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On the Centre of Crossed Modules of Lie Algebras

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Abstract. This paper studies the relationship between crossed modules of Lie algebras and their centres. We show that the homotopy of the centre of any crossed module $\partial : L_1 \to L_0$ of Lie algebras fits in an exact sequence involving cohomology of the homotopy Lie algebras $\pi_0(L_*)$ and $\pi_1(L_*)$.

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1. Introduction

Crossed modules (of groups) were introduced by J. H. C. Whitehead in the 1940's as a tool to study relative homotopy groups $\pi_2(X, A)$ [13]. It was discovered in the 60's that the category of crossed modules is isomorphic to the category of internal categories in the category of groups, see for example [9]. Thus crossed modules can be considered as simplifications of such internal categories. A similar simplification exists also for internal categories in the category of Lie algebras. The corresponding objects are known as crossed modules of Lie algebras (see [4], [8], [12]).

The aim of this work is to introduce the centre of a crossed module of Lie algebras. It is analogous to the centre of a crossed module (of groups) introduced by the author in [10] which is closely related to the Gottlieb group [5] of the classifying space and the Drinfeld centre of the corresponding monoidal category [6].

Recall that a crossed module of Lie algebras can be defined as a linear mapping $\partial : L_1 \to L_0$, where L_0 is a Lie algebra, L_1 is an L_0-module, i.e. we are given a bilinear map $L_0 \times L_1 \to L_1$, $(x, a) \mapsto x \cdot a$ such that

$$[x, y] \cdot a = x \cdot (y \cdot a) - y \cdot (x \cdot a),$$

∂ is a Lie module homomorphism (i.e $\partial(x \cdot a) = [x, \partial(a)]$) for which the relation

$$\partial(a) \cdot b + \partial(b) \cdot a = 0$$

holds. This definition is equivalent to the more commonly used Definition 2.1, see Lemma 2.4. The essential invariants of a crossed module $L_* = (L_1 \xrightarrow{\partial} L_0)$ are the Lie algebra $\pi_0(L_*) = \mathsf{Coker}(\partial)$ and the $\pi_0(L_*)$-module $\pi_1(L_*) = \ker(\partial)$. Recall

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also that a braided crossed module of Lie algebras is a linear map of vector spaces $\partial : L_1 \to L_0$ together with a bilinear map $L_0 \times L_0 \to L_1$, $(x,y) \mapsto \{x,y\}$ satisfying the identities (6), (12) and (13), see Proposition 2.5. Any braided crossed module is also a crossed module, where the Lie algebra structure on L_0 and the action of L_0 on L_1 are given by

$$[x,y] = \partial\{x,y\}, \quad x \cdot a = \{x, \partial(a)\}.$$

Now we state our main result.

Theorem. (i) *Let $\partial : L_1 \to L_0$ be a crossed module of Lie algebras. There exists a braided crossed module $\delta : L_1 \to \mathbf{Z}_0(L_*)$, where $\mathbf{Z}_0(L_*)$ is the collection of all pairs (x,ξ), where $x \in L_0$ and $\xi : L_0 \to L_1$ is a linear map satisfying the following identities*

$$\partial\xi(t) = [x,t], \quad \xi(\partial a) = x \cdot a, \quad \xi([s,t]) = s \cdot \xi(t) - t \cdot \xi(s).$$

Here $x,s,t \in L_0$ and $a \in L_1$. The linear map $\delta : L_1 \to \mathbf{Z}_0(L_*)$ is given by $\delta(c) = (\partial(c), \xi_c)$, where $\xi_c(t) = -t \cdot c$. Moreover, the structural bracket $\mathbf{Z}_0(L_*) \times \mathbf{Z}_0(L_*) \to L_1$ is given by $\{(x,\xi),(y,\eta)\} = \xi(y)$.

We call the braided crossed module $\delta : L_1 \to \mathbf{Z}_0(L_*)$ the centre of the crossed module $\partial : L_1 \to L_0$ *and denote it by $\mathbf{Z}_*(L_*)$.*

(ii) *Denote by z_0 the map $\mathbf{Z}_0(L_*) \to L_0$ given by $\mathsf{z}_0(x,\xi) = x$. Define an action of L_0 on $\mathbf{Z}_0(L_*)$ by $y \cdot (x,\xi) := ([y,x],\psi)$. Here $y \in L_0$, $(x,\xi) \in \mathbf{Z}_0(L_*)$ and $\psi(t) = t \cdot \xi(y)$. With this action, the map $\mathsf{z}_0 : \mathbf{Z}_0(L_*) \to L_0$ is a crossed module of Lie algebras which is denoted by $L_*//\mathbf{Z}_*(L_*)$.*

(iii) *Let L_* be a crossed module of Lie algebras. Then*

$$\pi_1(\mathbf{Z}_*(L_*)) \cong \mathsf{H}^0(\pi_0(L_*), \pi_1(L_*))$$

and one has an exact sequence

$$0 \to \mathsf{H}^1(\pi_0(L_*), \pi_1(L_*)) \to \pi_0(\mathbf{Z}_*(L_*)) \to \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)) \to \mathsf{H}^2(\pi_0(L_*), \pi_1(L_*)).$$

Here, for a Lie algebra M and an M-module A, one denotes by $\mathsf{Z}_A(M)$ the collection of all $m \in M$ such that $[m,x] = 0$ for all $x \in M$ (so m is central) and $m \cdot a = 0$ for all $a \in A$.

Part (i) of the Theorem is proved below as Proposition 3.2, while parts (ii) and (iii) are proved as Proposition 3.6 and Corollary 3.10 respectively.
To summarise parts (i) and ii), say that any crossed module $\partial : L_1 \to L_0$ of Lie algebras fits in a commutative diagram

$$\begin{array}{ccc} L_1 & \xrightarrow{\delta} & \mathbf{Z}_0(L_*) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \mathsf{z}_0} \\ L_1 & \xrightarrow{\partial} & L_0 \end{array}$$

where the top horizontal $L_1 \xrightarrow{\delta} \mathbf{Z}_0(L_*)$ and right vertical $\mathbf{Z}_0(L_*) \xrightarrow{\mathsf{z}_0} L_0$ arrows have again crossed module structures. Hence all four arrows in the square are crossed modules. In fact, the first one is even a braided crossed module. The pair of maps

$(\mathsf{z}_0, \mathsf{id}_{L_1})$ defines a morphism of crossed modules $\mathbf{Z}_*(L_*) \to L_*$. The crossed module $L_*//\mathbf{Z}_*(L_*)$ should be understood as the "homotopic cofibre" of $\mathbf{Z}_*(L_*) \to L_*$ since we have the following obvious exact sequence

$$0 \to \pi_1(\mathbf{Z}_*(L_*)) \to \pi_1(L_*) \to \pi_1(L_*//\mathbf{Z}_*(L_*)) \to$$

$$\to \pi_0(\mathbf{Z}_*(L_*)) \to \pi_0(L_*) \to \pi_0(L_*//\mathbf{Z}_*(L_*)) \to 0.$$

As we said, the construction of $\mathbf{Z}_*(L_*)$ and the above properties are parallel to the similar construction given in [10]. However the centre defined in [10] has one more important property. Let us recall this property, closely following [10]. Let $B(\mathbf{G}_*)$ denote the classifying space of $\mathbf{G}_*$. Next, for a topological space X denote by $\mathbf{Z}(X)$ the connected component of the space $Maps(X, X)$ of self continuous maps $X \to X$ containing the identity map id_X. Then the following assertion holds:

$$\mathbf{Z}(B\mathbf{G}_*) \cong B(\mathbf{Z}_*(\mathbf{G}_*)). \tag{1}$$

Recall also that if (X, x) is a pointed map, then the evaluation at x gives rise to a pointed map $ev_x : (\mathbf{Z}(X), \mathsf{id}_X) \to (X, x)$. Now apply the functor π_1 and denote by $G(X, x)$ the image of the induced map

$$\pi_1(\mathbf{Z}(X), \mathsf{id}_X) \to \pi_1(X, x).$$

The group $G(X, x)$ is known as the Gottlieb group [5]. As a consequence of isomorphisms we proved previously in [10], for $X = B(\mathbf{G}_*)$ the classifying space of a crossed module $\mathbf{G}_*$, we identified $G(X, x)$ as a specific subgroup of the so called Whitehead centre of X, which consists of those elements of $\pi_1(X)$ which are central and act trivially on $\pi_2(X)$.

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2. Crossed modules in Lie algebras

In this section we fix terminology and notation for (braided) crossed modules in Lie algebras [2], [8].

2.1. Definition of crossed and braided crossed modules of Lie algebras

In this paper we fix a field k of characteristic $\neq 2$. All vector spaces and linear maps are considered over k. Accordingly, the Lie algebras are defined over k.

Recall that if L is a Lie algebra, then a *(left) L-module* is a vector space V together with a bilinear map $L \times V \to V, (x, v) \mapsto x \cdot v$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v).$$

If additionally $V = M$ is also a Lie algebra and

$$x \cdot [m, n] = [x \cdot m, n] + [m, x \cdot n]$$

holds for all $x \in L$ and $m, n \in M$, then we say that L acts on the Lie algebra M by derivations.

Definition 2.1. A *precrossed module* of Lie algebras consists of a homomorphism of Lie algebras $\partial : L_1 \to L_0$ together with an action of the Lie algebra L_0 on L_1, denoted by $(x,a) \mapsto x \cdot a$, $x \in L_0$, $a \in L_1$. One requires that the following identity holds:

$$\partial(x \cdot a) = [x, \partial a] \tag{2}$$

for all $a \in L_1$ and $x \in L_0$. If additionally we have the identity

$$\partial(a) \cdot b = [a, b] \tag{3}$$

then $\partial : L_0 \to L_0$ is called a crossed module. If additionally the Lie algebra structure on L_0 is trivial and the action of L_0 on L_1 is also trivial, then L_* is called an abelian crossed module. ∎

If L_* is a precrossed module of Lie algebras, then both $Im(\partial)$ and $Ker(\partial)$ are ideals of L_0 and L_1 respectively. Thus $\pi_0(L_*)$ and $\pi_1(L_*)$ are Lie algebras. If additionally L_* is a crossed module, then π_1 is a central ideal of L_1 and hence $\pi_1(L_*)$ is an abelian Lie algebra and the action of L_0 on L_1 induces a $\pi_0(L_*)$-module structure on $\pi_1(L_*)$.

Definition 2.2. [2, Proposition 6.20] A *braided crossed module* (BCM) of Lie algebras L_* consists of a homomorphism of Lie algebras $\partial : L_1 \to L_0$ together with a bilinear map

$$L_0 \times L_0 \to L_1, (x,y) \mapsto \{x,y\},$$

such that the following identities hold:

$$\partial\{x,y\} = [x,y], \tag{4}$$

$$\{\partial a, \partial b\} = [a,b], \tag{5}$$

$$0 = \{\partial a, x\} + \{x, \partial a\}, \tag{6}$$

$$0 = \{x,[y,z]\} + \{z,[x,y]\} + \{y,[z,x]\}. \quad ∎ \tag{7}$$

Lemma 2.3. *Let L_* be a BCM of Lie algebras.*

(i) *Define the action of L_0 on L_1 by $x \cdot a := \{x, \partial(a)\}$. Then L_* is a crossed module of Lie algebras.*

(ii) *The Lie algebra $\pi_0(L_*)$ is abelian and the action of $\pi_0(L_*)$ on $\pi_1(L_1)$ is trivial.*

Proof. (i) First let us show that we really obtain an action of L_0 on the Lie algebra L_1. This requires checking the following two identities:

$$[u,v] \cdot a = u \cdot (v \cdot a) - v \cdot (u \cdot a) \tag{8}$$

and

$$x \cdot [a,b] = [x \cdot a, b] + [a, x \cdot b]. \tag{9}$$

In fact, we have

$$[u,v] \cdot a = \{[u,v], \partial(a)\}. \tag{10}$$

On the other hand, we also have

$$u \cdot (v \cdot a) = \{u, \partial(v \cdot a)\} = \{u, \partial\{v, \partial(a)\}\} = \{u, [v, \partial(a)]\}.$$

Similarly

$$v \cdot (u \cdot a) = \{v, [u, \partial(a)]\} = -\{v, [\partial(a), u]\}.$$

Now we can use (7) to write

$$u \cdot (v \cdot a) - v \cdot (u \cdot a) - [u,v] \cdot a = \{u, [v, \partial(a)]\} + \{v, [\partial(a), u] + \{\partial(a), [u,v]\}\} = 0$$

and the identity (8) follows.

For the identity (9), observe that we have

$$x \cdot [a,b] = \{x, \partial[a,b]\} = \{x, [\partial(a), \partial(b)]\}.$$

We also have

$$[x \cdot a, b] = \{\partial(x \cdot a), \partial(b)\} = \{\partial\{x, \partial(a)\}, \partial(b)\} = -\{\partial(b), [x, \partial(a)]\}.$$

Similarly, $$[a, x \cdot b] = -\{\partial(a), [\partial(b), x]\}$$
and the identity (9) also follows from (7).

We still need to check the relations (2) and (3) in our case. We have

$$\partial(x \cdot a) = \partial\{x, \partial(a)\} = [x, \partial(a)]$$

and (2) is proved. Finally we have

$$\partial(a) \cdot b = \{\partial(a), \partial(b)\} = [a,b]$$

and (3) is proved.

(ii) By (4), the Lie algebra $\pi_0(L_*)$ is abelian and by part (i) the action of L_0 on $\pi_1(L_*) = Ker(\partial)$ is trivial. ∎

2.2. Alternative definitions

Lemma 2.4. *A crossed module of Lie algebras can equivalently be defined as a linear map $\partial : L_1 \to L_0$, where L_0 is a Lie algebra, L_1 is a L_0-module and ∂ is a homomorphism of L_0-modules (that is, the equality (2) holds) satisfying additionally the following identity for all $a, b \in L_1$:*

$$\partial(a) \cdot b + \partial(b) \cdot a = 0. \tag{11}$$

Proof. By forgetting the Lie algebra structure on L_1, we see that any crossed module gives rise to a structure as described in the Lemma. Conversely, we can uniquely reconstruct the bracket on L_1 from such a structure by $[a,b] := \partial(a) \cdot b$. Our first claim is that ∂ respects the bracket:

$$\partial([a,b]) = \partial(\partial(a) \cdot b) = [\partial(a), \partial(b)].$$

Here we used the identity (2) for $x = \partial(a)$. Our second claim is that L_1 is a Lie algebra. In fact, we have

$$[a,b] + [b,a] = \partial(a) \cdot b + \partial(b) \cdot a = 0,$$

showing that the bracket is anticommutative. For the Jacobi identity we have

$$\begin{aligned}[a,[b,c]] + [c,[a,b]] + [b,[c,a]] &= [a,[b,c]] - [[a,b],c] - [b,[a,c]] \\ &= \partial(a) \cdot (\partial(b) \cdot c) - \partial([a,b]) \cdot c - \partial(b) \cdot (\partial(a) \cdot c).\end{aligned}$$

Since

$$\partial([a,b]) \cdot c = \partial(\partial a \cdot b) \cdot c = [\partial(a), \partial(b)] \cdot c = \partial(a) \cdot (\partial(b) \cdot c) - \partial(b) \cdot (\partial(a) \cdot c),$$

we see that $[a,[b,c]] + [c,[a,b]] + [b,[c,a]] = 0$. Thus, L_1 is also a Lie algebra and the second claim is proved.

It remains to show that L_0 acts on L_1 as a Lie algebra. Indeed, we have

$$x \cdot [a, b] - [x \cdot a, b] - [a, x \cdot b] = x \cdot (\partial(a) \cdot b) - (\partial(x \cdot a)) \cdot b - \partial(a) \cdot (x \cdot b).$$

Now we can observe that

$$(\partial(x \cdot a)) \cdot b = [x, \partial(a)] \cdot b = x \cdot (\partial(a) \cdot b) - \partial(a)(x \cdot b).$$

Comparing these expressions we see that $x \cdot [a, b] = [x \cdot a, b] + [a, x \cdot b]$ and the result follows. ■

Proposition 2.5. *A braided crossed module of Lie algebras can equivalently be defined as a linear map of vector spaces $\partial : L_1 \to L_0$ together with a bilinear map*

$$L_0 \times L_0 \to L_1, \quad (x, y) \mapsto \{x, y\}$$

satisfying the identity (6) and also the following identities

$$\partial\{x, x\} = 0, \tag{12}$$

$$\{u, \partial\{v, w\}\} + \{w, \partial\{u, v\}\} + \{v, \partial\{w, u\}\} = 0, \tag{13}$$

where $a \in L_1$ and $x, u, v, w \in L_0$.

Proof. By forgetting the Lie algebra structures on L_0 and L_1, we obtain a structure as described in the Proposition. In fact, we only need to check that the identities (12) and (13) hold. The identity (12) follows from the fact that $[x, x] = 0$ as in any Lie algebra and the identity (4). Quite similarly, (13) follows from (7) and the identity (4).

Conversely, assume $\partial : L_1 \to L_0$ is equipped with a structure as described in the Proposition. Then we can define brackets on L_0 and L_1 based on the identities (4) and (5). In this way one obtains Lie algebras L_0 and L_1. In fact the Jacobi identity in both cases is a consequence of (13). The antisymmetry of the bracket for L_0 follows from (12). The antisymmetry for L_1 follows from (6) by taking $x = \partial a$. Next, we have $\partial[a, b] = \partial\{\partial(b), \partial(a)\}$ by the definition of the bracket on L_1. The last expression is the same as $[\partial(a), \partial(b)]$ by the definition of the bracket on L_0. Thus, $\partial[a, b] = [\partial(a), \partial(b)]$. Hence, ∂ is a homomorphism of Lie algebras. By our constructions and assumptions, the identities (4), (5) and (6) hold. Finally, the identity (7) follows from (13). ■

3. The centre of a crossed module of Lie algebras

3.1. Definition and the first properties

The following is an analogue of the corresponding notion from [10].

Definition 3.1. Let $\partial : L_1 \to L_0$ be a crossed module of Lie algebras. Denote by $\mathbf{Z}_0(L_*)$ the set of all pairs (x, ξ), where $x \in L_0$ and $\xi : L_0 \to L_1$ is a linear map satisfying the following identities

$$\partial\xi(t) = [x, t], \tag{14}$$

$$\xi(\partial a) = x \cdot a, \tag{15}$$

$$\xi([s, t]) = s \cdot \xi(t) - t \cdot \xi(s). \tag{16}$$

Here $x, s, t \in L_0$ and $a \in L_1$.

Proposition 3.2. (i) *Let* $c \in L_1$. *Then the pair* $(\partial(c), \xi_c)$ *belongs to* $\mathbf{Z}_0(L_*)$, *where*

$$\xi_c(t) = -t \cdot c.$$

(ii) *The linear map* $\delta : L_1 \to \mathbf{Z}_0(L_*)$ *given by* $\delta(c) = (\partial(c), \xi_c)$, *together with the bilinear map* $\mathbf{Z}_0(L_*) \times \mathbf{Z}_0(L_*) \to L_1$ *given by* $\{(x, \xi), (y, \eta)\} = \xi(y)$, *is a braided crossed module of Lie algebras.*

Proof. (i) According to (2) we have

$$\partial \xi_c(t) = \partial(-t \cdot c) = -[t, \partial c] = [\partial c, t]$$

and the equality (14) holds. Next, we have

$$\xi_c(\partial(a)) = -\partial(a) \cdot c = \partial(c) \cdot a$$

and (15) follows. Here we used the equality (11). Finally, we have

$$\xi_c([s, t]) = -[s, t] \cdot c = -(s \cdot (t \cdot c)) + (t \cdot (s \cdot c)) = s \cdot \xi_c(t) - t \cdot \xi_c(s).$$

(ii) We will use the characterisation of a BCM given in Proposition 2.5. Thus we only need to check the three identities listed in Proposition 2.5. By definition we have

$$\partial(\{(x, \xi), (x, \xi)\}) = \partial(\xi(x)) = [x, x] = 0.$$

Here we used the identity (14). Hence the identity (12) holds. We also have

$$\{(x, \xi), \delta(a)\} + \{\delta(a), (x, \xi)\} = \xi(\partial(a)) + \xi_a(x) = x \cdot a - x \cdot a = 0.$$

This shows that the identity (6) holds. Now we will prove the validity of the identity (13). To this end, we take three elements in $\mathbf{Z}_0(L_*)$:

$$u = (x, \xi), \quad v = (y, \eta), \quad w = (z, \zeta).$$

We have $$\{v, w\} = \{(y, \eta), (z, \zeta)\} = \eta(z).$$

Hence $$\delta(\{v, w\}) = (\partial(\eta(z)), t \mapsto -t \cdot \eta(z)) = ([y, z], t \mapsto -t \cdot \eta(z)).$$

Thus, $$\{u, \delta\{v, w\}\} = \xi([y, z]). \tag{17}$$

Now we use the identity (16) to obtain

$$\{u, \delta\{v, w\}\} = y \cdot \xi(z) - z \cdot \xi(y).$$

Using the identity (15), we can rewrite $y \cdot \xi(z)$ as $\eta(\partial(\xi(z)))$ and $z \cdot \xi(y)$ as $\zeta(\partial(\xi(y)))$.

Thus, $$\{u, \delta\{v, w\}\} = \eta(\partial(\xi(z))) - \zeta(\partial(\xi(y))).$$

Then using the identity (14), we can rewrite this as

$$\{u, \delta\{v, w\}\} = \eta(\partial(\xi(z))) - \zeta(\partial(\xi(y))) = \eta([x, z]) - \zeta([x, y]).$$

Observe that by cyclically permuting the variables in equation (17), we obtain

$$\{w, \delta\{u, v\}\} = \zeta([x, y]) \quad \text{and} \quad \{v, \delta\{w, u\}\} = \eta([z, x]).$$

Thus we proved that

$$\{u, \delta\{v, w\}\} = -\{v, \delta\{w, u\}\} - \{w, \delta\{u, v\}\}$$

and (13) follows. ∎

Definition 3.3. Let L_* be a crossed module of Lie algebras. The BCM of Lie algebras

$$\mathbf{Z}_*(L_*) = (L_1 \xrightarrow{\delta} \mathbf{Z}_0(L_*))$$

constructed in Theorem 3.2 is called the *centre* of L_*. ■

3.2. The Lie algebra structure on $\mathbf{Z}_0(L_*)$

Corollary 3.4. *Let L_* be a crossed module of Lie algebras. Then the Lie algebra structures obtained by Proposition* 2.5 *on $\mathbf{Z}_0(L_*)$ and L_1 are given by*

$$[(x,\xi),(y,\eta)] = ([x,y],\theta) \quad \textit{and} \quad [a,b] = \partial(a)\cdot b,$$

where $\theta(t) = -t\cdot\xi(y)$. The map $\partial : L_1 \to L_0$ is a homomorphism of Lie algebras.

Proof. According to Proposition 2.5, we obtain Lie algebra structures on L_1 and $\mathbf{Z}_0(L_*)$ by putting

$$[a,b]_{new} := \{\delta(a),\delta(b)\} \ \text{ and } \ [(x,\xi),(y,\eta)] := \delta\{(x,\xi),(y,\eta)\}.$$

Here we used $[a,b]_{new}$ to denote the Lie algebra structure obtained from the bracket $\{-,-\}$. Now we prove that the two Lie algebra structures on L_1 coincide. In fact, we have

$$[a,b]_{new} = \{(\partial(a),\xi_a),(\partial(b),\xi_b)\} = \xi_a(\partial(b)) = -\partial(b)\cdot a = -[b,a] = [a,b].$$

Now we identify the bracket on $\mathbf{Z}_0(L_*)$. We have

$$[(x,\xi),(y,\eta)] = \delta(\xi(y)) = (\partial\xi(y),\xi_{\xi(y)}).$$

Since $\partial\xi(y) = [x,y]$ and $\xi_{\xi(y)}(t) = -t\cdot\xi(y)$, the result follows. ■

The following lemma shows that the same Lie algebra structure can be written in a slightly different form.

Lemma 3.5. *If (x,ξ) and (y,η) are elements of $\mathbf{Z}_0(L_*)$, then*

$$-t\cdot\xi(y) = \xi([y,t]) - \eta([x,t]) = t\cdot\eta(x).$$

Proof. We have

$$\begin{aligned}\xi([y,t]) - \eta([x,t]) &= y\cdot\xi(t) - t\cdot\xi(y) - \eta([x,t]) \\ &= \eta(\partial(\xi(t))) - t\cdot\xi(y) - \eta([x,t]) = -t\cdot\xi(y).\end{aligned}$$

Here we first used the equality (16), then (15) and finally (14). Quite similarly, we finish the proof by

$$\begin{aligned}\xi([y,t]) - \eta([x,t]) &= \xi([y,t]) - x\cdot\eta(t) + t\cdot\eta(x) \\ &= \xi([y,t]) - \xi(\partial(\eta(t))) + t\cdot\eta(x) = t\cdot\eta(x).\end{aligned}$$ ■

3.3. The crossed module $L_*//\mathbf{Z}_*(L_*)$

Proposition 3.6. (i) *Let $y \in L_0$ and $(x,\xi)\in\mathbf{Z}_0(L_*)$. Then $([y,x],\psi)\in\mathbf{Z}_0(L_*)$, where $\psi(t) = t\cdot\xi(y)$.*

(ii) *The rule $y\cdot(x,\xi) := ([y,x],\psi)$ defines an action of L_0 on $\mathbf{Z}_0(L_*)$. With this action, the map*

$$\mathbf{Z}_0(L_*) \to L_0; \quad (x,\xi)\mapsto x$$

is a crossed module of Lie algebras.

Proof. (i) We need to check that the pair $([y,x],\psi)$ satisfies the relations (14)–(16). We have

$$\partial\psi(t)=\partial(t\cdot\xi(y))=[t,\partial(\xi(y))]=[t,[x,y]]=[[y,x],t]$$

and the relation (14) holds. Next, we have

$$\begin{aligned}\psi(\partial(a))&=\partial(a)\cdot\xi(y)=\xi([\partial(a),y])+y\cdot\xi(\partial(a))=-\xi(\partial(y\cdot a))+y\cdot(x\cdot a)\\&=-x\cdot(y\cdot a)+y\cdot(x\cdot a)=[y,x]\cdot a\end{aligned}$$

and the relation (15) holds. Finally, we have

$$\psi([s,t])=[s,t]\cdot\xi(y)=s\cdot(t\cdot\xi(y))-t\cdot(s\cdot\xi(y))=s\cdot\psi(t)-t\cdot\psi(s)$$

and (16) holds.

(ii) Let us first check that the above formula defines an action. This requires checking of two identities. We have

$$u\cdot(v\cdot(x,\xi))=u\cdot([v,x],t\mapsto t\cdot\xi(v))=([u,[v,x]],t\mapsto t\cdot(u\cdot\xi(v))).$$

Similarly, $v\cdot(u\cdot(x,\xi))=([v,[u,x]],\ \ t\mapsto t\cdot(v\cdot\xi(u)))$. Hence, we obtain

$$\begin{aligned}&u\cdot(v\cdot(x,\xi))-v\cdot(u\cdot(x,\xi))\\&\qquad=([u,[v,x]],t\mapsto t\cdot(u\cdot\xi(v)))-([v,[u,x]],t\mapsto t\cdot(v\cdot\xi(u)))\\&\qquad=([u,[v,x]]-[v,[u,x]],t\mapsto(t\cdot(u\cdot\xi(v))-t\cdot(v\cdot\xi(u)))\\&\qquad=([[u,v],x],t\mapsto t\cdot\xi([u,v]))\end{aligned}$$

and thus $[u,v]\cdot(x,\xi)=u\cdot(v\cdot(x,\xi))-v\cdot(u\cdot(x,\xi))$.

For the second identity, observe that

$$\begin{aligned}y\cdot[(x,\xi),(x',\xi')]&=y\cdot([x,x'],t\mapsto -t\cdot\xi(x'))\\&=([y,[x,x']],t\mapsto -t\cdot(y\cdot\xi(x'))).\end{aligned}$$

On the other hand, $[y\cdot(x,\xi),(x',\xi')]+[(x,\xi),y\cdot(x',\xi')]$ is equal to

$$([[y,x],x'],t\mapsto -t\cdot(x'\cdot\xi(y)))+([x,[y,x']],t\mapsto -t\cdot\xi([y,x'])).$$

The first coordinate of this expression is

$$[[y,x],x']-[[y,x'],x]=[y,[x,x']],$$

while the second coordinate is equal to

$$\begin{aligned}&t\mapsto -t\cdot(x'\cdot\xi(y))-t\cdot\xi([y,x']))\\&\quad=-t\cdot(x'\cdot\xi(y))-t\cdot(y\cdot\xi(x'))+t\cdot(x'\cdot\xi(y))=-t\cdot(y\cdot\xi(x')).\end{aligned}$$

Comparing these expressions, we see that

$$y\cdot[(x,\xi),(x',\xi')]=[y\cdot(x,\xi),(x',\xi')]+[(x,\xi),y\cdot(x',\xi')].$$

Hence L_0 acts on $\mathbf{Z}_0(L_*)$.

In order to show that $\delta : \mathbf{Z}_0(L_*) \to L_0$ is a crossed module, it suffices to check the identities (2) and (11) thanks to Lemma 2.4. Since the image of $y\cdot(x,\xi) = ([y,x],\psi)$ in L_0 is $[y,x]$, the identity (2) holds for $\mathbf{Z}_0(L_*) \to L_0$. Finally for $A = (x,\xi)$, $A' = (x',\xi')$ we have

$$x \cdot A' + x' \cdot A = ([x,x'], t \mapsto t \cdot \xi'(x)) + ([x',x], t \mapsto t \cdot \xi(x')) = 0$$

thanks to Lemma 3.5. So the identity (11) also holds for $\mathbf{Z}_0(L_*) \to L_0$ and hence the result. ■

3.4. On homotopy groups of $\mathbf{Z}_*(L_*)$

Let L_* be a crossed module. In this section we investigate π_i of the crossed module $\mathbf{Z}_*(L_*)$, $i = 0, 1$. The case $i = 1$ is easy and the answer is given by the following lemma.

Lemma 3.7. *Let L_* be a crossed module. Then*

$$\pi_1(\mathbf{Z}_*(L_*)) \cong \mathsf{H}^0(\pi_0(L_*), \pi_1(L_*)).$$

Proof. By definition $a \in \pi_1(\mathbf{Z}_*(L_*))$ iff $\delta(a) = (0,0)$, thus when $\partial(a) = 0$ and $\xi_a(t) = 0$ for all $t \in L_0$. These conditions are equivalent to the conditions $a \in \pi_1(L_*)$ and $t \cdot a = 0$ for all $t \in L_0$ and hence the result. ■

To state our result on $\pi_0(\mathbf{Z}_*(L_*))$ we need to fix some notation.

For a Lie algebra L, we let $\mathsf{Z}(L)$ denote the centre of L, which is the set of elements $c \in L$ for which $[c,x] = 0$ for all $x \in L$. Moreover, if K is a Lie algebra on which L acts, we set

$$Ann_K(L) = \{x \in L | \, x \cdot k = 0 \text{ for all } k \in K\}.$$

The intersection of $\mathsf{Z}(L)$ and $Ann_K(L)$ is denoted by $\mathsf{Z}_K(L)$. Thus $c \in \mathsf{Z}_K(L)$ if and only if $c \cdot k = 0$ and $[x,c] = 0$ for all $x \in L$ and $k \in K$.

Let L_* be a crossed module. In this case we have defined

$$\mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)) \quad \text{and} \quad \mathsf{Z}_{L_1}(L_0).$$

We come back to the second one in the next section. Now we relate $\mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$ to $\pi_0(\mathbf{Z}_*(L_*))$. To this end denote the class of $x \in L_0$ in $\pi_0(L_*)$ by $\mathsf{cl}(x)$. Take now an element $(x,\xi) \in \mathbf{Z}_0(L_*)$. Accordingly, $\mathsf{cl}(x,\xi)$ denotes the class of (x,ξ) in $\pi_0(\mathbf{Z}_*(L_*))$.

Lemma 3.8. (i) *Let $(x,\xi) \in \mathbf{Z}_0(L_*)$. Then the class $\mathsf{cl}(x) \in \pi_0(L_*)$ belongs to*

$$\mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)).$$

Thus one has the well defined homomorphism $\mathbf{Z}_0(L_*) \xrightarrow{\omega'} \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$ given by $\omega'(x,\xi) := \mathsf{cl}(x)$.

(ii) *The composite map $L_1 \xrightarrow{\delta} \mathbf{Z}_0(L_*) \xrightarrow{\omega'} \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$ is trivial. Hence the map ω' induces a homomorphism*

$$\omega : \pi_0(\mathbf{Z}_*(L_*)) \to \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)).$$

(iii) *For a 1-cocycle* $\phi : \pi_0(L_*) \to \pi_1(L_*)$, *the pair* $(0, \tilde{\phi}) \in \mathbf{Z}_0(L_*)$, *where* $\tilde{\phi} : L_0 \to L_1$ *is the composite map*

$$L_0 \to \pi_0(L_*) \xrightarrow{\phi} \pi_1(L_*) \hookrightarrow L_1.$$

Moreover, the assignment $\phi \mapsto \mathsf{cl}(0, \tilde{\phi})$ *induces a homomorphism*

$$f : H^1(\pi_0(L_*), \pi_1(L_*)) \to \pi_0(\mathbf{Z}_*(L_*)).$$

(iv) *These maps fit in an exact sequence*

$$0 \to \mathsf{H}^1(\pi_0(L_*), \pi_1(L_*)) \xrightarrow{f} \pi_0(\mathbf{Z}_*(L_*)) \xrightarrow{\omega} \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)).$$

Proof. (i) In fact, according to the equation (14) of Definition 3.1, $[x, y] = \partial\xi(y)$ and hence $\mathsf{cl}([x, y]) = 0$ in $\pi_0(L_*)$. It follows that $\mathsf{cl}(x)$ is central in $\pi_0(L_*)$. Moreover, the equation (15) of Definition 3.1 tells us that $\xi(\partial a) = x \cdot a$. In particular, if $a \in \pi_1(L_*)$ (i.e. $\partial(a) = 0$) then $x \cdot a = 0$ and the result follows.

(ii) Take $a \in L_1$. By construction $\delta(a) = (\partial(a), \xi_a)$. Hence $\omega'\delta(a) = \mathsf{cl}(\partial(a)) = 0$.

(iii) Since ϕ is a 1-cocycle, $\tilde{\phi}$ satisfies the condition (16) of Definition 3.1. Next, the values of $\tilde{\phi}$ belong to $\pi_1(L_*)$, so $\partial\tilde{\phi} = 0$ and the condition (14) follows. Finally, $\tilde{\phi}(\partial a) = \phi(\mathsf{cl}(\partial a)) = \phi(0) = 0$ and (15) also holds. It remains to show that $\mathsf{B}^1(\pi_0(L_*), \pi_1(L_*))$ is sent to zero, i.e. that if $\phi(t) = t \cdot b$, for an element $b \in \pi_1(L_*)$, then $\mathsf{cl}(0, \tilde{\phi}) = 0$, but this follows from the fact that $\delta(b) = (0, \tilde{\phi})$.

(iv) Exactness at $\mathsf{H}^1(\pi_0(L_*), \pi_1(L_*))$: Assume $\phi : \pi_0(L_*) \to \pi_1(L_*)$ is a 1-cocycle such that $\mathsf{cl}(0, \tilde{\phi})$ is the trivial element in $\pi_0(\mathbf{Z}_*(L_*))$. That is, there exists a $c \in L_1$ such that $\delta(c) = (0, \tilde{\phi})$. Thus $\partial(c) = 0$ and $\tilde{\phi}(t) = -t \cdot c$.

So $c \in \pi_1(L_*)$ and the second equality implies that the class of ϕ is zero in $\mathsf{H}^1(\pi_0(L_*), \pi_1(L_*))$, proving that f is a monomorphism.

Exactness at $\pi_0(\mathbf{Z}_*(L_*))$: First take a cocycle $\phi : \pi_0(L_*) \to \pi_1(L_*)$. Then

$$\omega \circ f(\mathsf{cl}(\phi)) = \omega(\mathsf{cl}(0, \tilde{\phi})) = \mathsf{cl}(0) = 0.$$

Take now an element $(x, \xi) \in \mathbf{Z}_0(L_*)$ such that $\mathsf{cl}(x, \xi) \in \ker(\omega)$. Thus $x = \partial(a)$ for $a \in L_1$. Then $\mathsf{cl}(x, \xi) = \mathsf{cl}(y, \eta)$, where

$$(y, \eta) = (x, \xi) - \delta(a) = (x, \xi) - (\partial(a), t \mapsto -t \cdot a) = (x - \partial(a), t \mapsto \xi(t) + t \cdot a).$$

Since $y = x - \partial(a) = 0$, we see that $\partial\eta(t) = [y, t] = [0, t] = 0$ and

$$\eta(\partial c) = \xi(\partial(c)) + \partial(c) \cdot a = x \cdot c + \partial(c) \cdot a = 0$$

as $x = \partial(a)$. So $\eta = \tilde{\phi}$, where $\phi : \pi_0(L_*) \to \pi_1(L_*)$ is a 1-cocycle. Thus $f(\mathsf{cl}(\phi)) = \mathsf{cl}(0, \eta) = \mathsf{cl}(x, \xi)$ and exactness at $\pi_0(\mathbf{Z}_*(L_*))$ follows. ■

The map ω is not surjective in general. That is, for an element $x \in L_0$ such that $\mathsf{cl}(x) \in \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$, there is in general no linear map $\psi : L_0 \to L_1$ satisfying the conditions (14)-(16) of the Definition 3.1. However, there is a function satisfying (14) and (15), see the following Proposition. We will use this observation to extend the exact sequence constructed in Lemma 3.8.

Proposition 3.9. (i) *Take an element* $x \in L_0$ *such that* $\mathsf{cl}(x) \in \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$. *Then there exists a linear map* $\psi : L_0 \to L_1$ *such that* $\partial\psi(t) = [x,t]$ *and* $\psi(\partial a) = x \cdot a$ *for all* $t \in L_0$ *and* $a \in L_1$.

(ii) *The expression* $\bar{\theta}(s,t) := s \cdot \psi(t) - t \cdot \psi(s) - \psi([s,t])$ *is skew-symmetric, belongs to* $\pi_1(L_*)$ *and vanishes if* $s = \partial(a)$ *or* $t = \partial(a)$ *for some* $a \in L_1$. *Hence,* $\bar{\theta}$ *defines a map* $\Lambda^2\pi_0(L_*) \xrightarrow{\tilde{\theta}} \pi_1(L_*)$.

(iii) *The map* $\tilde{\theta}$ *is a* 2*-cocycle. That is, it satisfies the relation*

$$s \cdot \tilde{\theta}(t,r) - t \cdot \tilde{\theta}(s,r) + r \cdot \tilde{\theta}(s,t) = \tilde{\theta}([s,t],r) - \tilde{\theta}([s,r],t) + \tilde{\theta}([t,r],s).$$

(iv) *The class* $\theta \in \mathsf{H}^2(\pi_0(L_*), \pi_1(L_*))$ *is independent of the choice of* ψ. *The assignment* $x \mapsto \theta$ *defines a group homomorphism*

$$g : \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)) \to \mathsf{H}^2(\pi_0(L_*), \pi_1(L_*)).$$

Proof. (i) Since $\mathsf{cl}(x)$ is central in $\pi_0(L_*)$, for each $t \in L_0$ there exists an element $a_t \in L_1$ such that $\partial(a_t) = [x,t]$. We choose a linear splitting $L_0 = \mathsf{Im}(\partial) \oplus T$ and a linear basis (t_i) of T. The assignment $t_i \mapsto a_{t_i}$ can be extended as a linear map $\psi : T \to L_1$ for which $\partial(\psi(t)) = [x,t]$ holds for all $t \in T$. To define ψ on whole L_0, we first take $x \in \mathsf{Im}(\partial)$. In this case $x = \partial a$ and we can set $\psi(x) = x \cdot a$. This is well-defined since if $\partial a = \partial b$ we will have $a = b + c$, where $\partial c = 0$. It follows that $c \in \pi_1(L_*)$. By our assumption on x we also have $x \cdot c = 0$. It follows that $x \cdot a = x \cdot b$. This shows that ψ is well defined on $\mathsf{Im}(\partial)$.

Since L_0 is the direct sum of T and $\mathsf{Im}(d)$ we can extend ψ uniquely on L_0. It remains to check that $\partial(\psi(t)) = [x,t]$ holds for all $t \in L_0$. The required identity is linear on t and since it holds when $t \in T$, we can assume that $t = \partial(a)$. The result follows in this case from

$$\partial(\psi(t)) = \partial(\psi(\partial(a)) = \partial(x \cdot a) = [x, \partial(a)] = [x,t].$$

(ii) We have
$$\begin{aligned}\bar{\theta}(t,s) &= t \cdot \psi(s) - s \cdot \psi(t) - \psi([t,s]) \\ &= -(s \cdot \psi(t) - t \cdot \psi(s) - \psi([s,t]) = -\bar{\theta}(s,t).\end{aligned}$$

We also have
$$\begin{aligned}\partial(\bar{\theta}(s,t)) &= \partial(s \cdot \psi(t) - t \cdot \psi(s) - \psi([s,t])) \\ &= [s,[x,t]] - [t,[x,s]] - [x,[s,t]] = 0.\end{aligned}$$

Finally, let $t = \partial(a)$ for some $a \in L_1$. Then

$$\bar{\theta}(s, \partial(a)) = s \cdot \psi(\partial(a)) - \partial(a) \cdot \psi(s) - \psi([s, \partial(a)]).$$

We have $\psi(\partial(a)) = x \cdot a$ because ψ satisfies equation (15). Then by equation (11) and because ψ satisfies equation (14), we have $-\partial(a) \cdot \psi(s) = \partial(\psi(s)) \cdot a = [x,s] \cdot a$. Finally, $-\psi([s,\partial(a)]) = -\psi(\partial(s \cdot a)) = -x \cdot (s \cdot a)$ because ∂ is a Lie module homomorphism and ψ satisfies equation (15). Thus,

$$\bar{\theta}(s, \partial(a)) = s \cdot (x \cdot a) + [x,s] \cdot a - x \cdot (s \cdot a) = 0.$$

Hence, $\tilde{\theta} : \Lambda^2\pi_0(L_*) \to \pi_1(L_*)$ is a well-defined map.

(iii) The 2-cocycle condition follows from direct computation.

(iv) If ψ and ψ' both satisfy the conditions in (i), then $\psi - \psi'$ vanishes on $\mathsf{Im}(\partial)$ and takes values in $\ker(\partial) = \pi_1(L_*)$. In this way we obtain a well-defined map $h : \pi_0(L_*) \to \pi_1(L_*)$ such that $\psi - \psi'$ is equal to the composite map

$$L_0 \twoheadrightarrow \pi_0(L_*) \xrightarrow{h} \pi_1(L_*) \hookrightarrow L_1 .$$

From this the result follows, because $\bar{\theta} - \bar{\theta}' = d(h)$, where θ' is the function corresponding to ψ' as in (ii) and

$$d : \hom(\pi_0(L_*), \pi_1(L_*)) \to \hom(\Lambda^2 \pi_0(L_*), \pi_1(L_*))$$

is the coboundary map in the standard complex computing the Lie algebra cohomology. ■

Corollary 3.10. *Let L_* be a crossed module of Lie algebras. Then one has an exact sequence*

$$0 \to \mathsf{H}^1(\pi_0(L_0), \pi_1(L_*)) \xrightarrow{f} \pi_0(\mathbf{Z}_*(L_*)) \xrightarrow{\omega} \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*)) \xrightarrow{g} \mathsf{H}^2(\pi_0(L_*), \pi_1(L_*)).$$

Proof. According to Lemma 3.8 we only need to show exactness at $\mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$. To this end, take $(x, \xi) \in \mathbf{Z}_0(L_*)$. Since $\omega'(x, \xi) = \mathsf{cl}(x)$, we can choose $\psi = \xi$ for $g(\mathsf{cl}(x))$. Clearly $\bar{\theta} = 0$ for this ψ and hence $g \circ \omega = 0$.

Take now $x \in L_0$ such that $\mathsf{cl}(x) \in \mathsf{Z}_{\pi_1(L_*)}(\pi_0(L_*))$. Assume $g(\mathsf{cl}(x)) = 0$. We have to show that there exists a map $\xi : L_0 \to L_1$ such that $(x, \xi) \in \mathbf{Z}_0(L_*)$. According to part (i) of Proposition 3.9, we can choose a map $\psi : L_0 \to L_1$ such that the pair (x, ψ) satisfies all requirements in Definition 3.1 except perhaps the relation (16). By construction, $g(x)$ is the class of the 2-cocycle $\bar{\theta}$ in $\mathsf{H}^2(\pi_0(L_*), \pi_1(L_*))$. As this class is zero, $\bar{\theta}$ is a coboundary and therefore there exists a function $\phi : \pi_0(L_*) \to \pi_1(L_*)$ for which $\bar{\theta}(s,t) = s \cdot \phi(t) - t \cdot \phi(s) - \phi([s,t])$.

Since ϕ takes values in $\pi_1(L_*)$, the map $\psi'(t) = \psi(t) - \phi(t)$ also satisfies the conditions in (iii), thus we can replace ψ by ψ'. The above equation shows that the corresponding 2-cocycle $\bar{\theta}'$ vanishes, meaning that ψ' is a 1-cocycle. Thus $(x, \psi') \in \mathbf{Z}_0(L_*)$ and exactness follows. ■

3.5. Relation to nonabelian cohomology

In this section we relate our centre to the non-abelian cohomology of Lie algebras developed by D. Guin [4].

Let L_* be a crossed module. The group $\mathsf{H}^0(L_0, L_*)$ is defined by

$$\mathsf{H}^0(L_0, L_*) = \{a \in L_1 | \, \partial a = 0 \text{ and } x \cdot a = 0 \text{ for all } x \in L_0\}.$$

Clearly, $\mathsf{H}^0(L_0, L_*) = \mathsf{H}^0(\pi_0(L_*), \pi_1(L_*))$. It is a central subgroup of L_1.

In order to define the first cohomology group $\mathsf{H}^1(L_0, L_*)$, we first introduce the group $\mathsf{Der}_{L_0}(L_0, L_1)$. Elements of $\mathsf{Der}_{L_0}(L_0, L_1)$ are pairs (g, γ) (see [4, Definition 2.2.]), where $g \in L_0$ and $\gamma : L_0 \to L_1$ is a function for which two conditions hold:

$$\gamma([s,t]) = s \cdot \gamma(t) - t \cdot \gamma(s) \quad \text{and} \quad \partial\gamma(t) = [g,t].$$

Here $g, s, t \in L_0$. Comparing with Definition 3.1, we see that these are exactly the conditions 14 and 16 of Definition 3.1. Hence $\mathbf{Z}_0(L_*) \subset \mathsf{Der}_{L_0}(L_0, L_1)$.
We define a Lie algebra structure on $\mathsf{Der}_{L_0}(L_0, L_1)$ as follows.

If $(g,\gamma),(g',\gamma') \in \mathsf{Der}_{L_0}(L_0,L_1)$, then $([g,g'],\gamma*\gamma') \in \mathsf{Der}_{\mathbf{G}_0}(\mathbf{G}_0,\mathbf{G}_1)$ (see [4, Lemme 2.3.1]), where $\gamma*\gamma'$ is defined by

$$(\gamma*\gamma')(t) = \gamma(g'\cdot t) - \gamma'(g\cdot t).$$

Then $\mathbf{Z}_0(L_*)$ is a subalgebra of $\mathsf{Der}_{L_0}(L_0,L_1)$.
The group $\mathsf{H}^1(L_0,L_*)$ is defined as the quotient $\mathsf{Der}_{L_0}(L_0,L_1)/I$, where

$$I = \{(\partial(a)+c, \eta_a)\}$$

Here c is a central element of L_0, $a \in L_1$ and $\eta_a : L_0 \to L_1$ is defined by

$$\eta_a(t) = t\cdot a.$$

The fact that I is an ideal is proved in [4, Lemma 2.4]. The following fact is a direct consequence of the definition.

Lemma 3.11. *One has a commutative diagram with exact rows*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathsf{H}^0(L_0,L_*) & \longrightarrow & L_1 & \xrightarrow{\ \delta\ } & \mathsf{Der}_{L_0}(L_0,L_1) & \longrightarrow & \mathsf{H}^1(L_0,L_*) & \longrightarrow & 1\\
 & & \uparrow{\cong} & & \uparrow{Id} & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \pi_1(\mathbf{Z}_*(L_*)) & \longrightarrow & L_1 & \xrightarrow{\ \delta\ } & \mathbf{Z}_0(L_*) & \longrightarrow & \pi_0(\mathbf{Z}_*(L_*)) & \longrightarrow & 1
\end{array}$$

Remark 3.12. Guin's definition [4, Lemme 2.3.1] of the Lie algebra structure on $\mathsf{Der}_{L_0}(L_0,L_1)$ agrees with our definition of the Lie algebra structure on $\mathbf{Z}_0(L_*)$. Guin's definition directly translates to

$$[(x,\xi),(y,\eta)] = \xi([y,t]) - \eta([x,t]),$$

while from Proposition 3.2 we have

$$\{(x,\xi),(y,\eta)\} = \xi(y),$$

which lifts to
$$[(x,\xi),(y,\eta)] = -t\cdot\xi(y).$$

However,
$$\xi([y,t]) - \eta([x,t]) = -t\cdot\xi(y),$$

according to Lemma 3.5 and so the definitions agree.

Addendum

In [10] we proved that the centre of a crossed module is intimately related to the Drinfeld centre of a monoidal category [7]. Here we introduce the notion of a centre for Lie 2-algebras. We consider strict Lie 2-algebras according to Definition 40 in [1], which are the Lie analogues of categorical groups. By definition a (strict) Lie 2-algebra is a Lie algebra object in the category of all small categories. Thus a Lie 2-algebra is a category $\mathbf{L}$ whose set of objects is denoted by $\mathbf{L}_0$ and whose set of arrows is denoted by $\mathbf{L}_1$. Two bifunctors are given: the addition $+ : \mathbf{L}\times\mathbf{L} \to \mathbf{L}$

and bracket $[-,]:\mathbf{L}\times\mathbf{L}\to\mathbf{L}$. Also, for any $k\in K$ an endofunctor $\lambda_k:\mathbf{L}\to\mathbf{L}$ (the multiplication by a scalar) is given such that the all axioms of a Lie algebra hold strictly. For example, the distributivity law $k(x+y)=kx+ky$ implies the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{L}\times\mathbf{L} & \xrightarrow{+} & \mathbf{L} \\ {\scriptstyle \lambda_k\times\lambda_k}\downarrow & & \downarrow{\scriptstyle \lambda_k} \\ \mathbf{L}\times\mathbf{L} & \xrightarrow{+} & \mathbf{L}. \end{array}$$

It is well-known that the category of Lie 2-algebras and the category of crossed modules are equivalent, see Proposition 48 in [1]. We recall how to obtain Lie 2-algebras from crossed modules.

Let $\partial:X_1\to X_0$ be a linear map. Then one has the category $\mathsf{Cat}(X_*)$. Objects are elements of X_0 and a morphism from x to y is given by the diagram $x\xrightarrow{a}y$, where $a\in X_1$ satisfies the condition $\partial(a)+x=y$. Sometimes this morphism is also denoted by (x,a). Thus $(x,a):x\to\partial(a)+x$. The composition law in $\mathsf{Cat}(X_*)$ is induced by the addition in L_1. The identity morphism id_x of an object x is $x\xrightarrow{0}x$. So $\mathsf{id}_x=(x,0)$. For any $k\in K$ we have an endofunctor $\lambda_k:\mathsf{Cat}(X_*)\to\mathsf{Cat}(X_*)$ which on objects is given by $\lambda_k(x)=kx$ and on morphisms it is given by

$$\lambda_k(x\xrightarrow{a}y)=kx\xrightarrow{ka}ky.$$

We also have the bifunctor

$$+:\mathsf{Cat}(X_*)\times\mathsf{Cat}(X_*)\to\mathsf{Cat}(X_*)$$

which on objects is given by $(x,y)\mapsto x+y$ and on morphisms by

$$(x\xrightarrow{a}y)+(x'\xrightarrow{a'}y)'=x+x'\xrightarrow{a+a'}y+y'.$$

In this way we obtain a strict K-vector space object in the category of small categories.

In the case when instead of a linear map $\partial:X_1\to X_0$, a crossed module of Lie algebras is given, there is a bifunctor $[-,-]:\mathsf{Cat}(L_*)\times\mathsf{Cat}(L_*)\to\mathsf{Cat}(L_*)$, which on objects is given by the Lie algebra structure on L_0 and on morphisms it assigns to $x\xrightarrow{a}y$ and $x'\xrightarrow{a'}y'$ the morphism

$$[x,x']\xrightarrow{\partial(a)\cdot a'-x'\cdot a+x\cdot a'}[y,y'] .$$

In particular, we have the morphism $[x,x']\xrightarrow{x\cdot a'}[x,y']$, which is also denoted by $[x,x']\xrightarrow{[\mathsf{id}_x,a']}[x,y']$. In other words

$$[(x,a),(x',a')]=([x,x'],\partial(a)\cdot a'-x'\cdot a+x\cdot a').$$

For any $(x,\xi)\in\mathbf{Z}_0(L_*)$ and $y\in L_0$ one puts $\tau_y=-2\xi(y)$. Then τ_y defines a morphism $\tau_y:[x,y]\to[y,x]$.

In fact, the collection $(\tau_y)_{y\in L_0}$ defines a natural transformation of endofunctors

$$\tau : [x,-] \to [-,x]$$

for which additionally the following equality holds

$$\tau_{[y,z]} = y\cdot\tau_z - z\cdot\tau_y.$$

This equality can be rewritten as

$$\tau_{[y,z]} = [\mathsf{id}_y,\tau_z] - [\mathsf{id}_z,\tau_y]. \tag{18}$$

This suggests that we can define the centre of a Lie 2-algebra $\mathbf{L}$ to be the category whose objects are pairs (x,τ), where $x\in\mathbf{L}_0$ is an object of $\mathbf{L}$ and τ is a natural transformation $\tau:[x,-]\to[-,x]$, i.e., the collection of morphisms $(\tau_y:[x,y]\to[y,x])_{y\in\mathbf{L}_0}$ such that for any morphism $a:y\to z$ one has a commutative diagram

$$\begin{array}{ccc} [x,y] & \xrightarrow{\tau_y} & [y,x] \\ {\scriptstyle[\mathsf{id}_x,a]}\downarrow & & \downarrow{\scriptstyle[a,\mathsf{id}_x]} \\ [x,z] & \xrightarrow[\tau_z]{} & [z,x] \end{array} \quad .$$

One requires that additionally the equality (18) holds. A morphism $(x,\tau)\to(x',\tau')$ is given by a morphism $a:x\to x'$ for which the following diagram commutes

$$\begin{array}{ccc} [x,y] & \xrightarrow{\tau_y} & [y,x] \\ {\scriptstyle[a,\mathsf{id}_y]}\downarrow & & \downarrow{\scriptstyle[\mathsf{id}_y,a]} \\ [x',y] & \xrightarrow[\tau'_y]{} & [y,x']. \end{array}$$

Denote this category by $\mathcal{Z}(\mathbf{L})$ and call it the centre of the Lie 2-algebra $\mathbf{L}$. The bifunctor

$$[-,-]:\mathcal{Z}(\mathbf{L})\times\mathcal{Z}(\mathbf{L})\to\mathcal{Z}(\mathbf{L})$$

is defined by

$$[(x,\tau),(y,\eta)] = ([x,y],\theta)$$

where

$$\theta_z : [[x,y],z]\to[z,[x,y]]$$

is given by

$$\theta_z = \tau_{[y,z]} - \eta_{[x,z]}.$$

Here we used the facts that

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]] \quad \text{and} \quad [z,[x,y]] = [[y,z],x] - [[x,z],y].$$

Our computations show that $\mathcal{Z}(\mathbf{L})$ is again a Lie 2-algebra and in fact there is also braiding (see [11],[3] for braided Lie 2-algebras) given by

$$\tau_{[(x',\tau'),(x'',\tau'')]} = \tau'_y.$$

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