Georgian Mathematical Journal
Volume 11 (2004), Number 1, 177–185

POSITIVE SOLUTIONS OF A NEUTRAL DIFFERENCE EQUATION WITH POSITIVE AND NEGATIVE COEFFICIENTS

XIAN HUA TANG AND SUI SUN CHENG

Abstract. A neutral difference equation with positive and negative coefficients is a natural model for a population with depletion of the aged. In this paper, we obtain conditions which are sufficient for one such equation to support an eventually positive population.

2000 Mathematics Subject Classification: 34A10.
Key words and phrases: Neutral difference equation, positive solution.

1. Introduction

Let x_n denote the population size of a species in the time period n. The Malthus model asserts that the change $x_{n+1} - x_n$ is proportional to x_n, that is,

$$x_{n+1} - x_n = qx_n, \quad n = 0, 1, 2, \dots,$$

where q is a proportionality constant. Assuming that it takes m periods of time for the newborn to mature and be ready for reproduction, and that the growth rate is time dependent, a more general model may appear in the following form

$$x_{n+1} - x_n = q_n x_{n-m}, \quad n = 0, 1, 2, \dots .$$

When external factors (such as infectious diseases) cause depletion of the aged ones in the population, it is possible that the depleted population can be modeled in the form

$$x_{n-k+1} - x_{n-k} = p_n x_{n-\tau}, \quad n = 0, 1, 2, \dots,$$

where k and τ are nonnegative integers such that $\tau \geq k$. Therefore, if we denote $x_{n+1} - x_n$ by the standard notation Δx_n, the population now obeys the model

$$\Delta x_n - \Delta x_{n-k} = q_n x_{n-m} - p_n x_{n-\tau}, \quad n = 0, 1, \dots .$$

In this note, we will be concerned with a slightly more general neutral difference equation of the form

$$\Delta\left(x_n - c_n x_{n-k}\right) + p_n x_{n-\tau} - q_n x_{n-m} = 0, \quad n = 0, 1, 2, \dots, \tag{1}$$

where k, τ, m are nonnegative integers such that $k > 0$ and $\tau \geq m + 1$ (but not $\tau \geq k$). The assumption $\tau \geq m + 1$ is natural since the aged ones are more susceptible to depletion. We will also assume that the growth rate p_n and the depletion rate q_n are nonnegative for $n \geq 0$. It is also natural to assume that

ISSN 1072-947X / $8.00 / © Heldermann Verlag *www.heldermann.de*

the growth rate dominates the depletion rate in some sense. In this note, we will assume that

$$h_n \equiv p_n - q_{n-\tau+m} \geq 0, \quad n = 0, 1, 2, \dots .$$

Finally, we will assume that $\{c_n\}_{n=0}^{\infty}$ is a nonnegative sequence. To avoid trivial cases, we will also assume that $\{h_n\}_{n=0}^{\infty}$ has a positive subsequence.

There are several studies which are concerned with similar equations, see e.g. [1]–[4] and the references cited therein. In particular, in [1], [2], comparison and oscillation theorems are derived. Here we will deal with the question of the existence of eventually positive solutions. This question is important since for our model to be realistic, it must support a positive population for all large time periods. We will obtain two results which are sufficient for the existence of eventually positive solutions with stability requirements. The conditions we obtain are novel in the sense that they do not require

$$c_n + \sum_{i=n-\tau-m}^{n-1} q_i \leq 1$$

for all large n, which is stipulated in [1]–[4] and other works.

For the sake of convenience, we will set

$$\delta = \begin{cases} k & q_n = 0 \text{ for all large } n \\ \min\{k, m\} & \text{otherwise} \end{cases},$$

and

$$\rho = \begin{cases} \tau & q_n = 0 \text{ for all large } n \\ \max\{k, \tau\} & \text{otherwise} \end{cases}.$$

By a solution of (1), we mean a real sequence $\{x_n\}$ which is defined for $n \geq -\rho$ and satisfies (1). By writing (1) as a recurrence relation, it is easy to derive an existence and uniqueness theorem for its solutions. A solution $\{x_n\}$ of (1) will be called eventually positive if $x_n > 0$ for all large n.

2. Existence Criteria

One useful result for finding an eventually positive solution of (1) is the following comparison theorem.

Lemma 1. *Suppose*

$$c_n + \sum_{i=n-\tau+m}^{n-1} q_i + \sum_{i=n}^{n+\tau-1} h_i > 0 \tag{2}$$

for all large n. If the functional inequality

$$z_n \geq c_n z_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} + \sum_{i=n}^{\infty} h_i z_{i-\tau}, \quad n = 0, 1, 2, \dots, \tag{3}$$

has an eventually positive solution $\{z_n\}$, *then the functional equation*

$$x_n = c_n x_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i x_{i-m} + \sum_{i=n}^{\infty} h_i x_{i-\tau}, \quad n = 0, 1, 2, \ldots, \tag{4}$$

has an eventually positive solution $\{x_n\}$ *which satisfies* $0 < x_n \le z_n$ *for all large* n.

Proof. The idea of the proof is well known (see, e.g., [1, Theorem 1]), and therefore we will sketch it as follows. Suppose $z_n > 0$ and (2) holds for $n \ge N-\rho$ where $N \ge 0$. Let Ω be the set of all real sequences of the form $\{w_n\}_{n=N-\rho}^{\infty}$. Define an operator $T : \Omega \to \Omega$ by

$$(Tw)_n = 1, \; N-\rho \le n \le N-1,$$

and for $n \ge N$,

$$(Tw)_n = \frac{1}{z_n}\left\{c_n z_{n-k} w_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} w_{i-m} + \sum_{i=n}^{\infty} h_i z_{i-\tau} w_{i-\tau}\right\}.$$

Consider the following sequence $\left\{w^{(t)}\right\}$ of successive approximations: $w^{(0)} = \{1\}$, and $w^{(t+1)} = Tw^{(t)}$ for $t = 0, 1, 2, \ldots$. Clearly, in view of (3),

$$0 \le w_n^{(t+1)} \le w_n^{(t)} \le 1, \quad n \ge N, \quad t \ge 0.$$

Thus $\left\{w^{(t)}\right\}$ converges to some nonnegative sequence w^* which satisfies $w_n^* \le 1$ and

$$w_n^* z_n = c_n z_{n-k} w_{n-k}^* + \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} w_{i-m}^* + \sum_{i=n}^{\infty} h_i z_{i-\tau} w_{i-\tau}^*$$

for $n \ge N$. Clearly, the sequence $\{x_n\}$ defined by

$$x_n = w_n^* z_n, \quad n \ge N-\rho,$$

is an eventually nonnegative solution of (4). Since $w_n^* \le 1$ for $n \ge N$, we see that $x_n \le z_n$ for $n \ge N$. Finally, we may show that $x_n > 0$ for $n \ge N$ by means of condition (2). □

Now note that if $\{x_n\}$ is an eventually positive solution of (4), then by taking differences on both sides of (4), we see that it is also a solution of (1). Before we seek eventually positive solutions of (4), we prove a result for the following self-adjoint difference equation

$$\Delta(a_n \Delta y_n) + b_n y_n = 0, \quad n = 0, 1, 2, \ldots . \tag{5}$$

Lemma 2. *Suppose* $\{a_n\}_{n=0}^{\infty}$ *is a positive sequence and* $\{b_n\}_{n=0}^{\infty}$ *is a nonnegative sequence. If*

$$\sum_{i=n}^{\infty} b_i \le \frac{1}{4}\left\{\frac{1}{\sum_{i=0}^{n-1}(1/a_i)} + \frac{1}{\sum_{i=0}^{\infty}(1/a_i)}\right\}$$

for all large n, *then* (5) *has a solution* $\{y_n\}$ *which is eventually positive and eventually increasing.*

Proof. Set

$$w_n = \frac{1}{2\sum_{j=0}^{n-1}(1/a_j)},$$

then

$$\begin{aligned} w_n &= \sum_{i=n}^{\infty} \frac{w_i w_{i+1}}{a_i} + \frac{1}{4}\left\{\frac{1}{\sum_{i=0}^{n-1}(1/a_i)} + \frac{1}{\sum_{i=0}^{\infty}(1/a_i)}\right\} \\ &\geq \sum_{i=n}^{\infty} \frac{w_i w_{i+1}}{a_i} + \sum_{i=n}^{\infty} b_i \end{aligned}$$

for all large n. By means of the same technique described above, we may show that the functional equation

$$v_n = \sum_{i=n}^{\infty} \frac{v_i v_{i+1}}{a_i} + \sum_{i=n}^{\infty} b_i, \quad n = 0, 1, 2, \ldots, \tag{6}$$

also has an eventually nonnegative solution $\{v_n\}$. If $\sum_{i=n}^{\infty} b_i = 0$ for all large n, then by (5), there exists $c > 0$ such that $a_n \Delta y_n = c$. It follows that (5) has a solution $\{y_n\}$ which is eventually positive and eventually increasing. If $\sum_{i=n}^{\infty} b_i > 0$ eventually, then by (6), there exists a positive integer N such that $v_n > 0$ for $n \geq N$. Then by taking differences on both sides of (6) we see that

$$\Delta v_n + \frac{v_n v_{n+1}}{a_n} + b_n = 0, \quad n \geq N.$$

Let

$$y_n = \prod_{i=N}^{n-1}\left(1 + \frac{v_i}{a_i}\right), \quad n \geq N.$$

Then $y_n > 0$,

$$\Delta y_n = \frac{v_n}{a_n} y_n > 0,$$

and

$$\Delta(a_n \Delta y_n) = y_n \Delta v_n + v_{n+1}\Delta y_n = \left(\Delta v_n + \frac{v_n v_{n+1}}{a_n}\right) y_n = -b_n y_n$$

for $n \geq N$. □

Theorem 1. *Suppose that* (2) *holds for all large* n. *Suppose further that there is a positive nondecreasing sequence* $\{r_n\}$ *such that*

$$c_n r_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} \leq r_n, \tag{7}$$

and

$$\sum_{i=0}^{n-1} \frac{1}{r_i} \sum_{i=n}^{\infty} h_i r_{i-\tau} \leq \frac{\delta}{4}\left\{1 + \frac{\sum_{i=0}^{n-1}(1/r_i)}{\sum_{i=0}^{\infty}(1/r_i)}\right\}, \tag{8}$$

for all large n. *Then equation* (1) *has an eventually positive solution.*

Proof. Set

$$C_n = \frac{1}{r_{n-k}} \left\{ r_n - \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} \right\}, \tag{9}$$

for all large n, then in view of (7), $C_n \geq c_n$ for all large n. Next, note that by Lemma 2, (8) implies that the following self-adjoint difference equation

$$\Delta\left(r_n \Delta y_n\right) + \frac{1}{\delta} h_n r_{n-\tau} y_n = 0, \quad n = 0, 1, 2, \ldots,$$

has an eventually positive and eventually increasing solution $\{y_n\}$. Let $u_n = \Delta y_n$ for all large n. Then $u_n > 0$ and

$$\Delta u_n = \frac{\Delta(r_n \Delta y_n) - (\Delta y_{n+1}) \Delta r_n}{r_n} = \frac{-h_n r_{n-\tau} y_n/\delta - u_{n+1} \Delta r_n}{r_n} \leq 0$$

for all large n. Let N be a positive integer such that $u_n > 0, \Delta u_n \leq 0$ and (7) hold for $n \geq N$. Let us define a sequence $\{v_n\}$ by

$$v_n = \frac{1}{\delta} y_N, \quad N \leq n \leq N + \rho,$$

and

$$v_n = u_n + \frac{1}{r_n} \left(C_n r_{n-k} v_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} v_{i-m} \right), \quad n > N + \rho. \tag{10}$$

Then $v_n > 0$ for $n \geq N$ and

$$v_n = \frac{1}{\delta} y_N \leq \frac{1}{\delta} \left(\sum_{i=N}^{n-1} u_i + y_N \right) = \frac{1}{\delta} y_n, \ N \leq n \leq N + \rho. \tag{11}$$

For $N + \rho + 1 \leq n \leq N + \rho + \delta$,

$$\begin{aligned} v_n &= u_n + \frac{1}{r_n} \left(C_n r_{n-k} v_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} v_{i-m} \right) \\ &\leq u_n + \frac{1}{r_n} \left(C_n r_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} \right) \left(\frac{1}{\delta} \left(\sum_{i=N}^{n-\delta-1} u_i + y_N \right) \right) \\ &\leq \frac{1}{\delta} \sum_{i=N-\delta}^{n-1} u_i + \frac{1}{\delta} \left(\sum_{i=N}^{n-\delta-1} u_i + y_N \right) \\ &= \frac{1}{\delta} \left(\sum_{i=N}^{n-1} u_i + y_N \right) \\ &= \frac{1}{\delta} y_n. \end{aligned}$$

By induction, we then see that

$$v_n \leq \frac{1}{\delta} \left(\sum_{i=N}^{n-1} u_i + y_N \right) = \frac{1}{\delta} y_n, \quad N + \rho + j\delta + 1 \leq n \leq N + \rho + (j+1)\delta, \tag{12}$$

for $j = 0, 1, 2, \ldots$. Combining (11) and (12), we have

$$v_{n-\tau} \le \frac{1}{\delta} y_{n-\tau} < \frac{1}{\delta} y_n, \quad n \ge N + \tau,$$

so that

$$\Delta(r_n u_n) + h_n r_{n-\tau} v_{n-\tau} \le \Delta(r_n \Delta y_n) + \frac{1}{\delta} h_n r_{n-\tau} y_n \le 0, \quad n \ge N + \tau.$$

In view of (10), we may further write

$$\Delta \left(r_n v_n - C_n r_{n-k} v_{n-k} - \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} v_{i-m} \right) + h_n r_{n-\tau} v_{n-\tau} \le 0, \quad n \ge N + \rho.$$

If we now let $z_n = r_n v_n$, we obtain

$$\Delta \left(z_n - C_n z_{n-k} - \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} \right) + h_n z_{n-\tau} \le 0, \quad n \ge N + \rho,$$

which implies

$$\begin{aligned} z_n &\ge C_n z_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} + \sum_{i=n}^{\infty} h_i z_{i-\tau} \\ &\ge c_n z_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} + \sum_{i=n}^{\infty} h_i z_{i-\tau}, \end{aligned}$$

for $n \ge N + \rho$. The proof now follows from Lemma 1. □

As an immediate corollary [3, Theorem 4] by taking $r_n \equiv 1$, we see that equation (1) has an eventually positive solution provided condition (2),

$$c_n + \sum_{i=n-\tau+m}^{n-1} q_i \le 1, \tag{13}$$

and

$$n \sum_{i=n}^{\infty} h_i \le \frac{\delta}{4}$$

hold for all large n.

The techniques in the above proof can be modified to yield the following result.

Theorem 2. *Suppose* (2) *holds for all large* n. *Suppose further that there is positive nondecreasing sequence* $\{r_n\}$ *such that* (7) *holds for all large* n *and*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{1}{r_i} h_n r_{n-\tau} < \infty. \tag{14}$$

Then equation (1) *has an eventually positive solution* $\{x_n\}$ *which satisfies* $x_n \le r_n$ *for all large* n.

Proof. Note first that (14) implies

$$\sum_{n=0}^{\infty}\sum_{i=n}^{\infty}\frac{1}{r_n}h_i r_{i-\tau} < \infty.$$

Set

$$\Gamma_n = \frac{1}{r_n}\sum_{i=n}^{\infty} h_i r_{i-\tau},\ n \geq 0. \tag{15}$$

Then $\{\Gamma_n\}$ is a nonincreasing sequence. Choose an integer N such that

$$\frac{1}{\delta}\sum_{i=N}^{\infty}\sum_{j=i}^{\infty}\frac{1}{r_i}h_j r_{j-\tau} + \frac{1}{r_N}\sum_{i=N}^{\infty} h_i r_{i-\tau} \leq 1, \tag{16}$$

and (7) holds for $n \geq N$. Let us define a sequence $\{w_n\}$ by

$$w_n = \Gamma_N,\ N \leq n \leq N+\rho,$$

and

$$w_n = \Gamma_n + \frac{1}{r_n}\left(C_n r_{n-k} w_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} w_{i-m}\right), \quad n \geq N+\rho+1, \tag{17}$$

where C_n is defined by (9) for $n \geq n$. Note that $w_n > 0$ for $n \geq N$, and

$$w_n \leq \frac{1}{\delta}\sum_{i=N}^{n-1}\Gamma_i + \Gamma_N, \quad N \leq n \leq N+\rho.$$

By means of the same reasoning as used in the proof of Theorem 1, we may show that

$$w_n \leq \frac{1}{\delta}\sum_{i=N}^{n-1}\Gamma_i + \Gamma_N,\ n \geq N.$$

In view of (16),

$$0 < w_n \leq 1, \quad n \geq N.$$

Furthermore, in view of (17),

$$\begin{aligned} r_n w_n &= \sum_{i=n}^{\infty} h_i r_{i-\tau} + C_n r_{n-k} w_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} w_{i-m} \\ &\geq c_n r_{n-k} w_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} w_{i-m} + \sum_{i=n}^{\infty} h_i r_{i-\tau} w_{i-\tau} \end{aligned}$$

for $n \geq N+2\rho$. If we now set $z_n = r_n w_n$ for $n \geq N$, then $z_n > 0$, and

$$z_n \geq c_n z_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i z_{i-m} + \sum_{i=n}^{\infty} h_i z_{i-\tau}, \quad n \geq N+2\rho.$$

By Lemma 1, we see that

$$x_n = c_n x_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i x_{i-m} + \sum_{i=n}^{\infty} h_i x_{i-\tau}, \quad n \geq 0,$$

has an eventually positive solution $\{x_n\}$ which satisfies $0 < x_n \le z_n \le r_n$ for all large n. □

As an immediate corollary [3], by taking $r_n \equiv 1$, we see that if conditions (2) and (13) hold for all large n, and if

$$\sum_{i=0}^{\infty} ih_i < \infty,$$

then equation (1) has a bounded and eventually positive solution.

3. Examples

As an example, let us consider the equation

$$\Delta(x_n - x_{n-1}) + \frac{2x_{n-2}}{(n+3)(n+4)\phi_{n-2}} - \frac{x_{n-1}}{(n+3)(n+4)\phi_{n-1}} = 0, \quad n = 0, 1, 2 \dots, \tag{18}$$

where

$$\phi_n = \sum_{i=1}^{n+3} \frac{1}{i}, \quad n = -2, -1, \dots .$$

If we let $r_n = \phi_n$, then all the assumptions in Theorem 1 hold. In particular, condition (2) holds since

$$c_n r_{n-k} + \sum_{i=n-\tau+m}^{n-1} q_i r_{i-m} = \phi_{n-1} + \sum_{i=n-1}^{n-1} \frac{\phi_{i-1}}{(i+3)(i+4)\phi_{i-1}} \le \phi_n = r_n,$$

while (8) holds since

$$\begin{aligned}\sum_{i=0}^{n-1} \frac{1}{r_i} \sum_{i=n}^{\infty} h_i r_{i-\tau} &= \sum_{i=0}^{n-1} \frac{1}{\phi_i} \sum_{i=n}^{\infty} \frac{i\phi_{i-2}}{(i+2)(i+3)(i+4)\phi_{i-2}} \\ &\le \frac{1}{n+3} \sum_{i=0}^{n-1} \frac{1}{\phi_i} < \frac{1}{4} = \frac{\delta}{4}.\end{aligned}$$

Thus equation (18) has an eventually positive solution. It is interesting to note that $\{\phi_n\}$ is such a solution.

As another example, consider the equation

$$\Delta\left(x_n - \left(1 + \sin\frac{n\pi}{2}\right) x_{n-2}\right) + \left(\frac{1}{3} + \frac{1}{2^n}\right) x_{n-4} - \frac{1}{3} x_{n-1} = 0, \quad n = 0, 1, \dots .$$

If we take $\{r_n\} = \left\{3^{n/2}\right\}$, then it is easy to verify the assumptions in Theorem 2. Thus our equation has an eventually positive solution which is bounded by $3^{n/2}$ for all large n.

References

1. W. T. Li and S. S. Cheng, On a neutral difference equation with positive and negative coefficients. *Southeast Asian Bull. Math.* **22**(1998), No. 4, 407–418.
2. W. T. Li and S. S. Cheng, An oscillation theorem for a neutral difference equation with positive and negative coefficients. *Tamkang J. Math.* **30**(1999), No. 1, 39–45.
3. X. H. Tang, J. S. Yu, and D. H. Peng, Oscillation and nonoscillation of neutral difference equations with positive and negative coefficients. *Comput. Math. Appl.* **39**(2000), No. 7-8, 169–181.
4. M. P. Chen and B. G. Zhang, Oscillation and comparison theorems of difference equations with positive and negative coefficients. *Bull. Inst. Math. Acad. Sinica* **22**(1994), No. 4, 295–306.

(Received 29.04.2003)

Author's addresses:

Xian Hua Tang
Department of applied Mathematics
Central South Univeristy
Changsha, Hunan 410083
P. R. China

Sui Sun Cheng
Department of Mathematics
Tsing Hua University
Hsinchu, Taiwan 30043
R. O. China