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ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION TO A STOCHASTIC DIFFERENTIAL EQUATION IN A BANACH SPACE

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Abstarct. A sufficient condition is given for the existence of a solution to a stochastic differential equation in an arbitrary Banach space. The method is based on the concept of covariance operator and a special construction of the Itô stochastic integral in an arbitrary Banach space.

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Let X be a real separable Banach space, X^* its dual, $(\Omega, \mathcal{B}, P)$ a probability space, $\mathcal{B}(X)$ the Borel σ-algebra of X. A random element $\xi : \Omega \to X$ is called a Gaussian random element if $\langle \xi, x^* \rangle$ is a Gaussian random variable for all $x^* \in X^*$. The distribution of ξ, $\mu_\xi(B) = P\{\xi \in B\}$, $B \in \mathcal{B}(X)$ is uniquely determined by the mean $E\xi = \int_\Omega \xi(\omega) dP(\omega)$ and the covariance operator $R : X^* \to X, \langle Rx^*, y^* \rangle = E\langle \xi - E\xi, x^* \rangle \langle \xi - E\xi, y^* \rangle$, which is symmetric and positive ($\langle Rx^*, x^* \rangle \geq 0$ for all $x^* \in X^*$) linear operator. Symmetric and positive linear operators, which are covariance operators of Gaussian measures, are called Gaussian covariances. By the factorization lemma (see [1], [10], Lemmas 3.1.1, 3.1.2) the symmetric and positive linear operator $R : X^* \to X$ can be factorized through a separable Hilbert space $H, R = AA^*$, where $A : H \to X$ is a continuous linear operator.

A family of random elements $(W_t)_{t \in [0,1]}$, $W_t : \Omega \to X$, is called a Wiener process if

1) $W_0 = 0$ almost surely (a.s.);

2) $W_{t_{i+1}} - W_{t_i}$ $(i = 0, 1, \dots, n-1)$ are independent random elements for every $0 \leq t_0 < t_1 < \dots < t_n \leq 1$;

3) for every $t \in [0, 1]$, $W(t)$ is a Gaussian random element with mean zero and covariance operator tR, where $R : X^* \to X$ is a fixed Gaussian covariance.

If X is a finite-dimensional Hilbert space and R is the identity operator, then our definition of a Wiener process coincides with the usual definition of a finite-dimensional Wiener process. Our definition is a direct extension of the definition of a Wiener process for the Hilbert space case ([1], p. 113). It is known that for any Gaussian covariance R and an arbitrary Banach space X there exists a Wiener process and it has a.s. continuous sample paths (see [2]–[4]).

Let $(F_t)_{t \in [0,1]}$, $F_t \subset \mathcal{B}$ be an increasing family of σ-algebras such that

a) W_t is F_t-measurable for all $t \in [0, 1]$;

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b) $W_s - W_t$ is independent of F_t for $0 \le t < s \le 1$. F_0 contains all P-null sets in $\mathcal{B}$.

The main problem in studying a stochastic differential equation for the Banach space case is the definition of the stochastic integral of an operator-valued random function with respect to a Wiener process in a Banach space. The original definition of the Itô stochastic integral in the finite-dimensional case can be naturally extended to the Hilbert space case (see [1], [5]). Subsequently, by isolating the inequalities needed to construct Itô stochastic integrals in a Banach space it proved possible to extend the theory to special classes of Banach spaces with geometrical properties similar to those of a Hilbert space (see [6]–[8]).

We will define the stochastic integral of an operator-valued random function in an arbitrary Banach space. To this end, as the first step we define the Itô stochastic integral of a random function with values in a dual space. In what follows we fix a Wiener process $(W_t)_{t\in[0,1]}$ with values in a Banach space X associated with a given Gaussian covariance.

Definition 1 ([9])**.** A function $\varphi : [0,1] \times \Omega \to X^*$ is called nonanticipating with respect to $(F_t)_{t\in[0,1]}$ if the function $(t,\Omega) \to \langle \varphi(t,\omega), x\rangle$ from $([0,1]\times\Omega, \mathcal{B}$ $[0,1]\times\mathcal{B})$ into $(R^1, \mathcal{B}(R^1))$ is measurable for all $x \in X$, and the function $\omega \to \langle \varphi(t,\omega), x\rangle$ from $(\Omega,\mathcal{B})$ into $(R^1, \mathcal{B}(R'))$ is F_t-measurable for all $t \in [0,1]$ and $x \in X$.

We denote by $G_R(X^*)$ the class of nonanticipating functions φ for which

$$P_R(\varphi) = \left(\int\limits_0^1 \int\limits_\Omega \langle R\varphi(t,\omega), \varphi(t,\omega)\rangle dt\, dP \right)^{1/2} < \infty,$$

where $R : X^* \to X$ is a covariance operator of W_1, $G_R(X^*)$ is a linear space and P_R is a pseudonorm.

If $\varphi \in G_R(X^*)$ is a step-function, $\varphi(t,\omega) = \sum\limits_{j=0}^{n-1} \varphi_{t_i}(\omega)\chi_{[t_i,t_{i+1}]}(t)$, $0 = t_0 < \cdots < t_n = 1$, $\varphi_{t_i} : \Omega \to X^*$, $i = 0,\dots,n-1$, then the stochastic integral of φ with respect to $(W_t)_{t\in[0,1]}$ is naturally defined by the equality

$$\int\limits_0^1 \varphi(t,\omega) dW_t = \sum_{i=0}^{n-1} \langle \varphi_{t_i}(\omega), W_{t_{i+1}}(\omega) - W_{t_i}(\omega)\rangle.$$

The following lemma is true.

Lemma 1 ([9])**.** *For an arbitrary $\varphi \in G_R(X^*)$, there exists a sequence of step-functions $(\varphi_n)_{n\in N} \subset G_R(X^*)$ such that $\varphi_n \xrightarrow{P_R} \varphi$, and $\int_0^1 \varphi_n dW_t$ converges in $L_2(\Omega, \mathcal{B}, P)$ as $n \to \infty$.*

The limit of the sequence $\int_0^1 \varphi_n dW_t$ is called the stochastic integral of φ with respect to $(W_t)_{t\in[0,1]}$ and is denoted by $\int_0^1 \varphi\, dW_t$. It is easy to see that the above definition of the stochastic integral is correct and

$$E\Big(\int_0^1 \varphi\, dW_t\Big)^2 = \int_0^1 \int_\Omega \langle R\varphi,\, \varphi\rangle dt\, dP = P_R^2(\varphi).$$

Let $L(X,X)$ $(L(X^*,X^*))$ be the space of linear bounded operators from X to X (from X^* to X^*).

Definition 2. A function $\varphi : [0,1]\times\Omega \to L(X,X)$ is called nonanticipating with respect to $(\mathcal{F}_t)_{t\in[0,1]}$ if for all $x\in X$ and $x^*\in X^*$ the real-valued function $(t,\omega)\to\langle\varphi(t,\omega)x,x^*\rangle$ is measurable, and for all $t\in[0,1]$ the function $\omega\to\langle\varphi(t,\omega)x,x^*\rangle$ is F_t-measurable. By $G_R(L(X,X))$ we denote the class of all nonanticipating functions $\varphi : [0,1]\times\Omega\to L(X,X)$ such that

$$\int_0^1\int_\Omega \langle\varphi(t,\omega)R\varphi^*(t,\omega)x^*,x^*\rangle dt\, dP < \infty \quad \text{for all} \quad x^*\in X^*,$$

where $\varphi^*(t,\omega)$ is the dual of $\varphi(t,\omega)$ and $R : X^*\to X$ is the covariance operator of W_1.

Let $\varphi\in G_R(L(X,X))$ and $x^*\in X^*$, φ^*x^* map $[0,1]\times\Omega$ into X^* and $\varphi^*x^*\in G_R(X^*)$. Then we can define the stochastic integral $\int_0^1 \varphi^*x^*dW_t$. Consider the map $T_\varphi : X^*\to L_2(\Omega,\mathcal{B},P)$, $T_\varphi x^* = \int_0^1\varphi^*x^*dW_t$.

Proposition 1. *T_φ is a continuous linear operator, i.e., it is a random linear function (RLF).*

Proof. Let $R = AA^*$, $A : H\to X$ be a factorization of R. For any $x^*\in X^*$

$$\int_0^1\int_\Omega \langle\varphi(t,\omega)R\varphi^*(t,\omega)x^*,x^*\rangle\, dP\, dt = \int_0^1\int_\Omega \langle\varphi(t,\omega)AA^*\varphi^*(t,\omega)x^*,x^*\rangle\, dP\, dt,$$

$$\int_0^1\int_\Omega \big(A^*\varphi^*(t,\omega)x^*, A^*\varphi^*(t,\omega)x^*\big)_H\, dP\, dt = \int_0^1\int_\Omega \big\|A^*\varphi^*(t,\omega)x^*\big\|_H^2\, dP\, dt < \infty.$$

Therefore we have an operator $L : X^*\to L_2(\Omega\times[0,1],H)$, $Lx^* = A^*\varphi^*(t,\omega)x^*$. The closed graph theorem easily shows that L is continuous. Therefore

$$\sup_{\|x^*\|\le 1}\|T_\varphi x^*\|^2 = \sup_{\|x^*\|\le1}\int_0^1\int_\Omega \langle\varphi(t,\omega)R\varphi^*(t,\omega)x^*,x^*\rangle dP\, dt = \sup_{\|x^*\|\le 1}\|Lx^*\|^2 < \infty\,,$$

i.e., T_φ is continuous. □

Definition 3 ([9])**.** Let $\varphi\in G_R(L(X,X))$. The linear continuous map (RLF) $T_\varphi : X^*\to L_2(\Omega,\mathcal{B},P)$, $T_\varphi x^* = \int_0^1\varphi^*x^*dW_t$, $x^*\in X^*$ is called a generalized stochastic integral of an operator-valued random function φ with respect to $(W_t)_{t\in[0,1]}$.

Definition 4 ([9])**.** We say that a random element $\xi : \Omega \to X$ is the stochastic integral of φ (if such an element exists) if $\langle \xi, x^* \rangle = T_\varphi x^*$, and write $\xi = \int_0^1 \varphi \, dW_t$.

Therefore for all $\varphi \in G_R(L(X,X))$ the generalized stochastic integral $T_\varphi x^*$ exists, but the stochastic integral $\int_0^1 \varphi(t,\omega)\, dW_t$ does not always exist. The generalized stochastic integral as an RLF induces a cylindrical measure on X which cannot always be extended to a countably additive measure on the Borel σ-algebra $\mathcal{B}(X)$. Or there is no random element $\xi : \Omega \to X$ such that $T_\varphi x^* = \langle \xi, x^* \rangle$. Thus, the question of existence of a stochastic integral is reduced to the problem of extending a cylindrical measure to a countably additive measure or, equivalently, to the problem of decomposability of an RLF.

We need a notion of a positive symmetric linear operator $T : X^* \to X$ with a special property.

Definition 5. We say that a positive symmetric linear operator $T : X^* \to X$ belongs to $\mathcal{R}_2(X)$ if there exists $c > 0$ such that for all $n \in N$ and $x_1^*, \dots, x_n^*$ from X^*

$$\left(\sum_{i=1}^n \|Tx_i^*\|^2 \right)^{1/2} \le c \sup_{\langle Tx^*, x^* \rangle \le 1} \left(\sum_{i=1}^n \left| \langle Tx_i^*, x^* \rangle \right|^2 \right)^{1/2}.$$

We denote by $\Pi_2(T)$ the minimal c for which this inequality holds.

Operator $A : H \to X$ is called 2-absolutely summing if there is a constant $c \ge 0$ such that for all $n \in N$ and $h_1, \dots, h_n$ from H, $\left(\sum\limits_{i=1}^n \|Ah_i\|^2 \right)^{1/2} \le c \sup\limits_{\|h\| \le 1} \left(\sum\limits_{i=1}^n \langle h_i, h \rangle^2 \right)^{1/2}$. The minimal c for which this inequality is true is denoted by $\pi_2(A)$. By the factorization lemma it is easy to prove the following assertion.

Proposition 2. *A positive symmetric linear operator $T : X^* \to X$ belongs to $\mathcal{R}_2(X)$ if and only if in the factorization $T = AA^*$, $A : H \to X$ is a 2-absolutely summing operator and $\Pi_2(T) = \pi_2(A)$.*

From the properties of 2-absolutely summing operators follows

Proposition 3. *If $T \in \mathcal{R}_2(X)$ and $\langle T_1 x^*, x^* \rangle \le \langle Tx^*, x^* \rangle$ for all $x^* \in X^*$, then $T_1 \in \mathcal{R}_2(X)$ and $\Pi_2(T_1) \le \Pi_2(T)$. And if A_1, A_2 are linear bounded operators from X to X and $T_1, T_2 \in \mathcal{R}_2(X)$, then $A_1T_1A_1^* + A_2T_2A_2^* \in \mathcal{R}_2(X)$ and $\Pi_2(A_1T_1A_1^* + A_2T_2A_2^*) \le \|A_1\| \cdot \Pi_2(T_1) + \|A_2\| \cdot \Pi_2(T_2)$.*

We need the following

Lemma 2. *Let $\xi : \Omega \to X$ be a weak second order random element such that the correlation operator $R_\xi x^* := E\langle \xi, x^* \rangle \xi$ belongs to $\mathcal{R}_2(X)$. Then $E\|\xi\|^2 \le \Pi_2^2(R_\xi)$.*

Proof. By the factorization lemma, there exists $(x_k^*)_{k\in N} \subset X^*$ such that $\langle R_\xi x_k^*, x_j^*\rangle = \delta_{kj}$, $x_k := Rx_k^*$, $k = 1,\dots,n,\dots$. $R_\xi x^* = \sum_{k=1}^{\infty}\langle x^*, x_k\rangle x_k$. Let $R_\xi = AA^*$, $A : H \to X$, $(e_k)_{k\in N}$, $e_k = A^* x_k^*$, $k = 1, 2, \dots,$ be an orthonormal basis in H, $\eta_n := \sum_{k=1}^{n} e_k\langle \xi, x_k^*\rangle$; $\eta_n \in H$ and for all $h \in H$

$$E\langle \eta_n, h\rangle^2 = E\Big(\sum_{k=1}^{n}(h, e_k)\langle \xi(\omega), x_k^*\rangle\Big)^2 = \sum_{k=1}^{n}(h, e_k)^2$$

since

$$E\langle \xi(\omega), x_k^*\rangle\langle \xi(\omega), x_j^*\rangle = \langle R_\xi x_k^*, x_j^*\rangle = \delta_{kj}.$$

Thus $\eta_n : \Omega \to H$ is a weak second order random element and for all $h \in H$

$$E\langle \eta_n, h\rangle^2 \to \sum_{k=1}^{\infty}\langle h, e_k\rangle^2, \quad n \to \infty.$$

Then by Lemma 5.2 of [10] $A\eta_n = \sum_{k=1}^{n} Ae_k\langle \xi(\omega), x_k^*\rangle$ converges in $L_2(\Omega, X)$. For all $x^* \in X^*$

$$E\Big(\langle \xi(\omega), x^*\rangle - \sum_{k=1}^{n}\langle Ae_k, x^*\rangle\langle \xi(\omega), x_k^*\rangle\Big)^2$$

$$= \langle R_\xi x^*, x^*\rangle - 2\sum_{k=1}^{n}\langle Ae_k, x^*\rangle\langle R_\xi x_k^*, x^*\rangle + \sum_{k=1}^{n}\langle Ae_k, x^*\rangle^2$$

$$= \sum_{k=n+1}^{\infty}\langle Ae_k, x^*\rangle^2 \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $\sum_{k=1}^{\infty} Ae_k\langle \xi(\omega), x_k^*\rangle = \xi(\omega)$ a.s. Since $\eta \in H$ and $A : H \to X$ is 2-absolutely summing, by the Pietsch theorem (see [10], Theorem 2.2.2) there exists a probability measure ν on $B_H = \{h : \|h\| \le 1\}$ such that

$$\|A\eta_n(\omega)\|^2 \le \pi_2^2(A)\int\limits_{\|h\|\le 1}|(\eta_n(\omega), h)|^2 d\nu(h) \le \pi_2^2(A)\int\limits_{\|h\|\le 1}\|h\|^2 d\nu(h) \le \pi_2^2(A)$$

and $E\|\xi\|^2 = \lim\limits_{n\to\infty} E\|A\eta_n(\omega)\|^2 \le \pi_2^2(A)$. □

Let us state the following sufficient condition for the existence of a stochastic integral.

Theorem 1. *Let $\varphi \in G_R(L(X, X))$ and $L_\varphi : X^* \to X$ $L_\varphi x^* = \int_0^1\int_\Omega \varphi(t,\omega)R\,\varphi^*(t,\omega)x^*\,dt\,dP$ belong to $\mathcal{R}_2(X)$, where $\varphi^*(t,\omega) : X^* \to X$ is a dual operator of $\varphi(t,\omega)$ and R is the covariance operator of W_1. Then there exists* *the stochastic integral* $\int_0^1 \varphi(t,\omega)dW_t$ *and* $E\big\|\int_0^1 \varphi(t,\omega)dW_t\big\|^2 < \infty$; *the process* $\xi_t = \int_0^t \varphi(s,\omega)dW_s$ *has a.s. continuous sample paths.*

Proof. Let, as in the proof of Lemma 2, $(x_k^*)_{k\in N} \subset X^*$ be such that $\langle R_\varphi x_k^*, x_j^*\rangle = \delta_{kj}$, $R_\varphi = AA^*$, $A : H \to X$, $A^*x_k^* = e_k$, $k = 1,2,\dots$, be an orthonormal basis in H, $x_k := Rx_k^*$, $k = 1,2,\dots$. Then, as in the proof of Lemma 2, we can prove that $I_t(\omega) := \int_0^t \varphi(s,\omega)dW_s = \sum_{k=1}^\infty x_k \int_0^t \varphi^*(s,\omega)x_k^* dW_s$ and convergence is understanding in the sense of $L_2(\Omega, X)$. Let

$$\zeta_n(t) := \sum_{k=1}^n x_k \int\limits_0^t \varphi^*(s,\omega)x_k^* dW_s.$$

By Lemma 2.2 of [9], for any k, there exists a sequence of step-functions $(\psi_n^*)_{n\in N} \subset G_R(X^*)$ such that $\psi_n^* \xrightarrow{P_R} \varphi^* x_k^*$ and $\int_0^1 \psi_n^*(t,\omega)dW_t \to \int_0^1 \varphi^*(t,\omega)x_k^* dW_t$ in $L_2(\Omega,\mathcal{B},P)$ as $n \to \infty$. Therefore, using the one-dimensional technique, it is easy to prove that the process $\int_0^t \varphi^*(s,\omega)x_k^* dW_s$ has a.s. continuous sample paths. Now we use the method applied in [11] (Theorem 2.1.6). Let $(y_n^*)_{n\in N} \subset X^*$ be a total subset in X^*. There exists $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$ such that $\langle I_t(\omega), y_n\rangle$ and $\zeta_n(t,\omega)$, $n = 1,2,\dots$, are continuous for all $\omega \in \Omega_0$. For $t_m \to t$, $m \to \infty$, and $\omega \in \Omega_0$, $\liminf_{t_m\to t} \langle I_{t_m}(\omega) - \zeta_n(t_m,\omega), y_k^*\rangle| \ge |\langle I_t(\omega) - \zeta_n(t,\omega), y_k^*\rangle|$. Hence we have $\liminf_{t_m\to t} \langle I_{t_m}(\omega) - \zeta_n(t_m,\omega)\rangle| \ge \|\langle I_t(\omega) - \zeta_n(t,\omega)\rangle|$. Then $\sup_{t\in[0,1]} \|I_t(\omega) - \zeta_n(t,\omega)\| = \sup_{t\in Q} \|I_t(\omega) - \zeta_n(t,\omega)\|$, $\omega \in \Omega_0$, where Q is a set of rational numbers in $[0,1]$. Since $\|I_t - \zeta_n(t)\|$ is a submartingale, for all $\varepsilon > 0$, $P\{\sup_{t\in Q} \|I_t - \zeta_n(t)\| > \varepsilon\} \le \varepsilon^{-1} E\|I_1 - \zeta_n(1)\| \to 0$, $n \to \infty$. There exists $(n_k)_{k\in N}$ such that $\lim_{n_k\to\infty} \sup_{t\in[0,1]} \|I_t - \zeta_{n_k}(t)\| = 0$ a.s. Therefore I_t has a.s. continuous sample paths. □

Let us state now a theorem on the existence and uniqueness of a strong solution to a stochastic differential equation in a Banach space.

Theorem 2. *Consider the stochastic differential equation*

$$d\xi(t) = a(t,\xi(t))dt + B(t,\xi(t))dW_t, \tag{1}$$

where $a : [0,1] \times X \to X$ *and* $B : [0,1] \times X \to L(X,X)$ *are such that*

1) *for all* $x \in X$, $B(\cdot,\cdot)x$ *and* $a(\cdot,\cdot)x$ *are measurable with respect to the* σ*-algebra* $\mathcal{B}[0,1] \times \mathcal{B}(X)$.

2) *There exist* $K > 0$, $\overline{R} \in \mathcal{R}_2(X)$ *and bounded linear operators* $A_i : X \to X$, $i = 1,2,\dots$, *such that* $\sum_{i=1}^\infty \|A_i\| < \infty$ *and for all* $t \in [0,1]$, $x, y \in X$, $x^* \in X^*$,

$$\langle a(t,x), x^*\rangle^2 + \langle B(t,x)RB^*(t,x)x^*, x^*\rangle \le K^2\big(\langle \overline{R}x^*, x^*\rangle + \sum_{i=1}^\infty \langle A_i x, x^*\rangle^2\big),$$

$$\langle a(t,x)-a(t,y),x^*\rangle^2+\langle(((B(t,x)-B(t,y))R(B^*(t,x)-B^*(t,y))x^*,x^*\rangle$$
$$\le K^2\Bigg(\sum_{i=1}^{\infty}\langle A_i(x-y),x^*\rangle^2\Bigg).$$

3) *There is a $\mathcal{F}_0$-measurable random element $\xi_0:\Omega\to X$ with $\langle R_0x^*,x^*\rangle := E\langle\xi_0,x^*\rangle^2$, $R_0:X^*\to X$ belonging to $\mathcal{R}_2(X)$.*

Then there exists a strong solution $(\xi_t)_{t\in[0,1]}$ to (1) with continuous sample paths, $\xi(0)=\xi_0$, $\sup\limits_{0\le t\le 1}E\|\xi_t\|^2<\infty$, and if $\xi_1(t)$ and $\xi_2(t)$ are two solutions, $\xi_1(0)=\xi_2(0)=\xi_0$, then $P\big\{\sup\limits_t\|\xi_1(t)-\xi_2(t)\|=0\big\}=1$.

Proof. Let $\xi_0(t)=\xi_0$ and

$$\langle\xi_n(t),x^*\rangle=\langle\xi_0,x^*\rangle+\int_0^t\langle a(s,\xi_{n-1}(s)),x^*\rangle ds+\int_0^t B^*(s,\xi_{n-1}(s))x^*dW(s). \quad (2)$$

We have

$$E\langle\xi_n(t),x^*\rangle^2\le 3E\langle\xi_0,x^*\rangle^2+3tE\big(\int_0^t\langle a(s,\xi_{n-1}(s)\big),x^*\rangle^2ds$$
$$+3E\int_0^t\langle B(s,\xi_{n-1}(s))RB^*(s,\xi_{n-1}(s))x^*,x^*\rangle ds\le 3E\langle\xi_0,x^*\rangle^2$$
$$+3K^2E\int_0^t(\langle\overline{R}x^*,x^*\rangle+\sum_{i=1}^{\infty}\langle A_i\xi_{n-1}(s),x^*\rangle^2)ds$$
$$+3E\langle\xi_0,x^*\rangle^2+3\cdot tK^2\langle\overline{R}x^*,x^*\rangle+3K^2E\int_0^t\sum_{i=1}^{\infty}\langle A_i\xi_{n-1}(s),x^*\rangle^2ds$$

When $n=1$,

$$E\langle\xi_1(t),x^*\rangle^2\le 3\langle R_0x^*,x^*\rangle+3K^2t\langle\overline{R}x^*,x^*\rangle+3K^2t\sum_{i=1}^{\infty}\langle A_iR_0A_i^*x^*,x^*\rangle.$$

When $n=2$,

$$E\langle\xi_2(t),x^*\rangle^2\le 3\langle R_0x^*,x^*\rangle+3K^2t\langle\overline{R}x^*,x^*\rangle+3K^2E\int_0^t\sum_{i=1}^{\infty}\langle\xi_1(s),A_i^*x^*\rangle^2ds$$
$$\le 3\langle R_0x^*,x^*\rangle+3K^2t\langle\overline{R}x^*,x^*\rangle+3K^2\int_o^t\sum_{i_2=1}^{\infty}(3\langle R_0A_{i_2}^*x^*,A_{i_2}^*x^*\rangle$$
$$+3K^2s\langle\overline{R}A_{i_2}^*x^*,A_{i_2}^*x^*\rangle+3K^2s\sum_{i_1=1}^{\infty}\langle A_{i_1}R_0A_{i_1}^*A_{i_2}x^*,A_{i_2}x^*\rangle)ds$$

$$= 3\langle R_0x^*, x^*\rangle + 3^2K^2t\sum_{i_2=1}^{\infty}\langle A_{i_2}R_0A_{i_2}^*x^*, x^*\rangle$$
$$+ (3K^2)^2\cdot\frac{t^2}{2}\sum_{i_2=1}^{\infty}\sum_{i_1=1}^{\infty}\langle A_{i_2}A_{i_1}R_0A_{i_1}^*A_{i_2}^*x^*, x^*\rangle$$
$$+ 3K^2t\langle\overline{R}x^*, x^*\rangle + (3K^2)^2\cdot\frac{t^2}{2}\sum_{i_2=1}^{\infty}\langle A_{i_2}\overline{R}A_{i_2}^*x^*, x^*\rangle.$$

It is obvious that

$$E\langle\xi_n(t), x^*\rangle^2 \le 3\sum_{m=0}^{n}\frac{(3K^2t)^m}{m!}\sum_{i_1=1}^{\infty}\cdots\sum_{i_m=1}^{\infty}\langle A_{i_m}\cdots A_{i_1}R_0A_{i_1}^*\cdots A_{i_m}^*x^*x^*\rangle$$
$$+\sum_{m=1}^{n}\frac{(3K^2t)^m}{m!} < \sum_{i_1=1}^{\infty}\cdots\sum_{i_{m-1}=1}^{\infty}\langle A_{i_{m-1}}\cdots A_{i_1}\overline{R}A_{i_1}^*\cdots A_{i_{m-1}}^*x^*, x^*\rangle.$$

We will show that random element ξ_n exists for all $n \in N$. To this end, we have to prove that $\langle R_nx^*, x^*\rangle := E\langle\xi_n, x^*\rangle^2$, $x^* \in X^*$ belongs to $\mathcal{R}_2(X)$ (see [10], Theorem VI.5.3). It is enough to show that $\sum\limits_{i_1=1}^{\infty}\cdots\sum\limits_{i_m=1}^{\infty} < A_{i_m}\cdots A_{i_1}R_0A_{i_1}^*\cdots A_{i_m}^*$ and, analogously, we obtain $\sum\limits_{i_1=1}^{\infty}\cdots\sum\limits_{i_{m-1}=1}^{\infty} < A_{i_{m-1}}\cdots A_{i_1}\overline{R}A_{i_1}^*\cdots A_{i_{m-1}}^*$ belongs to $\mathcal{R}_2(X)$.

$$\Pi_2\Big(\sum_{i_1=1}^{\infty}\cdots\sum_{i_m=1}^{\infty}A_{i_1}\cdots A_{i_m}R_0A_{i_1}^*\cdots A_{i_m}^*\Big)$$
$$\le\sum_{i_1=1}^{\infty}\cdots\sum_{i_m=1}^{\infty}\Pi(A_{i_1}\cdots A_{i_m}R_0A_{i_1^*}\cdots A_{i_m}^*)$$
$$\le\sum_{i_1=1}^{\infty}\cdots\sum_{i_m=1}^{\infty}\|A_{i_1}\|\cdot\|A_{i_2}\|\cdots\|A_{i_m}\|\Pi_2(R_0) = \Big(\sum_{i=1}^{\infty}\|A_i\|\Big)^m\Pi_2(R_0).$$

Obviously,

$$\Pi_2(R_n) \le 3^{\frac12}\sum_{m=0}^{n}\Big(\frac{(3k^2\cdot t)^m}{m!}\Big)^{\frac12}\Big(\sum_{i=1}^{\infty}\|A_i\|\Big)^m\Pi_2(R_0)$$
$$+\sum_{m=1}^{n}\Big(\frac{(3k^2\cdot t)^m}{m!}\Big)^{\frac12}\Big(\sum_{i=1}^{\infty}\|A_i\|^2\Big)^{m-1}\cdot\Pi_2(\overline{R}) < \infty.$$

Thus, equality (2) makes sense.

Analogously, for any $x^* \in X^*$

$$E\langle\xi_{n+1}(t)-\xi_n(t), x^*\rangle^2 \le 2tE\int_0^t\langle a(s,\xi_n(s)) - a(s,\xi_{n-1}(s)), x^*\rangle^2ds$$

$$+2E\int_0^t \langle [B(s,\xi_n(s))-B(s,\xi_{n-1}(s))]R[B^*(s,\xi_n(s))-B^*(s,\xi_{n-1}(s))]x^*,x^*\rangle$$

$$\leq 2K^2E\int_0^t \sum_{i=1}^{\infty}\langle A_i(\xi_n(s)-\xi_{n-1}(s)),x^*\rangle^2 ds;$$

$$E(\langle\xi_1(t),x^*\rangle-\langle\xi_0(t),x^*\rangle)^2=E\Big(\int_0^t\langle a(s,\xi_0(s)),x^*\rangle ds+\int_0^t B^*(s,\xi_0(s))x^*dW_s\Big)^2$$

$$\leq 2\Big(E\int_0^t(\langle a(s,\xi_0(s),x^*\rangle^2+B(s,\xi_0(s))RB^*(s,\xi_0(s))x^*,x^*\rangle\Big)ds$$

$$\leq 2K^2t\Big(\langle\overline{R}x^*,x^*\rangle+\sum_{i=1}^{\infty}\langle A_iR_0A_i^*x^*,x^*\rangle\Big);$$

$$E\langle\xi_2(t)-\xi_1(t),x^*\rangle^2\leq 2K^2E\int_0^t\Big(\sum_{i_2=1}^{\infty}\langle\xi_1(s)-\xi_0(s),A_{i_2}^*x^*\rangle^2\Big)ds$$

$$\leq\frac{(2K^2t)^2}{2}\Big(\sum_{i_2=1}^{\infty}\langle A_{i_2}\overline{R}A_{i_2}^*x^*,x^*\rangle+\sum_{i_2=1}^{\infty}\sum_{i_1=1}^{\infty}\langle A_{i_2}A_{i_1}R_0A_{i_1}^*A_{i_2}^*x^*,x^*\rangle\Big).$$

It is easy to prove that

$$E\big(\langle\xi_n(t)-\xi_{n-1}(t),x^*\rangle^2\leq\frac{(2K^2t)^n}{n!}\Big(\sum_{i_n=1}^{\infty}\cdots\sum_{i_2=1}^{\infty}\langle A_{i_n}\cdots A_{i_2}\overline{R}A_{i_2}^*\cdots A_{i_n}^*x^*,x^*\rangle$$

$$+\sum_{i_n=1}^{\infty}\cdots\sum_{i_1=1}^{\infty}\langle A_{i_n}\cdots A_{i_1}R_0A_{i_1}^*\cdots A_{i_n}^*x^*,x^*\rangle\Big).$$

Denote the correlation operator of a random element $\xi_n(t)-\xi_{n-1}(t)$ by $R(n,t)$. It is easy to show that

$$\Pi_2(R(n,t))\leq\Big(\frac{(2k^2t)^n}{n!}\Big)^{\frac{1}{2}}\Big[\Big(\sum_{i=1}^{\infty}\|A_i\|\Big)^{n-1}\Pi_2(\overline{R})+\Big(\sum_{i=1}^{\infty}\|A_i\|\Big)^n\cdot\Pi_2(R_0)\Big].$$

According to Lemma 2, we have

$$E\|\xi_n(t)-\xi_{n-1}(t)\|^2\leq\Pi_2^2(R(n,t))$$

$$\leq\frac{(2k^2t)^n\cdot\Big(\sum\limits_{i=1}^{\infty}\|A_i\|\Big)^{2(n-1)}}{n!}\Big(2\Pi_2^2(\overline{R})+2\Big(\sum_{i=1}^{\infty}\|A_i\|\Big)^2\times\Pi_2^2(R_0)\Big)=C_1\cdot\frac{C^{n-1}}{n!}.$$

We also have that for all $x^*\in X^*$

$$E\langle a(s,\xi_n)-a(s,\xi_{n-1}),x^*\rangle^2\leq K^2E\sum_{i=1}^{\infty}\langle\xi_n-\xi_{n-1},A_i^*x^*\rangle^2.$$

Therefore $E\|a(s,\xi_n) - a(s,\xi_{n-1})\|^2 \leq C_2 \cdot \frac{C^{n-1}}{n!}$. Since $\int_0^t (B(s,\xi_n(s)) - B(s,\xi_{n-1}(s)))\, dW_s$ is a martingale with a.s. continuous sample paths, we have

$$E\sup_t \left\| \int_0^t (B(s,\xi_n(s)) - B(s,\xi_{n-1}(s)))\, dW_S \right\|^2$$

$$\leq 4E\left\| \int_0^1 (B(s,\xi_n(s)) - B(s,\xi_{n-1}(s)))\, dWs \right\|^2 \leq 4C_2 \cdot \frac{C^{n-1}}{n!}.$$

Therefore

$$E\sup_t \left\|\xi_{n+1}(t) - \xi_n(t)\right\|^2 \leq 2\int_0^1 E\left\|a(s,\xi_n(s)) - a(s,\xi_{n-1}(s))\right\|^2 ds$$

$$+2E\sup_{0\leq t\leq 1} \left\| \int_0^1 (B(s,\xi_n(s)) - B(s,\xi_{n-1}(s)))\, dW_S \right\|^2$$

$$\leq 2C_2 \cdot \frac{C^{n-1}}{n!} + 8C_2 \cdot \frac{C^{n-1}}{n!} = 10C_2 \cdot \frac{C^{n-1}}{n!}.$$

That is,

$$\sum_{n=1}^{\infty} P\Big\{ \sup_t \|\xi_{n+1}(t) - \xi_n(t)\| > \frac{1}{n^2} \Big\} \leq \sum_{n=1}^{\infty} \frac{10C_2 \cdot C^{n-1}}{n!} \cdot n^4 < \infty.$$

By the Borell–Cantelli lemma, the sum $\xi_0 + \sum\limits_{n=0}^{\infty} (\xi_{n+1}(t) - \xi_n(t))$ converges a.s. uniformly in t to a random element which we denote by $(\xi_t)_{t\in[0,1]}$. From (2), as $n \to \infty$, we have

$$\langle \xi(t), x^* \rangle = \langle \xi_0, x^* \rangle + \int_0^t \langle a(s,\xi(s)), x^* \rangle ds + \int_0^t B^*(s,\xi(s)) x^* dW_S$$

a.s. for all $x^* \in X^*$. Therefore $(\xi_t)_{t\in[0,1]}$ is a solution to the stochastic differential equation (1). Obviously, we have $\xi_n(t) \to \xi(t)$ in $L_2(\Omega, X)$ as $n \to \infty$ and $\sup\limits_t E\|\xi_t\|^2 < \infty$.

Now let $\xi_1(t)$ and $\xi_2(t)$ be two solutions to (1) with $\xi_1(0) = \xi_2(0) = \xi_0$. Then we have

$$E\langle \xi_1(t) - \xi_2(t), x^* \rangle^2$$

$$\leq (2K^2)^n E \sum_{i_1=1}^{\infty} \cdots \sum_{i_n=1}^{\infty} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \langle \xi_1(s) - \xi_2(s), A_{i_1}^* \cdots A_{i_n}^* x^* \rangle^2 ds\, dt_{n-1} \cdots dt_1$$

$$\leq \frac{(2k^2t)^n}{n!}\left(\sum_{i=1}^{\infty}\|A_i\|^2\right)^n\cdot\|x^*\|^2\cdot\sup_t E\|\xi_1(t)-\xi_2(t)\|^2.$$

Since n is arbitrary, we have $E\langle\xi_1(t)-\xi_2(t),x^*\rangle^2=0$ for all $x^*\in X^*$. We can choose a countable total set $\{x_n^*,\ \ n\in N\}$. Therefore $\xi_1(t)=\xi_2(t)$ a.s. for all t, and since $\xi_1(t)$ and $\xi_2(t)$ have continuous sample paths, we get $P\{\sup_t\|\xi_1(t)-\xi_2(t)\|=0\}=1$. □ □

Remark. Since $P\{W_t\in\overline{RX^*},\ t\in[0,1]\}=1$ (see [12]), we can consider the operators $B(t,x)$, $t\in[0,1]$, $x\in X$, mapping from $\overline{RX^*}$ to X.

Finally we give a simple example: let $A_{0i}:X\to X,\ i=1,2,\dots$ be linear operators such that $\sum\limits_{i=1}^{\infty}\|A_{0i}\|<\infty$, and $\alpha_{0i}(\cdot):[0,1]\to R$ be such that $\sum\limits_{i=1}^{\infty}\alpha_{0i}^2(t)\leq c$ for any $c>0$. Let $a(t,x)=\sum\limits_{i=1}^{\infty}\alpha_{0i}(t)A_{0i}x$. Then for all $x^*\in X^*$, $\langle a(t,x),x^*\rangle^2\leq\Big(\sum\limits_{i=0}^{\infty}\alpha_{0i}(t)\langle A_{0i}x,x^*\rangle\Big)^2\leq c\sum\limits_{i=1}^{\infty}\langle A_{0i}x,x^*\rangle^2$.

Consider now $\langle B(t,x)RB^*(t,x)x^*,x^*\rangle$, $x^*\in X^*$. By the factorization lemma we can choose $(a_k)_{k\in N}\subset X$, $(a_k^*)_{k\in N}\subset X^*$ such that $\langle a_k,a_j^*\rangle=\delta_{kj}$, $Rx^*=\sum\limits_{k=1}^{\infty}\langle a_k,x^*\rangle a_k$ and $\overline{RX^*}=\overline{X_0}$, where $X_0=\Big\{\sum\limits_{k=1}^{\infty}\lambda_k a_k,\sum\limits_{k=1}^{\infty}\lambda_k^2<\infty\Big\}$. Then

$$\langle B(t,x)RB^*(t,x)x^*,x^*\rangle=\sum_{k=1}^{\infty}\langle a_k,B^*(t,x)x^*\rangle^2=\sum_{k=1}^{\infty}\langle B(t,x)a_k,x^*\rangle^2.$$

Let $A_{ki}:X\to X$ $k=1,2,\dots$, $i=1,2,\dots$, be linear operators such that $\sum\limits_{k=1}^{\infty}\sum\limits_{i=1}^{\infty}\|A_{ki}\|<\infty$, and $\alpha_{ki}(\cdot):[0,1]\to R$, $k=1,2,\dots$, $i=1,2,\dots$, be such that $\sum\limits_{i=1}^{\infty}\alpha_{ki}^2<c$. Let $B(t,x)a_k=\sum\limits_{i=1}^{\infty}\alpha_{ki}(t)A_{ki}x$, $k=1,2,\dots$. It is easy to see that $B(t,x)$ can be extended as a bounded linear operator in $\overline{RX^*}$. For all $x^*\in X^*$ we have

$$\langle B(t,x)RB^*(t,x)x^*,x^*\rangle=\sum_{k=1}^{\infty}\Big(\langle\sum_{i=1}^{\infty}\alpha_{ki}(t)A_{ki}x,x^*\rangle\Big)^2\leq c\sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\langle A_{ki}x,x^*\rangle^2.$$

Therefore, for operators A_{ki}, $k=1,2,\dots$, $i=1,2,\dots$, $a(t,x)$ and $B(t,x)$ satisfy the conditions of Theorem 2.

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