alpha}_2$. Let $\ell \in \{0, 1, \ldots, n\}$. The Dynkin type of $\overline{S}(\mathsf{v}_\ell)$ is A_n and $\overline{\alpha}_1 + \overline{\alpha}_2 \notin 2P_{\mathsf{v}_\ell}$ because $\langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle = \langle \overline{\alpha}_2^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle = 1$. Proposition 3.6 now yields that Ξ is $G(\mathsf{v}_\ell)$-adapted if and only if Ξ satisfies (AL3) at v_ℓ.
With Lemma 3.5(d) and the fact that

$$\textstyle\bigcup_{\ell=0}^n \overline{S}(\mathsf{v}_\ell)^+ \setminus \overline{S}(\mathsf{v}_0)^+ = \{\overline{\alpha}_0 + \overline{\alpha}_1, \overline{\alpha}_n + \overline{\alpha}_0\},$$

the lemma follows because the coroots of $G = G(\mathsf{v}_\ell)$ listed in (AL3) are $\overline{\alpha}_0^\vee, \overline{\alpha}_2^\vee, \ldots, \overline{\alpha}_{n-1}^\vee$ when ℓ is odd, and $\overline{\alpha}_1^\vee, \overline{\alpha}_3^\vee, \ldots, \overline{\alpha}_n^\vee$ when ℓ is even. ■

The next lemma says that in this case all $G(\mathsf{v}_0)$-adapted lattices which do not satisfy (AL1) at v_0 are $K\tau$-admissible.

Lemma 4.10. *Suppose Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n \geq 3$ odd and let Ξ be a $G(\mathsf{v}_0)$-adapted subgroup of P_{v_0} that does not satisfy* (AL1) *at v_0. Then*

$$\begin{aligned} \Xi =& \langle \overline{\alpha}_2 + \overline{\alpha}_3, \overline{\alpha}_3 + \overline{\alpha}_4, \ldots, \overline{\alpha}_{n-1} + \overline{\alpha}_n, e\omega_{n-1}, r\omega_{n-1} + \omega_n \rangle_{\mathbb{Z}}, \\ & \textit{for some } e, r \in \mathbb{N} \textit{ with } e | \frac{n+1}{2} \textit{ and } 0 \leq r \leq e-1 \end{aligned} \tag{14}$$

and Ξ is $K\tau$-admissible.

Proof. Equation (14) follows from Remark 3.7(d). By Equation (11) we have

$$-(\overline{\alpha}_0+\overline{\alpha}_1)=(\overline{\alpha}_2+\overline{\alpha}_3)+(\overline{\alpha}_4+\overline{\alpha}_5)+\cdots+(\overline{\alpha}_{n-1}+\overline{\alpha}_n)$$

and $$-(\overline{\alpha}_n+\overline{\alpha}_0)=(\overline{\alpha}_1+\overline{\alpha}_2)+(\overline{\alpha}_3+\overline{\alpha}_4)+\cdots+(\overline{\alpha}_{n-2}+\overline{\alpha}_{n-1}).$$

Since $\overline{S}(\mathsf{v}_0)^+\subset\Xi$, because Ξ satsifies (AL3) at v_0, these two equations imply that $\overline{\alpha}_0+\overline{\alpha}_1,\overline{\alpha}_n+\overline{\alpha}_0\in\Xi$. By Lemma 4.9, what remains is to show that the even coroots $\overline{\alpha}_0^\vee,\overline{\alpha}_2^\vee,\ldots,\overline{\alpha}_{n-1}^\vee$ are part of a $\mathbb{Z}$-basis of Ξ^*. To do so, we will apply the elementary divisors theorem, see e.g. [17, Theorem 5.2, p. 234]. We first recall that $\overline{\alpha}_0^\vee=-(\overline{\alpha}_1^\vee+\overline{\alpha}_2^\vee+\cdots+\overline{\alpha}_n^\vee)$ by Equation (12).

Next we give a name to the basis elements of Ξ in Equation (14), that is, for every $i\in\{1,2,\ldots,n\}$ we set

$$\sigma_i=\begin{cases}\overline{\alpha}_{i+1}+\overline{\alpha}_{i+2} & \text{if } 1\le i\le n-2;\\ e\omega_{n-1} & \text{if } i=n-1;\\ r\omega_{n-1}+\omega_n & \text{if } i=n\end{cases}$$

and we consider the matrix A with n rows and $d:=\frac{n+1}{2}$ columns whose (i,j)-th entry is

$$A_{ij}=\left\langle\sigma_i\,\middle|\,\overline{\alpha}_{2(j-1)}^\vee\right\rangle.$$

Put differently, the columns of A give the coordinates of the coroots $\overline{\alpha}_0^\vee,\overline{\alpha}_2^\vee,\ldots,\overline{\alpha}_{n-1}^\vee$ in the basis of Ξ^* that is dual to the basis $(\sigma_i)_{i=1}^n$ of Ξ. For example, for $n=7$ we have

$$A=\begin{pmatrix}0&1&-1&0\\0&-1&1&0\\0&0&1&-1\\0&0&-1&1\\-1&0&0&1\\-e&0&0&e\\-r-1&0&0&r\end{pmatrix}.$$

We need to show that the greatest common divisor of all $d\times d$-minors of A is 1. To do so, we consider the $d\times d$-submatrix M of A consisting of rows $1,3,5,\ldots,n-2$ and n of A. For example for $n=7$, we have

$$M=\begin{pmatrix}0&1&-1&0\\0&0&1&-1\\-1&0&0&1\\-r-1&0&0&r\end{pmatrix}.$$

Elementary computations show that $\det(M)=\pm1$ and then the elementary divisors theorem implies that the even coroots are part of a $\mathbb{Z}$-basis of Ξ^*. ■

Together with Lemma 4.2 we have proven

Lemma 4.11. *Suppose Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n\ge3$ odd. The subgroups Ξ of $\mathfrak{a}$ that are $K\tau$-admissible are those satisfying $2\overline{S}(\mathsf{v}_0)\subset\Xi\subset2\Lambda_\tau$ and those in Equation* (14).

We have justified entry (3) for $n\ge3$ in Theorem 2.12 and shown that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_n^{(1)}$ with $n\ge3$ odd.

Case: Φ_τ is of type $\mathsf{B}_n^{(1)}$ with $n \geq 3$.

Lemma 4.12. *Suppose Φ_τ is of type $\mathsf{B}_n^{(1)}$ with $n \geq 3$ and let Ξ be a $G(\mathsf{v}_0)$-adapted subgroup of P_{v_0} that does not satisfy (AL1) at v_0. Then Ξ is not $G(\mathsf{v}_n)$-adapted, and therefore not $K\tau$-admissible.*

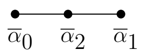
Proof. For this affine root system Φ_τ, $G(\mathsf{v}_0)$ is of type B_n, with $n \geq 3$. As Ξ does not satisfy (AL1) at v_0, it follows that it satisfies (AL4) or (AL5) at v_0 which implies that

$$\Xi \not\subset 2P_{\mathsf{v}_n}. \tag{15}$$

Indeed, this holds in both cases because we have $\overline{\alpha}_1 + \overline{\alpha}_2 \in \Xi$, $\overline{\alpha}_1 \in \overline{S}(\mathsf{v}_n)$ and $\langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle = 1$ is odd.

If $n \geq 4$, then $G(\mathsf{v}_n)$ is of type $\mathsf{D_n}$, which means, by Proposition 3.6, that the only $G(\mathsf{v}_n)$-adapted lattices are those satisfying (AL1). By (15) it follows that Ξ is not $G(\mathsf{v}_n)$-adapted.

If $n = 3$, then $G(\mathsf{v}_n) = G(\mathsf{v}_3)$ is of type A_3 and its Dynkin diagram is



By (15), Ξ does not satisfy (AL1) at v_3. We show that it also doesn't satisfy (AL3) at v_3, which then implies by Proposition 3.6 that Ξ is not $G(\mathsf{v}_3)$-adapted, as there are no other adapted lattices for a group of type A_3.

If Ξ satisfies (AL4) at v_0, then $\Xi = \langle \overline{\alpha}_1 + \overline{\alpha}_2, \overline{\alpha}_2 + \overline{\alpha}_3, 2\overline{\alpha}_3 \rangle_{\mathbb{Z}}$. Since the greatest common divisor of the 2×2-minors of the matrix

$$\begin{pmatrix} \langle \overline{\alpha}_0^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle \\ \langle \overline{\alpha}_0^\vee \,|\, \overline{\alpha}_2 + \overline{\alpha}_3 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_2 + \overline{\alpha}_3 \rangle \\ \langle \overline{\alpha}_0^\vee \,|\, 2\overline{\alpha}_3 \rangle & \langle \overline{\alpha}_1^\vee \,|\, 2\overline{\alpha}_3 \rangle \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}$$

is 2, the elementary divisors theorem tells us, that the coroots $\overline{\alpha}_0^\vee$ and $\overline{\alpha}_1^\vee$ are not part of a basis of the dual lattice Ξ^*. Consequently, Ξ does not satisfy (AL3) at v_3 in this case.

If Ξ satisfies (AL5) at v_0, then $\Xi = \langle \overline{\alpha}_1, \overline{\alpha}_2, \overline{\alpha}_3 \rangle_{\mathbb{Z}}$. Using the Dynkin diagram of $\mathsf{B}_3^{(1)}$, one computes the matrix

$$\begin{pmatrix} \langle \overline{\alpha}_0^\vee \,|\, \overline{\alpha}_1 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_1 \rangle \\ \langle \overline{\alpha}_0^\vee \,|\, \overline{\alpha}_2 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_2 \rangle \\ \langle \overline{\alpha}_0^\vee \,|\, \overline{\alpha}_3 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_3 \rangle \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -1 & -1 \\ 0 & 0 \end{pmatrix}.$$

As the greatest common divisor of the 2×2-minors of this matrix is 2, it follows again that $\{\overline{\alpha}_0^\vee, \overline{\alpha}_1^\vee\}$ is not part of a basis of Ξ^*, so that once again Ξ does not satisfy (AL3) at v_3. ∎

Lemma 4.12 and Lemma 4.2 establish the following

Lemma 4.13. *Suppose Φ_τ is of type $\mathsf{B}_n^{(1)}$ with $n \geq 3$. The subgroups Ξ of $\mathfrak{a}$ that are $K\tau$-admissible are those that satisfy $2\overline{S}(\mathsf{v}_0) \subset \Xi \subset 2\Lambda_\tau$.*

This shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{B}_n^{(1)}$ with $n \geq 3$.

Case: Φ_τ is of type $\mathsf{C}_n^{(1)}$ with $n \geq 2$.

Here $G(\mathsf{v}_0)$ is of type C_n. By Proposition 3.6 (and Remark 3.7(b)), the $G(\mathsf{v}_0)$-adapted lattices are

$$2\langle\overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}, 2P_{\mathsf{v}_0} \text{ and } P_{\mathsf{v}_0} = \Lambda_\tau \text{ for all } n \geq 2, \text{ and} \tag{16}$$

$$\text{in addition } \langle\overline{\alpha}_1 + \overline{\alpha}_2, 2\overline{\alpha}_1\rangle_{\mathbb{Z}} \text{ and } \langle\overline{\alpha}_1, \overline{\alpha}_2\rangle_{\mathbb{Z}} \text{ when } n = 2. \tag{17}$$

We first deal with the lattices in (16). The first two, $2\langle\overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ and $2P_{\mathsf{v}_0}$, are $K\tau$-admissible by Lemma 4.2 and it was shown in [15, Proposition 2.7.3] that Λ_τ is $K\tau$-admissible. This shows

Lemma 4.14. *If Φ_τ is of type $\mathsf{C}_n^{(1)}$ with $n \geq 3$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are Λ_τ, $2\Lambda_\tau$ and $2\langle\overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$.*

This justifies the entry (4) for $n \geq 3$ in Theorem 2.12 and shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{C}_n^{(1)}$ with $n \geq 3$.

Lemma 4.15. *If Φ_τ is of type $\mathsf{C}_2^{(1)}$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are*

$$\Lambda_\tau, 2\Lambda_\tau \text{ and } 2\langle\overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}.$$

Proof. Because the argument before Lemma 4.14 also applies to the case $n = 2$ we only need to consider the two lattices in Equation (17). We show that neither of them is $K\tau$-admissible using Lemma 3.3(c). We first show that

$$\Xi = \langle\overline{\alpha}_1 + \overline{\alpha}_2, 2\overline{\alpha}_1\rangle_{\mathbb{Z}} = \langle\omega_2, 4\omega_1 - 2\omega_2\rangle_{\mathbb{Z}} = \langle 4\omega_1, \omega_2\rangle_{\mathbb{Z}}$$

is not $G(\mathsf{v}_1)$-adapted, using Proposition 3.10. Note that $G(\mathsf{v}_1)$ is of type $\mathsf{A}_1 \times \mathsf{A}_1$. Writing $\widetilde{\omega}_0$ and $\widetilde{\omega}_2$ for the fundamental weights of $G(\mathsf{v}_1)$ we compute, using Equation (12), that $\Xi = \langle -4\widetilde{\omega}_0, -\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle_{\mathbb{Z}}$. An easy computation shows $\Sigma^N(\Xi) = \emptyset$ and it follows that $\overline{S}_\Xi = \{\overline{\alpha}_0, \overline{\alpha}_2\}$. Because

$$\det\begin{pmatrix} \langle\overline{\alpha}_0^\vee \,|\, -4\widetilde{\omega}_0\rangle & \langle\overline{\alpha}_2^\vee \,|\, -4\widetilde{\omega}_0\rangle \\ \langle\overline{\alpha}_0^\vee \,|\, -\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle & \langle\overline{\alpha}_2^\vee \,|\, -\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle \end{pmatrix} = \det\begin{pmatrix} -4 & 0 \\ -1 & 1 \end{pmatrix} = -4$$

$\{\overline{\alpha}_0^\vee, \overline{\alpha}_2^\vee\}$ is not a basis of Ξ^*, and therefore condition (1) of Proposition 3.10 is not satisfied.

Next, we show that $\Xi = \langle\overline{\alpha}_1, \overline{\alpha}_2\rangle_{\mathbb{Z}} = \langle 2\omega_1, \omega_2\rangle_{\mathbb{Z}}$ is not $G(\mathsf{v}_1)$-adapted. Again writing $\widetilde{\omega}_0, \widetilde{\omega}_2$ for the fundamental weights of $G(\mathsf{v}_1)$ we find that

$$\Xi = \langle -2\widetilde{\omega}_0, -\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle_{\mathbb{Z}}. \tag{18}$$

Here too $\Sigma^N(\Xi) = \emptyset$ and therefore $\overline{S}_\Xi = \{\overline{\alpha}_0, \overline{\alpha}_2\}$. Because

$$\det\begin{pmatrix} \langle\overline{\alpha}_0^\vee \,|\, -2\widetilde{\omega}_0\rangle & \langle\overline{\alpha}_2^\vee \,|\, -2\widetilde{\omega}_0\rangle \\ \langle\overline{\alpha}_0^\vee \,|\, -\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle & \langle\overline{\alpha}_2^\vee \,|\, -\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle \end{pmatrix} = \det\begin{pmatrix} -2 & 0 \\ -1 & 1 \end{pmatrix} = -2$$

$\{\overline{\alpha}_0^\vee, \overline{\alpha}_2^\vee\}$ is not a basis of Ξ^*, and once again condition (1) of Proposition 3.10 is not satisfied. ∎

We have justified entry (4) for $n = 2$ in Theorem 2.12 and shown that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{C}_2^{(1)}$.

Case: Φ_τ is of type $\mathsf{A}_2^{(2)}$.

Here $G(\mathsf{v}_0) \cong \mathrm{SL}(2)$ and it is well known (or can be read from Proposition 3.6) that the $G(\mathsf{v}_0)$-adapted lattices are

$$\mathbb{Z}\omega_1,\ 2\mathbb{Z}\omega_1 \text{ and } 4\mathbb{Z}\omega_1. \tag{19}$$

For this affine root system, Equation (12) reads

$$2\overline{\alpha}_0^\vee + 4\overline{\alpha}_1^\vee = 0$$

so that $-2\overline{\alpha}_1^\vee = \overline{\alpha}_0^\vee$. This implies that $\langle \overline{\alpha}_0^\vee \,|\, \omega_1 \rangle = -2$, so that

$$\omega_1 = -2\widetilde{\omega}_0$$

where $\widetilde{\omega}_0$ is the fundamental weight of $G(\mathsf{v}_1)$. Since $G(\mathsf{v}_1) \cong \mathrm{SL}(2)$ the only $G(\mathsf{v}_1)$-adapted lattices are

$$\mathbb{Z}\widetilde{\omega}_0,\ 2\mathbb{Z}\widetilde{\omega}_0 \text{ and } 4\mathbb{Z}\widetilde{\omega}_0. \tag{20}$$

and so of the three subgroups of $\mathfrak{a}$ in (19) only

$$\mathbb{Z}\omega_1 = 2\mathbb{Z}\widetilde{\omega}_0 \text{ and } 2\mathbb{Z}\omega_1 = 4\mathbb{Z}\widetilde{\omega}_0$$

are $K\tau$-admissible. We have thus proved:

Lemma 4.16. *If Φ_τ is of type $\mathsf{A}_2^{(2)}$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are Λ_τ and $2\Lambda_\tau$.*

This justifies the entries (6) and (7) for $n = 1$ in Theorem 2.12 and shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_2^{(2)}$.

Remark 4.17. The $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_2^{(2)}$ are already contained in [14, §11, Example 2], see also [15, page 515].

Case: Φ_τ is of type $\mathsf{A}_{2n}^{(2)}$ with $n \geq 2$.

Here $G(\mathsf{v}_0)$ is of type C_n. By Proposition 3.6 (and Remark 3.7(b)), the $G(\mathsf{v}_0)$-adapted lattices are

$$2\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}, 2P_{\mathsf{v}_0} \text{ and } P_{\mathsf{v}_0} = \Lambda_\tau \text{ for all } n \geq 2, \text{ and} \tag{21}$$

$$\text{in addition } \langle \overline{\alpha}_1 + \overline{\alpha}_2, 2\overline{\alpha}_1\rangle_{\mathbb{Z}} \text{ and } \langle \overline{\alpha}_1, \overline{\alpha}_2\rangle_{\mathbb{Z}} \text{ when } n = 2. \tag{22}$$

We first deal with the lattices in (21). It was shown in [15, Proposition 2.7.3] that Λ_τ is $K\tau$-admissible. Since $k(\alpha_n) = 1$ and $k(\alpha_0) = 2 > 1$, Lemma 4.1(c) tells us that $2\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ is a proper subgroup of $2\langle \overline{S}(\mathsf{v}_n)\rangle_{\mathbb{Z}}$. As $\overline{S}(\mathsf{v}_n)$ is of type B_n and $\Xi = 2\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ does not satisfy (AL1), (AL4) or (AL5), nor (AL6) when $n = 2$, for $G = G(\mathsf{v}_n)$, it follows from Proposition 3.6, that $2\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ is not $G(\mathsf{v}_n)$-adapted and therefore also not $K\tau$-admissible. This leaves the $G(\mathsf{v}_0)$-adapted lattice $2P_{\mathsf{v}_0}$. It follows from Lemma 4.1(a) that

$$2P_{\mathsf{v}_0} \subset 2P(\mathsf{v}_k) \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

Since $\langle \overline{\alpha}_i^\vee \,|\, \overline{\alpha}_\ell \rangle \in \mathbb{Z}$ for all $i, \ell \in \{0, 1, 2, \ldots, n\}$ by Definition 2.2(b), we have $\langle \overline{\alpha}_i^\vee \,|\, 2\overline{\alpha}_\ell \rangle \in 2\mathbb{Z}$ and consequently that

$$2\overline{S}(\mathsf{v}_k) \subset 2P_{\mathsf{v}_0} \quad \text{for all } k \in \{1, 2, \ldots, n\}.$$

By Proposition 3.11 and Lemma 3.3(c) we obtain that $2P_{\mathsf{v}_0} = 2\Lambda_\tau$ is $K\tau$-admissible. We have proved:

Lemma 4.18. *If Φ_τ is of type $\mathsf{A}_{2n}^{(2)}$ with $n \geq 3$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are Λ_τ and $2\Lambda_\tau$.*

This justifies the entries (6) and (7) for $n \geq 3$ in Theorem 2.12 and shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_{2n}^{(2)}$ with $n \geq 3$.

Lemma 4.19. *If Φ_τ is of type $\mathsf{A}_4^{(2)}$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are*

$$\Lambda_\tau,\ 2\Lambda_\tau \text{ and } \langle\overline{\alpha}_1, \overline{\alpha}_2\rangle_{\mathbb{Z}}$$

Proof. Because the argument before Lemma 4.18 also applies to the case $n = 2$ we only need to consider the two lattices in Equation (22). We fist show that $\langle\overline{\alpha}_1 + \overline{\alpha}_2, 2\overline{\alpha}_1\rangle_{\mathbb{Z}} = \langle\omega_2, 4\omega_1 - 2\omega_2\rangle_{\mathbb{Z}}$ is not $G(\mathsf{v}_2)$-adapted. Indeed, writing $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$ for the fundamental weights of $G(\mathsf{v}_2)$ we compute, using Equation (12), that $\langle\overline{\alpha}_1 + \overline{\alpha}_2, 2\overline{\alpha}_1\rangle_{\mathbb{Z}} = \langle 4\widetilde{\omega}_1, 2\widetilde{\omega}_0\rangle_{\mathbb{Z}}$. As this lattice does not satisfy (AL1), (AL4), (AL5) or (AL6) at v_2, it is not $G(\mathsf{v}_2)$-adapted.

Next, we show that $\Xi = \langle\overline{\alpha}_1, \overline{\alpha}_2\rangle_{\mathbb{Z}} = \langle 2\omega_1, \omega_2\rangle_{\mathbb{Z}}$ is $K\tau$-admissible, by showing that it is $G(\mathsf{v}_1)$- and $G(\mathsf{v}_2)$-adapted. Writing $\widetilde{\omega}_0, \widetilde{\omega}_1$ for the fundamental weights of $G(\mathsf{v}_2)$ we compute, using Equation (12), that $\Xi = 2P_{\mathsf{v}_2}$, which is a $G(\mathsf{v}_2)$-adapted lattice by (AL1). To show that Ξ is $G(\mathsf{v}_1)$-adapted, we use Proposition 3.10. If we now write $\widetilde{\omega}_0, \widetilde{\omega}_2$ for the fundamental weights of $G(\mathsf{v}_1)$ we find that $\Xi = \langle -4\widetilde{\omega}_0, -2\widetilde{\omega}_0 + \widetilde{\omega}_2\rangle_{\mathbb{Z}}$. Straightforward computations show that $\Sigma^N(\Xi) = \{2\overline{\alpha}_0\}$ and $\overline{S}_\Xi = \{\overline{\alpha}_2\}$ and then also that the three conditions in Proposition 3.10 for Ξ to be $G(\mathsf{v}_1)$-adapted are satisfied. ■

This justifies the entries (5), (6) and (7) for $n = 2$ in Theorem 2.12 and shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_4^{(2)}$.

Case: Φ_τ is of type $\mathsf{A}_{2n-1}^{(2)}$ with $n \geq 3$.

Lemma 4.20. *Suppose Φ_τ is of type $\mathsf{A}_{2n-1}^{(2)}$ with $n \geq 3$ and let Ξ be a $G(\mathsf{v}_0)$-adapted subgroup of P_{v_0} that does not satisfy* (AL1) *at v_0. Then Ξ is not $G(\mathsf{v}_n)$-adapted, and therefore not $K\tau$-admissible.*

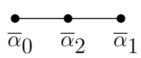
Proof. For this affine root system Φ_τ, $G(\mathsf{v}_0)$ is of type C_n, with $n \geq 3$. As Ξ does not satisfy (AL1) at v_0, it follows that it satisfies (AL6), that is, $\Xi = P_{\mathsf{v}_0}$. This implies that

$$\Xi \not\subset 2P_{\mathsf{v}_n}. \tag{23}$$

Indeed, $\omega_1 \in \Xi$, $\overline{\alpha}_1 \in \overline{S}(\mathsf{v}_n)$ and $\langle\overline{\alpha}_1^\vee \,|\, \omega_1\rangle = 1$, which is odd.

If $n \geq 4$, then $G(\mathsf{v}_n)$ is of type $\mathsf{D_n}$, which means, by Proposition 3.6, that the only $G(\mathsf{v}_n)$-adapted lattices are those satisfying (AL1). By (23), it follows that Ξ is not $G(\mathsf{v}_n)$-adapted.

If $n = 3$, then $G(\mathsf{v}_n) = G(\mathsf{v}_3)$ is of type A_3 and its Dynkin diagram is



By (23), Ξ does not satisfy (AL1) at v_3. We show that it also doesn't satisfy (AL3) at v_3, which then implies by Proposition 3.6 that Ξ is not $G(\mathsf{v}_3)$-adapted, as there are no other adapted lattices for a group of type A_3. Recall that $\omega_1, \omega_2, \omega_3$ are the fundamental weights of $G(\mathsf{v}_0)$. For the root system Φ_τ of type $\mathsf{A}_5^{(2)}$, Equation (12) becomes

$$\overline{\alpha}_0^\vee + \overline{\alpha}_1^\vee + 2\overline{\alpha}_2^\vee + 2\overline{\alpha}_3^\vee = 0 \quad \text{or equivalently} \quad \overline{\alpha}_0^\vee = -\overline{\alpha}_1^\vee - 2\overline{\alpha}_2^\vee - 2\overline{\alpha}_3^\vee.$$

Using this formula, one computes the matrix

$$\begin{pmatrix} \langle \overline{\alpha}_0^\vee \,|\, \omega_1 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \omega_1 \rangle \\ \langle \overline{\alpha}_0^\vee \,|\, \omega_2 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \omega_2 \rangle \\ \langle \overline{\alpha}_0^\vee \,|\, \omega_3 \rangle & \langle \overline{\alpha}_1^\vee \,|\, \omega_3 \rangle \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -2 & 0 \\ -2 & 0 \end{pmatrix}.$$

As the greatest common divisor of the (2×2)-minors of this matrix is 2, the elementary divisors theorem tells us, that the coroots $\overline{\alpha}_0^\vee$ and $\overline{\alpha}_1^\vee$ are not part of a basis of the dual lattice Ξ^*. Consequently, Ξ does not satisfy (AL3) at v_3. ∎

Lemma 4.20 and Lemma 4.2 establish the following

Lemma 4.21. *Suppose Φ_τ is of type $\mathsf{A}_{2n-1}^{(2)}$ with $n \geq 3$. The subgroups Ξ of $\mathfrak{a}$ that are $K\tau$-admissible are those satisfying $2\overline{S}(\mathsf{v}_0) \subset \Xi \subset 2\Lambda_\tau$.*

This shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{A}_{2n-1}^{(2)}$ with $n \geq 3$.

Case: Φ_τ is of type $\mathsf{D}_{n+1}^{(2)}$ with $n \geq 2$.

Here $G(\mathsf{v}_0)$ is of type B_n. By Proposition 3.6 (and Remark 3.7(b)), the $G(\mathsf{v}_0)$-adapted lattices which do not satisfy (AL1) at v_0 are

$$\langle \overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\} \rangle_{\mathbb{Z}} \text{ and } \langle \overline{S}(\mathsf{v}_0) \rangle_{\mathbb{Z}} \text{ for all } n \geq 2, \text{ and} \tag{24}$$

$$\text{in addition } P_{\mathsf{v}_0} = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}} \text{ when } n = 2. \tag{25}$$

We will use that Equation (11) and Equation (12) become

$$\overline{\alpha}_0 + \overline{\alpha}_1 + \cdots + \overline{\alpha}_n = 0, \tag{26}$$

and

$$\overline{\alpha}_0^\vee + 2\overline{\alpha}_1^\vee + 2\overline{\alpha}_2^\vee + \cdots + 2\overline{\alpha}_{n-1}^\vee + \overline{\alpha}_n^\vee = 0 \tag{27}$$

for the affine root system Φ_τ of type $\mathsf{D}_{n+1}^{(2)}$.

Lemma 4.22. *Suppose Φ_τ is of type $\mathsf{D}_{n+1}^{(2)}$ with $n \geq 2$. If n is even, then $\langle \overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\} \rangle_{\mathbb{Z}}$ is not $G(\mathsf{v}_n)$-adapted and therefore not $K\tau$-admissible.*

Proof. Set $\Xi = \langle \overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\} \rangle_{\mathbb{Z}}$. Like $G(\mathsf{v}_0)$, the group $G(\mathsf{v}_n)$ has Dynkin type B_n. First note that $\langle \overline{\alpha}_1^\vee \,|\, \overline{\alpha}_1 + \overline{\alpha}_2 \rangle = 1$ (when $n = 2$ too). As $\overline{\alpha}_1 + \overline{\alpha}_2 \in \Xi$ and $\overline{\alpha}_1 \in \overline{S}(\mathsf{v}_n)$, this implies that $\Xi \not\subset 2P_{\mathsf{v}_n}$ and so Ξ does not satisfy (AL1) at v_n. Next we show that

$$\overline{\alpha}_0 + \overline{\alpha}_1 \notin \Xi, \tag{28}$$

since this implies that Ξ does not satisfy (AL4) or (AL5) at v_n and that it does not satisfy (AL6) when $n = 2$. Since $\overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\}$ is a basis of Ξ, we know that $\overline{\alpha}_n \notin \Xi$.

Because n is even and because $\overline{S}(\mathsf{v}_0)^+ \in \Xi$, this implies (28) thanks to Equation (26). As we have shown that Ξ cannot be any of the $G(\mathsf{v}_n)$-adapted lattices listed in Proposition 3.6, the lemma follows. ■

We now show that the subgroup of $\mathfrak{a}$ in (25) is not $K\tau$-admissible.

Lemma 4.23. *Suppose Φ_τ is of type $\mathsf{D}_3^{(2)}$. Then $\Lambda_\tau = \langle \omega_1, \omega_2 \rangle_{\mathbb{Z}}$ is not $G(\mathsf{v}_1)$-adapted and thefore not $K\tau$-admissible.*

Proof. Here $G(\mathsf{v}_1)$ is of type $\mathsf{A}_1 \times \mathsf{A}_1$. Writing $\widetilde{\omega}_0$ and $\widetilde{\omega}_2$ for the fundamental weights of $G(\mathsf{v}_1)$ we find, using Equation (27), that $\Xi = \langle -2\widetilde{\omega}_0, \widetilde{\omega}_2 - \widetilde{\omega}_0 \rangle_{\mathbb{Z}}$, which is exacly the lattice we encountered in Equation (18) in the proof of Lemma 4.15. We showed there that it follows from Proposition 3.10 that Ξ is not $G(\mathsf{v}_1)$-adapted. ■

Next we show that the remaining subgroups of $\mathfrak{a}$ in Equation (24) are $K\tau$-admissible.

Lemma 4.24. *Suppose Φ_τ is of type $\mathsf{D}_{n+1}^{(2)}$ with $n \geq 2$. If n is odd, then $\langle \overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\} \rangle_{\mathbb{Z}}$ is a $K\tau$-admissible subgroup of $\mathfrak{a}$.*

Proof. Set $\Xi = \langle \overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\} \rangle_{\mathbb{Z}}$. First we observe that Equation (26) and the fact the n is odd imply

$$2\overline{\alpha}_0 = -2(\overline{\alpha}_1 + \overline{\alpha}_2) - 2(\overline{\alpha}_3 + \overline{\alpha}_4) - \cdots - 2(\overline{\alpha}_{n-2} + \overline{\alpha}_{n-1}) - 2\overline{\alpha}_n;$$

and
$$\overline{\alpha}_0 + \overline{\alpha}_1 = -(\overline{\alpha}_2 + \overline{\alpha}_3) - (\overline{\alpha}_4 + \overline{\alpha}_5) - \cdots - (\overline{\alpha}_{n-1} + \overline{\alpha}_n).$$

Consequently
$$2\overline{\alpha}_0, \overline{\alpha}_0 + \overline{\alpha}_1 \in \Xi \tag{29}$$

and $\Xi = \langle \overline{S}(\mathsf{v}_n)^+ \cup \{2\overline{\alpha}_0\} \rangle_{\mathbb{Z}}$. This shows that Ξ satisfies (AL4) at v_n and consequently is $G(\mathsf{v}_n)$-adapted.

We now fix $\ell \in \{1, 2, \ldots, n-1\}$ and check that Ξ is $G(\mathsf{v}_\ell)$-adapted using Proposition 3.10. Observe that $G(\mathsf{v}_\ell)$ has Dynkin type $\mathsf{B}_\ell \times \mathsf{B}_{n-l}$. To begin, we note that

$$\langle \overline{\alpha}_k^\vee \,|\, \overline{\alpha}_{k-1} + \overline{\alpha}_k \rangle = \langle \overline{\alpha}_k^\vee \,|\, \overline{\alpha}_k + \overline{\alpha}_{k+1} \rangle = 1 \quad \text{for all } k \in \{1, 2, \ldots, n-1\}. \tag{30}$$

Since $\overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\} \subset \Xi$ it follows from Equations (29) and (30) that

$$\Sigma^N(\Xi) = (\{2\overline{\alpha}_0, \overline{\alpha}_0 + \overline{\alpha}_1, 2\overline{\alpha}_n\} \cup \overline{S}(\mathsf{v}_0)^+) \setminus \{\overline{\alpha}_{\ell-1} + \overline{\alpha}_\ell, \overline{\alpha}_\ell + \overline{\alpha}_{\ell+1}\}.$$

An elementary, if somewhat lengthy computation then shows that $\overline{S}_\Xi = \emptyset$. Consequently the three conditions in Proposition 3.10 are trivially justified. Thanks to Lemma 3.5(d) we can conclude that Ξ is $K\tau$-admissible. ■

Lemma 4.25. *Suppose Φ_τ is of type $\mathsf{D}_{n+1}^{(2)}$ with $n \geq 2$.*
Then $\langle \overline{S}(\mathsf{v}_0) \rangle_{\mathbb{Z}} = \langle \overline{\alpha}_1, \overline{\alpha}_2, \ldots, \overline{\alpha}_n \rangle_{\mathbb{Z}}$ is a $K\tau$-admissible subgroup of $\mathfrak{a}$.

Proof. By Lemma 3.5(d) it suffices to show that $\langle \overline{S}(\mathsf{v}_0) \rangle_{\mathbb{Z}}$ is $G(\mathsf{v}_\ell)$-adapted for all $\ell \in \{1, 2, \ldots, n\}$. We begin with $\ell = n$. Using Equation (26) one directly sees, that

$$\langle \overline{S}(\mathsf{v}_0) \rangle_{\mathbb{Z}} = \langle \overline{\alpha}_0, \overline{\alpha}_1, \ldots, \overline{\alpha}_{n-1} \rangle_{\mathbb{Z}} = \langle \overline{S}(\mathsf{v}_n) \rangle_{\mathbb{Z}}.$$

Consequently, $\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ satisfies (AL5) at v_n and is therefore $G(\mathsf{v}_n)$-adapted. We now fix $\ell \in \{1, 2, \dots, n-1\}$. Then $\overline{\Phi_\tau(\mathsf{v}_\ell)}$ is of type $\mathsf{B}_\ell \times \mathsf{B}_{n-\ell}$ and $G(\mathsf{v}_\ell) = G_1 \times G_2$, where G_1 has type B_ℓ and set of simple roots $\{\overline{\alpha}_0, \overline{\alpha}_1, \dots, \overline{\alpha}_{\ell-1}\}$, and G_2 has type $\mathsf{B}_{n-\ell}$ and set of simple roots $\{\overline{\alpha}_{\ell+1}, \overline{\alpha}_{\ell+2}, \dots, \overline{\alpha}_n\}$. Using Equation (26) once again, one directly sees that $\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}} = \mathcal{X}_1 \oplus \mathcal{X}_2$, where

$$\mathcal{X}_1 = \langle \overline{\alpha}_0, \overline{\alpha}_1, \dots, \overline{\alpha}_{\ell-1}\rangle_{\mathbb{Z}} \quad \text{and} \quad \mathcal{X}_2 = \langle \overline{\alpha}_{\ell+1}, \overline{\alpha}_{\ell+2}, \dots, \overline{\alpha}_n\rangle_{\mathbb{Z}}.$$

Consequently, $\mathcal{X}_1$ is G_1-adapted and $\mathcal{X}_2$ is G_2-adapted by (AL5) and therefore $\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ is $G(\mathsf{v}_\ell)$-adapted. ■

We have shown the following

Lemma 4.26. *If Φ_τ is of type $\mathsf{D}^{(2)}_{n+1}$ with $n \geq 2$, then the $K\tau$-admissible subgroups of $\mathfrak{a}$ are: $2\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$, $2\Lambda_\tau$ and $\langle \overline{S}(\mathsf{v}_0)\rangle_{\mathbb{Z}}$ and, in addition, $\langle \overline{S}(\mathsf{v}_0)^+ \cup \{2\overline{\alpha}_n\}\rangle_{\mathbb{Z}}$ when n is odd.*

This justifies entries (8) and (9) in Theorem 2.12 and shows that Theorem 2.12 contains all $K\tau$-admissible subgroups of $\mathfrak{a}$ when Φ_τ is of type $\mathsf{D}^{(2)}_{n+1}$ with $n \geq 2$.

References

[1] J. Adams, J.-S. Huang, D. A. Vogan, Jr.: *Functions on the model orbit in E_8*, Represent. Theory 2 (1998) 224–263.

[2] A. Alekseev, A. Malkin, E. Meinrenken: *Lie group valued moment maps*, J. Differential Geom. 48 (1998) 445–495.

[3] A. Alekseev, E. Meinrenken, C. Woodward: *Duistermaat-Heckman measures and moduli spaces of flat bundles over surfaces*, Geom. Funct. Anal. 12 (2002) 1–31.

[4] I. N. Bernstein, I. M. Gel′fand, S. I. Gel′fand: *Models of representations of Lie groups*, Trudy Sem. Petrovsk. (1976), no. Vyp. 2, 3–21.

[5] P. Boalch, D.Yamakawa: *Twisted wild character varieties*, arXiv:1512.08091v1 [math. AG] (2015).

[6] N. Bourbaki: *Éléments de Mathématique. Fasc. XXXIV. Groupes et Algèbres de Lie. Chapitre IV: Groupes de Coxeter et Systèmes de Tits. Chapitre V: Groupes Engendrés par des Réflexions. Chapitre VI: Systèmes de Racines*, Actualités Scientifiques et Industrielles No. 1337, Hermann, Paris (1968).

[7] P. Bravi, J. Gandini, A. Maffei: *Projective normality of model varieties and related results*, Represent. Theory 20 (2016) 39–93.

[8] R. Camus: *Variétés Sphériques Affines Lisses*, Thèse de Doctorat, Institut Fourier, Grenoble (2001).

[9] I. M. Gel′fand, A. V. Zelevinskiĭ: *Models of representations of classical groups and their hidden symmetries*, Funktsional. Anal. i Prilozhen. 18/3 (1984) 14–31.

[10] I. M. Gel′fand, A. V. Zelevinsky: *Representation Models for classical Groups and their Higher Symmetries*, Astérisque (1985), Numéro Hors Serie, 117–128, The mathematical heritage of Élie Cartan (Lyon, 1984).

[11] V. Guillemin, S. Sternberg: *Multiplicity-free spaces*, J. Differential Geom. 19 (1984) 31–56.

[12] V. G. Kac: *Infinite-Dimensional Lie Algebras*, 3rd ed., Cambridge University Press, Cambridge (1990).

[13] F.Knop: *Automorphisms of multiplicity free Hamiltonian manifolds*, J. Amer. Math. Soc. 24 (2011) 567–601.

[14] F. Knop: *Multiplicity free quasi-Hamiltonian manifolds*, arXiv: 1612.03843v2 [math. SG] (2016).

[15] F. Knop: *Classification of multiplicity free quasi-Hamiltonian manifolds*, Pure Appl. Math. Q. 20 (2024) 471–523.

[16] F. Knop, K. Paulus: *(Quasi-)Hamiltonian manifolds of cohomogeneity one*, Math. Zeitschrift 305 (2023), art.no. 29, 27p.

[17] S. Lang: *Algebra*, 2nd ed., Advanced Book Program, Addison-Wesley, Reading (1984).

[18] I. V. Losev: *Proof of the Knop conjecture*, Ann. Inst. Fourier (Grenoble) 59 (2009) 1105–1134.

[19] D. Luna: *La variété magnifique modèle*, J. Algebra 313 (2007) 292–319.

[20] I. G. Macdonald: *Affine root systems and Dedekind's η-function*, Invent. Math. 15 (1972) 91–143.

[21] I. G. Macdonald: *Affine Hecke Algebras and Orthogonal Polynomials*, Cambridge Tracts in Mathematics vol. 157, Cambridge University Press, Cambridge (2003).

[22] E. Meinrenken: *Convexity for twisted conjugation*, Math. Res. Lett. 24 (2017) 1797–1818.

[23] S. Mohrdieck, R. Wendt: *Integral conjugacy classes of compact Lie groups*, Manuscripta Math. 113 (2004) 531–547.

[24] T. Oda: *Convex Bodies and Algebraic Geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15, Springer, Berlin (1988).

[25] K. Paulus: *Some Momentum Polytopes for Multiplicity Free Quasi-Hamiltonian Manifolds*, Doctoral Thesis, FAU Erlangen-Nürnberg (2018).

[26] K. Paulus, G. Pezzini, B. Van Steirteghem: *On some families of smooth affine spherical varieties of full rank*, Acta Math. Sin. (Engl. Ser.) 34 (2018) 563–596.

[27] G. Pezzini, B. Van Steirteghem: *Combinatorial characterization of the weight monoids of smooth affine spherical varieties*, Trans. Amer. Math. Soc. 372 (2019) 2875–2919.

[28] È. B. Vinberg, B. N. Kimel'fel'd: *Homogeneous domains on flag manifolds and spherical subgroups of semisimple Lie groups*, Funktsional. Anal. i Prilozhen. 12/3 (1978) 12–19, 96.

[29] R. Wendt: *Weyl's character formula for non-connected Lie groups and orbital theory for twisted affine Lie algebras*, J. Funct. Anal. 180 (2001) 31–65.

Kay Paulus, math@kaypaulus.de.

Bart Van Steirteghem, Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany; bartvs@math.fau.de.

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