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NORMALIZATION OF INTERIOR BOUNDARIES

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Abstract. It is proved that any closed piecewise-continuous curve can be converted by means of a small deformation to a normal curve such that the initial curve and the obtained one may have an interior extension only simultaneously.

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One of the basic requirements in the papers dealing with various aspects of the theory of interior extension of closed curves is that the curve be normal [1]–[5]. This requirement is rather cumbersome both for theoretical research and for application of the developed constructions. In the paper [6] the special case of violation of normality under homotopic deformation is considered. In the present paper it is proved that for any closed (not necessarily normal) curve satisfying certain conditions there exists a close normal curve such that these two curves may have an interior extension only simultaneously.

1. Under parametrization (representation) of a curve Γ we mean any mapping $\gamma : [a,b] \to \mathbb{R}^2$ which belongs to the equivalence class Γ. Closed curves are assumed to be parametrized by the mappings of the unit circle C and are written in the form $w = \gamma(\zeta) = \gamma(e^{it}) = \gamma(t)$, $t \in [0, 2\pi]$. The symbol $\langle\gamma\rangle$ or $\langle\Gamma\rangle$ denotes the set of values γ, while the notation $\gamma[t_1, t_2]$ stands for the restriction of γ to the interval $[t_1, t_2]$. The class of mappings containing $\gamma[t_1, t_2]$ is denoted by $\Gamma[A, B]$, where $A = \gamma(t_1)$, $B = \gamma(t_2)$. If the terminal point of the curve Γ_1 coincides with the initial point of the curve Γ_2, then under $\Gamma_1 \cdot \Gamma_2$ we understand the curve represented by the mapping

$$\gamma = \begin{cases} \gamma_1(t), & t \in [a,b], \\ \gamma_2(t), & t \in [b,c], \end{cases}$$

where $\gamma_1 \in \Gamma_1$, $\gamma_2 \in \Gamma_2$.

A curve is called locally simple [7] if, for any of its parametrization, there exists $\delta(\gamma) > 0$ such that the restriction of γ on $[t - \delta(\gamma), t + \gamma(\delta)]$ is injective. In this paper we consider piecewise-smooth, locally simple curves.

Let $U(w)$ denote a neighborhood of the point w, and $D(w, \rho)$ its circular neighborhood of radius ρ. Taking into account the properties of the class of curves under consideration, it is not difficult to establish the existence of numbers δ_0, $0 < \delta_0 \le \delta(\gamma)$, and $\rho_0 > 0$ such that $\langle\gamma[t - \delta_0, t + \delta_0]\rangle$ divides $D(\gamma(t), \rho_0)$ into two components. We say that the arc $\gamma[t - \delta_0, t + \delta_0]$ and the neighborhood $D(\gamma(t), \rho_0)$ are distinguished and denote them by $\gamma[t] = \gamma[w]$ and $D[\gamma(t)]$,

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respectively. The noncircular neighborhood with the same property is denoted by $U[\gamma(t)]$.

We will require that the curves under consideration satisfy the following additional condition: $\mathbb{C}\setminus\langle\Gamma\rangle$ consists of a finite number of components. We call this condition the condition E. It is easy to verify that when this condition is fulfilled, then values ρ_0 and δ_0 can be chosen so that any two arcs $\gamma[t_1]$ and $\gamma[t_2]$, where $\gamma(t_1)=\gamma(t_2)=w$, either have a unique common point w or coincide on some arc. Note that, for a locally simple curve, the set $\gamma^{-1}(w)$ is finite for any point $w\in\langle\Gamma\rangle$ since otherwise the infinite set of points $\gamma^{-1}(w)$ would have a condensation point, in any neighborhood of which the mapping γ would not be injective.

We write $(\gamma(t_1],\gamma[t_2])\in P_\Gamma(w)$, where $t_1,t_2\in\gamma^{-1}(w)$, if $\langle\gamma[t_1]\rangle$ separates the components of the set $\langle\gamma[t_2]\rangle\setminus\{w\}$ in $D[w]$. If however $\langle\gamma[t_1]\rangle$ does not separate the components of the set $\langle\gamma[t_2]\rangle\setminus\{w\}$ in $D[w]$, then we write $(\gamma[t_1],\gamma[t_2])\in Q_\Gamma(w)$. It is evident that for each w the sets $P_\Gamma(w)$ and $Q_\Gamma(w)$ are finite.

We write $(\gamma[a,b],\gamma[c,d])\in\mathcal{R}(\Gamma)$ if:

1) the intersection of the representation class of arcs $\Gamma[\gamma(a),\gamma(b)]$ and $\Gamma[\gamma(c),\gamma(d)]$ or arcs $\Gamma[\gamma(a),\gamma(b)]$ and $\Gamma^{-1}[\gamma(c),\gamma(d)]$ is not empty, and

2) condition 1) is not fulfilled for any $\Gamma[\gamma(a'),\gamma(b')]$ and $\Gamma[\gamma(c'),\gamma(d')]$, where $a'<a<b<b'$ and $c'<c<d<d'$.

Condition E implies that the sets $P(\Gamma)=\bigcup\limits_{w\in\langle\Gamma\rangle}P_\Gamma(w)$, $Q(\Gamma)=\bigcup\limits_{w\in\langle\Gamma\rangle}Q_\Gamma(w)$ and $\mathcal{R}(\Gamma)$ are finite. Indeed, let $w_j\in\langle\Gamma\rangle$, $j=1,2$, $w_1\neq w_2$ and $t_1,t_1^*\in\gamma^{-1}(w_1)$, $t_2,t_2^*\in\gamma^{-1}(w_2)$. Then $\mathbb{C}\setminus(\langle\gamma[t_1,t_2]\rangle\cup\langle\gamma[t_1^*,t_2^*]\rangle)$ consists of two components at least and the assumption that the set of points $w\in\langle\Gamma\rangle$ such that $\operatorname{card}(p(w)\cup Q(w))\geq 1$ is infinite would contradict the condition E. Let now $(L',L'')\in\mathcal{R}(\Gamma)$ and $(L_2',L_2'')\in\mathcal{R}(\Gamma)$. Denote by t_k' and t_k'', $k=1,2$, the values of the parameter that correspond to the initial points of the arcs L_k' and L_k'', $k=1,2$, respectively, and by T_k' and T_k'', $k=1,2$, the terminal points of the same arcs. Assume, for definiteness, that $\gamma(t_1')=\gamma(t_1'')$ and $\gamma(t_2')=\gamma(t_2'')$ (therefore $\gamma(T_1')=\gamma(T_1'')$ and $\gamma(T_2')=\gamma(T_2'')$). Then $\mathbb{C}\setminus(\langle\gamma[t_1',T_2']\cup\gamma[T_1'',t_2'']\rangle)$ consists of two components at least and the assumption that the set $\mathcal{R}(\Gamma)$ is infinite would again contradict the condition E.

If $\operatorname{card}P_\Gamma(w)=1$, $Q_\Gamma(w)=0$, then a point w is called a point of simple self-intersection. A curve is called normal [8] if all points for which $\operatorname{card}\gamma^{-1}(w)>1$ are points of simple self-intersection. Denote $V(\Gamma)=\{w\in\Gamma;\operatorname{card}(P_\Gamma(w)\cup Q_\Gamma(w))>1\}$.

2. A continuous mapping of a domain B is called interior if it is open and the preimage of any point is totally disconnected [9]. Any interior mapping F can be represented as $F=\Phi\circ H$, where H is a homeomorphism of the domain B and Φ is a holomorphic function. If, in this representation, the derivative of the function Φ has, at the point $H(z_0)$, zero of order m, $m\geq 1$, then the point z_0 is called a critical point of order m of the function F. In that case, the mapping F realizes the $m+1$-sheeted ramified covering of some neighborhood of the point $F(z_0)$ by the neighborhood of the point z_0 [10].

Denote by $D^+[\gamma(t)]$ ($D^-[\gamma(t)]$) the component of the set $D[\gamma(t)] \setminus \langle\gamma[t]\rangle$ that lies to the left (to the right) of $\gamma[t]$, and let $D_\pm[\gamma(t)]$ stand for $D^+[\gamma(t)] \cup \gamma[t]$ and $D^-[\gamma(t)] \cup \gamma[t]$, respectively.

Assume that F is the interior mapping of the domain B and the restriction of F on some simple arc l of the boundary of B is a homeomorphism. M. Morse [7] introduced the notion of a partial branch element of the function F^{-1} at a point $w_0 \in F(\zeta_0)$, $\zeta_0 \in C$. Rephrasing this notion for the mapping F, we say that a point ζ_0 is a boundary critical point of multiplicity m if, after extending $F(z)$ from some half-neighborhood $U_+(\zeta_0)$ to $U(\zeta_0)$ by means of the function $F_-(z)$ mapping homeomorphically $U_-(\zeta_0)$ on $U_-(w)$, we obtain the $m+1$-sheeted ramified covering

$$F^*(z) = \begin{cases} F(z), & z \in U_+(\zeta_0), \\ F_-(z), & z \in U_-(\zeta_0) \end{cases} \tag{1}$$

of some neighborhood $U'(w_0)$ by the neighborhood $U'(\zeta_0)$.

A closed curve Γ is called an interior boundary if there exists a continuous mapping F of the closed unit circle $\overline{\mathbb{D}}$ into $\mathbb{R}$ which is interior in $\mathbb{D}$ and such that its restriction on C gives a parametrization of the curve Γ. This fact can also be formulated as follows: F is an interior extension of the curve Γ. The problem of the existence of a given closed normal curve was solved by various approaches in [1] and [2].

As is known [7], if the locally simple curve is an interior boundary, then its interior extension may have only a finite number of critical points. Denote the set of images of critical points of the function F by $W(F)$.

The following two remarks are obvious:

a) Let w_1 and w_2 be points of some domain G. Then the number of components $G \cap \Gamma$ separating w_1 and w_2 in G is finite since otherwise the curve length would be infinite.

b) Let the finite set X of points and the finite set Y of piecewise-smooth arcs be given in the domain G. Any two points of the set $\overline{G}$ can be connected in G by a simple piecewise smooth curve l having only a finite number of common points with the set of points of arcs of the set Y and, moreover, each common point is a point of simple intersection and $\langle l\rangle \cap X = \varnothing$. If the connected points lie on the domain boundary, then all points of the curve l except for end points are assumed to lie in G. The above-described property of the curve l will be written in the form $l \in N(G, X, Y)$.

3. Lemma. *Let F be an interior extension of the curve Γ, z_0 be a critical point $F(z_0) = w_0$. Then there exists $\varepsilon_0 > 0$ such that for any w_*, $|w_* - w_0| < \varepsilon_0$, there exists an interior extension F_0 of the curve Γ such that z_0 is a critical point of the same multiplicity for the function F_0 and $F_0(z_0) = w_*$.*

Making an appropriate choice of the homeomorphism H in the representation of the mapping F and replacing $\gamma(t)$ by $\gamma(t) - w_0$, we can assume that $z_0 = 0$ and $F(0) = 0$.

Let $z_0 = 0$ be an interior critical point and $\widetilde{U}(0)$ be its neighborhood containing no other critical points and let the mapping F be a ramified covering of the circle $D(0, \varepsilon_0)$ by the domain $\widetilde{U}(0)$. Denote by φ the mapping of the circle $D(0, \varepsilon_0)$:

$$\tau = w - \frac{w_*}{\varepsilon_0}\,(\varepsilon_0 - |w|),$$

where $w_* \in D(0, \varepsilon_0)$.

From the inequality

$$|\tau| = |w| + \frac{|w_*|}{\varepsilon_0}\,(\varepsilon_0 - |w|) \leq \varepsilon$$

it follows that φ maps $D(0, \varepsilon_0)$ onto itself and every point on the circle boundary is fixed. Furthermore, for w', $w'' \in D(0, \varepsilon_0)$, $w' \neq w''$, we have

$$|\varphi(w') - \varphi(w'')| \geq |w' - w''|\Big(1 - \frac{|w_*|}{\varepsilon_0}\Big)$$

which implies that φ is a homeomorphic mapping of $D(0, \varepsilon_0)$ onto itself. Consider the mapping

$$F_0(z) = \begin{cases} \varphi \circ F(z), & z \in \widetilde{U}(0), \\ F(z), & z \in \overline{B} \setminus \overline{\widetilde{U}}(0). \end{cases}$$

By the properties of the mapping φ we conclude that F_0 satisfies the conditions of the lemma.

Let now z_0 be a boundary critical point. Then, after extending $F(z)$ from $U_+(0)$ onto entire $U(0)$, we construct the function $F^*(z)$ by formula (1). Having done this and taking the restriction of F^* to $\overline{B}$, we obtain $F_0(z)$ with the required properties.

Corollary. *If the curve Γ has an interior extension, then it has an interior extension F_0 such that $W(F_0) \cap \langle\Gamma\rangle = \varnothing$ and F_0 takes different values at different critical points.*

We call such an interior extension good.

4. Let the interior extension F of the curve Γ homeomorphically map $U_+(\zeta_0)$ onto $U_+[w]$, $w_0 = \gamma(\zeta_0) = \gamma(e^{it_0})$. Then the curve Γ_0 obtained from Γ by replacing the arc $\Gamma[a, b]$, $a, b \in U_+[w_0]$, $a = \gamma(t')$, $b = \gamma(t'')$, $t' < t_0 < t''$, by a simple arc $l[a, b]$, $l[a, b] \in N(U_+[w_0], W(F), \Gamma)$ is also an interior boundary. Indeed, since the restriction of F to $U_+(\zeta_0)$ is a homeomorphism, the curve $F^{-1} \circ l$ is a simple curve with end points $e^{it'}$ and $e^{it''}$ and therefore the Jordan curve $C[e^{it'}, e^{it''}](F^{-1} \circ l)^{-1}$ bounds the simply connected subdomain D_0 of the unit circle. Hence the restriction of F to $\mathbb{D} \setminus D_0$ is an interior extension of the curve Γ_0. It is obvious that if F is a good interior extension of the curve Γ, then its restriction on $\mathbb{D} \setminus D_0$ is a good extension of the curve Γ_0. Analogously, the statement is also true when $\langle l[a, b]\rangle \subset U_-[w_0]$ and, moreover, in that case it is not required that the restriction of F on $U_+(\zeta_0)$ be a homeomorphism. In other words, the point ζ_0 can be a boundary critical point. Indeed, after extending the mapping F from $U_+(\zeta_0)$ onto $U(\zeta_0)$ by the homeomorphism $F^- : U_-(\zeta_0) \to U_-(w_0)$, where $\widetilde{U}_-(w)$ is the domain bounded by $l[a,b] \cdot \Gamma^{-1}[a,b]$, we obtain the mapping

$$\widetilde{F}(z) = \begin{cases} F(z), & z \in \mathbb{D}, \\ F^-(z), & z \in \widetilde{U}_-(\zeta_0) \end{cases}$$

which is an interior extension of the curve Γ_0.

The above method of replacing the curve $\Gamma[a,b]$ by the curve $l[a,b]$ will be called the local normalization of the initial curve. From the above reasoning it follows that the curve obtained by means of local normalization of the curve under consideration is an interior boundary if and only if the initial curve is such. After parametrizing the arc at each stage of local normalization by the variable t, $t \in [t', t'']$, $\gamma(e^{it'}) = a$, $\gamma(e^{it''}) = b$, we obtain the curve Γ_* represented by the mapping $\gamma_*(t)$, $|\gamma_*(t) - \gamma(t)| < \varepsilon_0$, $t \in [0, 2\pi]$, where $\varepsilon_0 < \operatorname{dist}(W(F_0), \langle\Gamma\rangle)$.

5. Let $(L', L'') \in \mathcal{R}(\Gamma)$. Cover L' by a finite number of distinguished neighborhoods U_j, $j = \overline{1,n}$, numbered successively so that the initial point $\gamma(t_1)$ of the arc L' belong to U_1, the terminal point $\gamma(t_n)$ to the neighborhood U_n and $\left(\bigcup_{k=1}^{n} U_k\right) \cap W(F) = \varnothing$. Take two values t' and t'' such that $t' < t_1 < t''$, $\gamma(t') \in (U_1)_+$, $\gamma(t'') \in (U_1)_+ \cap (U_2)_+$ and connect them by a simple arc l_1, $l_1 \in N((U_1)_+, V(\Gamma), \Gamma)$, parametrized by a variable which ranges within the same segment $[t', t'']$. Let $\Gamma_1 = \Gamma[\gamma(0), \gamma(t')] \cdot l_1 \cdot \Gamma[\gamma(t''), \gamma(2\pi)]$. Choose two points $w_1' \in \langle l_1(t_1')\rangle$ and $w_1'' = \gamma(t_1'') \in (U_2)_+ \cap (U_3)_+$, $t_1' < t''$, $t_1'' > t''$, connect them by a simple arc l_2, $l_2 \in N(U_2, V(\Gamma_1), \Gamma_1)$ and denote by $\Gamma_2 = \Gamma[\gamma(0), \gamma(t')] \cdot l_1[t', t_1'] \cdot l_2[t_1', t_1''] \cdot \Gamma[\gamma(t''), \gamma(2\pi)]$. Continuing the above-described procedure of replacing the arc L', we obtain, after n steps, the curve $\Gamma_n = \Gamma[\gamma(0), \gamma(t')] \cdot l_1[t', t_1'] \cdot l_2[t_1', t_2'] \cdots l_n[t_{n-1}', t_n''] \cdot \Gamma[\gamma(t_n''), \gamma(2\pi)]$, where $t_n'' > t_n$, $\gamma(t_n'') \in (U_n)_+$. For Γ_n we have the inequality $\operatorname{card} \mathcal{R}(\Gamma_n) < \operatorname{card} \mathcal{R}(\Gamma)$. After making an analogous replacement for all pairs of the set $\mathcal{R}(\Gamma)$, we obtain the curve $\widetilde{\Gamma}$ for which $\mathcal{R}(\widetilde{\Gamma}) = \varnothing$. It is easy to verify that the representation $\widetilde{\gamma}(t)$ of the obtained curve satisfies the inequality $|\widetilde{\gamma}(t) - \gamma(t)| < \varepsilon_0$. When the curve Γ is converted by deformation to the curve $\widetilde{\Gamma}$, the number $\operatorname{card} P(\Gamma)$ may increase, but from the condition E and the remarks a) and b) it follows that $\operatorname{card} P(\widetilde{\Gamma}) < \infty$.

Let now $w_1 \in V(\widetilde{\Gamma})$. After performing the local normalization of the curve $\widetilde{\Gamma}$ in $U[w_1]$, we obtain a new curve $\widetilde{\Gamma}_1$, for which we will have $\operatorname{card}(P(\widetilde{\Gamma}_1) \cup Q(\widetilde{\Gamma}_1)) < \operatorname{card}(P(\widetilde{\Gamma}) \cup Q(\widetilde{\Gamma}))$. Applying the procedure of local normalization as many times as needed, we obtain a normal curve Γ_N. Moreover, since at each step of local normalization, the new and the old curve have an interior extension simultaneously, the normal curve Γ_N is an interior boundary if and only if the curve Γ is such. Moreover, Γ_N has a representation $\gamma_{{}_N}$ such that $|\gamma_{{}_N}(t) - \gamma(t)| < \varepsilon_0$.

Since the value ε_0 depends on a good extension, a question arises whether for an arbitrarily given $\varepsilon > 0$ it is possible to construct a normal curve Γ_ε

having the same properties as Γ_N and satisfying in particular the condition $|\gamma_\varepsilon(t)-\gamma(t)|<\varepsilon$, $t\in[0,2\pi]$. The answer to this question is positive. Indeed, let $D_1, D_2, \dots, D_k$ be the bounded components $\mathbb{C}\setminus\langle\Gamma\rangle$ containing branch points of the inverse function of a good extension. In each component D_j we take as many points $z_1^{(j)}, z_2^{(j)}, \dots, z_{m_j}^{(j)}$, $j=\overline{1,k}$, as there are branch points $a_1^{(j)}, a_2^{(j)}, \dots, a_q^{(j)}$ of the function F_0^{-1} which are contained in this component. Further, we connect each point $z_s^{(j)}$, $s=\overline{1,m_j}$, with one of the points $a_q^{(j)}$, $q=\overline{1,m_j}$, by disjoint simple arcs passing in D_j. Since Γ_N is a normal curve, whose interior extension has critical points mapped into points $a_q^{(j)}$, $q=\overline{1,m_j}$, $j=\overline{1,k}$, the curve Γ_N has an assemblage [4] with the initial points of paths at the branch points of the function F_0^{-1}. In that case, repeating the arguments of Theorem 2 from [11], we make sure that the curve Γ_N has an assemblage with the initial points of paths at the points $z_s^{(j)}$, $s=\overline{1,m_j}$, $j=\overline{1,k}$, which is equivalent to the existence of a good interior extension of the initial curve Γ with branch points of the inverse function at the points $z_s^{(j)}$, $s=\overline{1,m_j}$, $j=\overline{1,k}$. Then it is obvious that as ε_0 we can take a number $\operatorname{dist}\Big(\bigcup_{j=1}^{k}\bigcup_{s=1}^{m_j}\{z_s^{(j)}\},\langle\Gamma\rangle\Big)$ not depending on a concrete interior extension of the curve Γ.

Thus we have proved the following

Theorem. *For any closed piecewise smooth curve Γ represented by a mapping $\gamma(e^{i\vartheta}), 0\le\vartheta\le 2\pi$, and any $\varepsilon>0$ there exists a normal curve Γ_N represented by a mapping $\gamma_N(e^{i\vartheta})$ such that $|\gamma(e^{i\vartheta})-\gamma_N(e^{i\vartheta})|<\varepsilon$, $0\le\vartheta\le 2\pi$, and both curves Γ and Γ_N may have an interior extension only simultaneously.*

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