

# Decision and Search in Non-Abelian Cramer-Shoup Public Key Cryptosystem

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*Dedicated to Benjamin Fine on the occasion of his 60th birthday.*

Received: March 29, 2009

Revised manuscript received: August 17, 2009

A method for non-abelian Cramer-Shoup cryptosystem is presented. The role of decision and search is explored, and the platform of solvable / polycyclic group is suggested. In the process we review recent progress in non-abelian cryptography and post some open problems that naturally arise from this path of research.

## 1. Introduction

The field of combinatorial group theory began with decision problems of Max Dehn from 1912, known as the *word problem*, the *conjugacy problem* and the *isomorphism problem*. These fields have developed close connections to topology, logic and computer science. Word problem: Let  $G$  be a group given by a finite presentation. Does there exist an algorithm to determine if an arbitrary word  $w$  in the generators of  $G$  whether or not  $w =_G 1$ ? Conjugacy problem: Let  $G$  be a group given by a finite presentation. Does there exist an algorithm to determine if an arbitrary pair of words  $u$  and  $v$  in the generators of  $G$  whether or not  $u$  and  $v$  define conjugate elements of  $G$ ? By the celebrated theorem of Novikov [25] and Boone [7], there are groups for which these questions are undecidable: they cannot be answered algorithmically. Nevertheless, because of the practical

\*The first author was supported by a grant from research foundation of CUNY (PSC-CUNY) and the Faculty Fellowship Publication Program Award from City University of New York.

importance of these problems, a lot of effort is devoted to the development of methods for investigating finitely presented groups. The study of designing such algorithms and implementing them, is computational group theory. These topics have generated much attention and proved of importance in modern cryptography, namely algebraic cryptography, initiated in 1999 by I. Anshel, M. Anshel and D. Goldfeld [1]. Algebraic key establishment protocols based on the difficulty of solving equations over algebraic structures are described as a theoretical basis for constructing public-key cryptosystems. These applications rely strongly on complexity of decision problems in combinatorial group theory. Needless to say, complexity problems are some of the most important problems in mathematics.

### 1.1. Motivation

Key exchange problems are of central interest in cryptology [24]. The basic aim of key exchange problems is that two people who can only communicate via an insecure channel want to find a common secret key. Key exchange methods are usually based on one-way functions; that is, functions which are easy to compute, while their inverses are difficult to determine (see [26, 15]). Here ‘easy’ and ‘difficult’ can mean that the complexities or the practicality of the methods are far away from each other; ideally, the one-way function has a polynomial complexity and its inverse has an exponential complexity. Many of the known one-way functions have a common problem: it is often easy to find a one-way function with a polynomial complexity, but showing that there is no inverse function with similar complexity or practicality is usually the difficult part of the project, since the best inverse function might just not have been discovered yet. Hence it is of interest to investigate new one-way functions. The decision problems in combinatorial group theory have shown much potential for this purpose. Another reason is that, using Shors algorithm the discrete log problem and prime factorization problem admit polynomial-time quantum algorithm. That leaves the current cryptosystem in danger if the quantum computers are to be built! Non-abelian (a.k.a. Non-commutative) group theorists have been working in the field of non-commutative cryptography for about ten years, but a class of groups which provides a provably secure basis for the non-commutative protocols is not known yet. The groups of the authors interest are mainly polycyclic groups. These are natural generalizations of cyclic groups but are much more complex in their structure. Hence their algorithmic theory is more difficult.

In the last decade the Braid group cryptography has been of great interest to many researchers in the field and has been investigated by private and public sectors. No such investigation has been done for solvable / polycyclic groups.

The reason for developing a non-abelian Cramer-Shoup cryptosystem is to strengthen the non-commutative ElGamal cryptosystem [20]. We think that it should be secured against CCA1 and CCA2. The tested modification would be tested while such research can go on for developing the non-commutative ElGamal [20]. It has been pointed out to us that the goal of Cramer-Shoup was to defend ElGamal

against chosen ciphertext attacks. We do not know if the non-commutative ElGamal [20] can be attacked using chosen ciphertext attack. However in the meantime a C-S like cryptosystem can be developed to defend the non-commutative ElGamal public key cryptosystem [20].

## 2. Applications of Non-commutative Group Theory in Cryptography

In 1984, Wagner et al. [32] proposed an approach to design of public-key cryptosystems based on groups and semigroups with undecidable word problem. In 2005, Birget et al. [6] pointed out that Wagner's idea is actually not based on word problem, but on, generally easier, premise problem. And finally, Birget et al. proposed a new public-key cryptosystem which is based on finitely presented groups with hard word problem. In 1999, Anshel et al. [1] proposed a compact algebraic key establishment protocol. The foundation of their method lies in the difficulty of solving equations over algebraic structures, in particular non-commutative groups. In their pioneering paper, they also suggested that braid groups may be good alternative platforms for PKC (public-key cryptography). Subsequently, Ko et al. [22] and Dehornoy [9] developed the theory and practice of braid-based cryptography. The security foundation is that the conjugator search problem (CSP) is intractable when the system parameters, such as braid index and the canonical length of the working braids, are selected properly. In 2002, certain homomorphic cryptosystems were constructed for the first time for non-abelian groups due to Grigoriev and Ponomarenko [17]. Shortly afterwards, they [18] extended their method to arbitrary nonidentity finite groups based on the difficulty of the membership problem for groups of integer matrices. In 2004, Eick and Kahrobaei [12] proposed a new cryptosystem based on polycyclic groups. In 2005 Baumslag, Fine and Xu proposed public key cryptosystem using the modular group [3], [4]. In 2005, Shpilrain and Ushakov [28] suggested that R. Thompson's group may be a good platform for constructing public-key cryptosystems. In their contribution, the key assumption is the intractability of the decomposition problem, which is more general than the conjugator search problem, defined over R. Thompson's group, also a infinite non-abelian group given by finite presentation. In [8], Cao et al. propose a new method for designing public key cryptosystems based on general non-commutative rings. The key idea of their proposal is that for a given non-commutative ring, they can define polynomials and take them as the underlying work structure. In 2006 Kahrobaei and Khan, proposed a non-commutative key-exchange scheme which generalizes the classical ElGamal Cipher [20]. This scheme is closer to the spirit of ElGamal and the they proposed polycyclic groups for such protocol.

Recently, Gilman, et al. in [14] study the algorithmic security of the Anshel-Anshel-Goldfeld (AAG) key exchange scheme and show that contrary to prevalent opinion, the computational hardness of AAG depends on the structure of the chosen subgroups, rather than on the conjugacy problem of the ambient braid group. Proper choice of these subgroups produces a key exchange scheme which is resistant to all known attacks on AAG.

In recent work [16], Grigoriev et al., show new constructions of cryptosystems based on group invariants and suggest methods to make such cryptosystems secure in practice. In their paper they introduce a new notion of cryptographic security, a provable break, and prove that cryptosystems based on matrix group invariants and also a variation of the Anshel-Anshel-Goldfeld key agreement protocol for modular groups are secure against provable worst-case break unless  $NP \subset RP$ .

### 2.1. Non-Commutative public key cryptosystem using Conjugacy

The following is the public key cryptosystem problem proposed by Kahrobaei and Khan [20].

Let  $G$  be a finitely presented non-abelian group having solvable word problem. Let  $S, T < G$  be finitely generated proper subgroups of  $G$ , for which the subgroup  $[S, T]$  (i.e. the subgroup generated by  $\{[s, t] \mid s \in S, t \in T\}$ ) is the trivial subgroup consisting of just the identity element of  $G$ . Now suppose two parties, Alice and Bob, wish to establish a session key over an unsecured network.

Bob takes  $s \in S$ ,  $b \in G$  and publishes  $b$  and  $c = b^s$  as his public keys, keeping  $s$  as his private key. If Alice wishes to send  $x \in G$  as a session key to Bob, she first chooses a random  $t \in T$  and sends

$$E = x^{(c^t)}$$

to Bob, along with the header

$$h = b^t.$$

Bob then calculates  $(b^t)^s = (b^s)^t = c^t$  with the header. He can now compute

$$E' = (c^t)^{-1}$$

which allows him to decrypt the session key,

$$\left(x^{(c^t)}\right)^{E'} = \left(x^{(c^t)}\right)^{(c^t)^{-1}} = x.$$

The element  $x \in G$  can now be used as a session key.

The feasibility of this scheme rests on the assumption that products and inverses of elements of  $G$  can be computed efficiently. To deduce Bob's private key from public information would require solving the equation  $c = b^s$  for  $s$ , given the public values  $b$  and  $c$ . This is called the *conjugacy search problem* for  $G$ . Thus the security of this scheme rests on the assumption that there is no fast algorithm for solving the conjugacy search problem for the group  $G$ .

### 2.2. Non-Commutative public key cryptosystem using Power Conjugacy

The following is the public key cryptosystem problem which was proposed by Kahrobaei and Khan in [20] What if the conjugacy search problem is tractable?

The next paradigm embellishes conjugacy-based key exchange to address this possibility. Bob takes  $s \in S$ ,  $g \in G$  and  $n \in \mathbb{N}$  and publishes  $v = g^n$  and  $w = s^{-1}gs$  as his public keys. Note that the centralizer of  $g$  is trivial. Bob keeps  $n \in \mathbb{N}$  and  $s \in S$  as his private keys. Note that  $v$  and  $w$  satisfy  $w^n = s^{-1}vs$ . If Alice wishes to send  $x \in G$  to Bob, she first chooses a random  $m \in \mathbb{N}$  and  $t \in T$ . To encrypt  $x$ , Alice computes

$$E = x^{-1}t^{-1}(v)^m tx = x^{-1}t^{-1}g^{mn}tx$$

and sends it to Bob along with the header

$$h = t^{-1}w^m t = t^{-1}s^{-1}g^m st.$$

Bob receives  $E$  and  $h$ , and computes

$$E' = sh^n s^{-1} = t^{-1}g^{mn}t.$$

Note that  $E = x^{-1}E'x$ , so if Bob can solve the conjugacy search problem, he can obtain  $x \in G$ , which can then serve as the common secret that can be used as a symmetric session key for secure communication.

The feasibility of this scheme rests on the assumption that products and inverses of elements of  $G$  can be computed efficiently, and that the conjugacy problem is solvable. To deduce Bob's private key from public information would require solving the equation  $w^n = s^{-1}g^n s$  for  $n$  and  $s$ , given the public values  $g^n$  and  $w$ . This is called the *power conjugacy search problem* for  $G$ . Thus the security of this scheme rests on the assumption that there is no fast algorithm for solving the power conjugacy search problem for the group  $G$ .

### 3. Cramer-Shoup cryptosystem

Cramer-Shoup cryptosystem is a generalization of ElGamal Key exchange problems, it is provably secure against adaptive chosen ciphertext attack. Moreover, the proof of security relies only on a standard intractability assumption, namely, the hardness of the Diffie-Hellman decision problem in underlying group (see [31], [30]). A hash function  $H$  whose output can be interpreted as a number in  $\mathbb{Z}_q$  (where  $q$  is a large prime number). It should be hard to find collisions in  $H$ . In fact, with a fairly minor increase in cost and complexity, one can eliminate  $H$  altogether.

#### 3.1. Definition of provably secure against adaptive chosen ciphertext attack

The right formal, mathematical definition of security against active attacks evolved in a sequence of papers by Naor and Yung, Rackoff and Simon, Dolev, Dwork and Naor. The notion is called chosen ciphertext security or equivalently non-malleability. The intuitive thrust of this definition is that even if an adversary can get arbitrary ciphertexts of his choice decrypted, he still gets no partial information about other encrypted messages. (for more information see [31], [30])

### 3.2. The Cramer-Shoup Scheme

**Secret Key:** random  $x_1, x_2, y_1, y_2, z \in \mathbb{Z}_q$ .

**Public Key:**

$$\begin{aligned} g_1, g_2 &\text{ in } G \text{ (but not 1)} \\ c &= g_1^{x_1} g_2^{x_2}, \quad d = g_1^{y_1} g_2^{y_2} \\ h &= g_1^z. \end{aligned}$$

**Encryption of  $m \in G$ :**  $(u_1, u_2, e, v)$ , where

$$\begin{aligned} u_1 &= g_1^r, \quad u_2 = g_2^r, \quad e = h^r m, \quad v = c^r d^{r\alpha}, \\ r &\text{ in } \mathbb{Z}_q \text{ is random, and } \alpha = H(u_1, u_2, e). \end{aligned}$$

**Decryption of  $(u_1, u_2, e, v)$ :**

$$\begin{aligned} \text{If } v &= u_1^{x_1 + \alpha y_1} u_2^{x_2 + \alpha y_2}, \quad \text{where } \alpha = H(u_1, u_2, e) \\ \text{then } m &= e / u_1^z \\ \text{else } &\text{“reject”}. \end{aligned}$$

## 4. Non-commutative Cramer-Shoup Cryptosystem

The objective of this section is to propose non-commutative Cramer-Shoup cryptosystem and analyze its security. As it is pointed out in [31], [30], with a fairly minor increase in cost and complexity, one can eliminate the hash function altogether and that is what we are doing in this scheme. Can our protocol be extended so that our C-S like cryptosystem can incorporate hash functions? Here we have in mind the winner of the SHA 3 hash function competition sponsored by NIST.

Let  $G$  be a non-abelian group such that every element has a normal form, and the conjugacy search problem is hard.

**Secret Key:** random  $x_1, x_2, y_1, y_2, z \in G$ .

**Public Key:**

$$\begin{aligned} g_1, g_2 &\text{ in } G \text{ (but not 1), such that } [g_2^{x_2}, g_1^{y_1}] = 1 \\ c &= g_1^{x_1} g_2^{x_2}, \quad \text{where } g_1^{x_1} = x_1^{-1} g_1 x_1 \text{ and } g_2^{x_2} = x_2^{-1} g_2 x_2 \\ d &= g_1^{y_1} g_2^{y_2}, \quad \text{where } g_1^{y_1} = y_1^{-1} g_1 y_1 \text{ and } g_2^{y_2} = y_2^{-1} g_2 y_2 \\ h &= g_1^z = z^{-1} g_1 z. \end{aligned}$$

**Encryption of  $m \in G$ :**  $(u_1, u_2, e, v)$ , where

$$u_1 = g_1^r = r^{-1} g_1 r, \quad u_2 = g_2^r = r^{-1} g_2 r, \quad e = m^{h^r}, \quad v = c^r d^r.$$

Suppose  $r$  be a random element in  $G$  such it commutes with  $x_1, x_2, y_1, y_2$  and  $z$ .

**Decryption of  $(u_1, u_2, e, v)$ :**

$$\begin{aligned} \text{If } v &= u_1^{x_1} u_1^{y_1} u_2^{x_2} u_2^{y_2} \\ \text{then } m &= e^{(u_1^z)^{-1}} = m^{h^r u_1^{z^{-1}}} = m^{u_1^z (u_1^z)^{-1}} \\ \text{else } &\text{“reject”}. \end{aligned}$$

Note that the authors currently experimenting the search for the secure randomly generated  $r$ .

## 5. Polycyclic groups new platform for cryptology

Polycyclic groups are a natural generalization of cyclic groups, but they are much more complex in their structure than cyclic groups. Hence their algorithmic theory is more difficult and thus it seems promising to investigate classes of polycyclic groups as candidates to have a more substantial platform perhaps more secure. Recall that a group is called polycyclic if there exists a polycyclic series through the group; that is, a subnormal series of finite length with cyclic factors. There are two different natural representations for these groups which can be used for computations: polycyclic presentations and matrix groups over the integers. We refer to [29], [27], [13], [11], [5] and [19] for background and a more detailed introduction to polycyclic groups.

In particular polycyclic groups are linear. In this setting, both group multiplication and the word problem are efficiently solvable, since matrix multiplication for such groups is solvable in polynomial time. One can show that for a subgroup of a general linear group, if two elements are conjugate then they have the same Jordan normal form. Using this lemma one conclude that the search conjugacy problem in any subgroup of the General Linear group is solvable. However the complexity is not known but is conjectured to be exponential. Kharlampovich [21] showed that there is a finitely presented solvable group with an undecidable word problem. It follows by a theorem of Arzhantseva-Osin in [2] that the word problem is in NP for any finitely generated metabelian group. For polycyclic groups, some decision problems are known to be difficult but not provably so [10, 23].

A large growth rate would imply a large key space for the set of all possible keys, thus making the brute force search of this space intractable. Ideally, we would like to use groups which exhibit provably exponential growth. A large class of polycyclic groups are known to have an exponential growth rate, namely those which are not virtually nilpotent, by results of Wolf and Milnor in 1968. Using these observations about the polycyclic groups, we conjecture they are the best candidate for the non-abelian Cramer-Shoup cryptosystem.

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