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OSCILLATION CRITERIA OF COMPARISON TYPE FOR SECOND ORDER DIFFERENCE EQUATIONS

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Abstract. In this paper we investigate the oscillatory character of the second order nonlinear difference equations of the forms

$$\Delta(c_{n-1}\Delta(x_{n-1}+p_n x_{\sigma_n}))+q_n f(x_{\tau_n})=0, \qquad n=1,2,\dots$$

and the corresponding nonhomogeneous equation

$$\Delta(c_{n-1}\Delta(x_{n-1}+p_n x_{\sigma_n}))+q_n f(x_{\tau_n})=r_n, \qquad n=1,2,\dots$$

via comparison with certain second order linear difference equations where the function f is not necessarily monotonic. The results of this paper are essentially new and can be extended to more general equations.

1. Introduction

In this article we are concerned with nonlinear difference equations of the forms

$$\Delta(c_{n-1}\Delta(x_{n-1}+p_n x_{\sigma_n}))+q_n f(x_{\tau_n})=0, \qquad n=1,2,\dots \tag{1.1}$$

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and

$$\Delta(c_{n-1}\Delta(x_{n-1} + p_n x_{\sigma_n})) + q_n f(x_{\tau_n}) = r_n, \qquad n = 1, 2, \ldots \tag{1.2}$$

where Δ is the forward difference operator, i.e., $\Delta u_n = u_{n+1} - u_n$, $\{c_n\}$, $\{p_n\}$, $\{q_n\}$, $\{r_n\}$, $\{\sigma_n\}$ and $\{\tau_n\}$ are sequences of real numbers such that $c_n > 0$ for $n \geq 0$, $\{p_n\}$ and $\{q_n\}$ are eventually nonnegative sequences, σ_n and τ_n are integers such that $\lim_{n\to\infty} \sigma_n = \infty = \lim_{n\to\infty} \tau_n$, and $f \in C(R, R)$ such that $uf(u) > 0$ for $u \neq 0$.

By a solution of equation (1.1) (or (1.2)) we mean a sequence $\{x_n\}$ which satisfies equation (1.1) (or (1.2)) for all $n \geq \min\{0, \inf_{s\geq 0} \sigma_s, \inf_{s\geq 0} \tau_s\}$. A nontrivial solution $\{x_n\}$ of any of the above equations is said to be oscillatory if for every integer $N > 0$ there exists $n \geq N$ such that $x_n x_{n+1} \leq 0$, otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

The oscillatory behaviour of solutions of difference equations has been the subject of intensive literature during the past few years. For example, we refer the reader to the papers [1]–[3], [5]–[15] and the references cited therein where several particular cases of (1.2) have been discussed. In most cases the function f was assumed as a nondecreasing function (see [1], [6]–[7], [11]–[15].)

Therefore our main purpose is to investigate the oscillatory behavior of (1.1) and (1.2) via comparison with certain linear equations particularly when f is not assumed to be a nondecreasing function. Some of the results of this paper are the discrete analogue of some of the results which appeared in [4].

Throughout this paper it will be assumed that $f(u) = G(u)H(u)$ for all $u \in R$ where $G : R \to (0, \infty)$ is nondecreasing on $(-\infty, 0)$ and nonincreasing on $(0, \infty)$ while H is nondecreasing function on R. This condition, according to [9, Lemma 4], is equivalent to saying that the function f is of bounded variation on every $[a, b] \subset (-\infty, 0) \cup (0, \infty)$. For convenience, we consider the following notations for any real sequence $\{\Lambda_n\}$:

$$\Lambda_n^+ = \max\{\Lambda_n, 0\}, \quad \Lambda_n^- = \min\{\Lambda_n, 0\}, \qquad n = 0, 1, \ldots$$

and

$$I_a(\Lambda) = \{n \geq 0 : \Lambda_n - 1 > a\}, \qquad a \quad \text{is positive integer.}$$

Moreover, we will make use of a new sequence $\{C_n\}$ defined by

$$C_n = \sum_{i=N}^{n} c_i^{-1}, \quad n \geq N \geq 0.$$

2. The homogeneous case

This section is devoted to study equation (1.1). The following conditions will be needed:

$$\sum_{i=N}^{\infty} c_i^{-1} = \infty, \tag{2.1}$$

$$c_n \geq c_{h_n} \qquad \text{for all} \quad n \geq 0, \tag{2.2}$$

where

$$h_n = \min\{n, \tau_n\}, \quad \Delta h_n \geq 0 \qquad \text{for all} \quad n \geq 0, \tag{2.3}$$

$$0 \leq p_n < 1, \qquad n \geq 0, \tag{2.4}$$

and

$$\sigma_n \leq n-1, \qquad n \geq 0. \tag{2.5}$$

Theorem 2.1. Suppose that the conditions *(2.1)–(2.5)* are satisfied. If for every $d \geq 1$, $k > 0$ and all large N $(N \geq 0)$, the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^1 u_n = 0, \qquad \text{for all} \quad n \in I_N(h) \tag{2.6}$$

where

$$Q_n^1 = \frac{G(dC_{\tau_n - 1})}{dC_{h_n - 1}}\, H(k(1 - p_{\tau_n + 1})),$$

is oscillatory, then equation (1.1) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1), one can assume that $x_n > 0$ eventually (since the case $x_n < 0$ eventually can be treated similarly). So, there exists an integer $N_1 > 0$ such that $x_n > 0$, $x_{\tau_n} > 0$ and $x_{\sigma_n} > 0$, for all $n \geq N_1$. Now, define

$$u_{n-1} = x_{n-1} + p_n x_{\sigma_n}, \qquad n \geq N_1. \tag{2.7}$$

Equation (1.1) implies

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n f(x_{\tau_n}) = 0, \tag{2.8}$$

then

$$\Delta(c_{n-1}\Delta u_{n-1}) \leq 0, \qquad n \geq N_1. \tag{2.9}$$

Fromthe above inequality, we conclude that $c_{n-1}\Delta u_{n-1}$ decreases monotonically, for $n > N_1$, which implies that Δu_{n-1} is eventually of one sign. If Δu_{n-1} is eventually negative, then one can find a suitable integer $N_2 \geq N_1$ such that $\Delta u_{N_2 - 1} < 0$. Summing (2.9) from N_2 to n, we obtain

$$c_n \Delta u_n \leq c_{N_2 - 1}\Delta u_{N_2 - 1}.$$

or

$$\Delta u_n \le \frac{c_{N_2-1}\Delta u_{N_2-1}}{c_n}, \qquad n \ge N_2.$$

Summing the preceding inequality from N_2 to ∞, in view of (2.1) we get $\lim_{n\to\infty} u_n = -\infty$ which by (2.7) leads to a contradiction to the positivity assumption of x_n. Thus Δu_{n-1} is eventually positive i.e., there exists an integer $N_3 \ge N_1$ such that $\Delta u_n > 0$ for $n > N_3$. From (2.9), we obtain

$$u_n \le u_N + (c_{N-1}\Delta u_{N-1}) \sum_{i=N}^{n-1} \frac{1}{c_i}, \qquad n > N, \quad \text{for all} \quad N > N_3,$$

which yields

$$u_n \le d_1 \sum_{i=N}^{n-1} \frac{1}{c_i} = d_1 C_{n-1}, \quad \text{for all} \quad n > N, \text{ and some } d_1 \ge 1. \tag{2.10}$$

Using (2.7) and (2.10), we get

$$x_n \le d_1 C_{n-1} \quad \text{or} \quad x_{\tau_n} \le d_1 C_{\tau_n - 1}, \qquad n \in I_N(h). \tag{2.11}$$

Define the sequence $\{W_n\}$ by

$$W_n = \frac{c_{n-1}\Delta u_{n-1}}{u_{h_n-1}}, \qquad n \in I_N(h).$$

Then

$$\Delta W_n = \frac{\Delta(c_{n-1}\Delta u_{n-1})}{u_{h_n}} - \frac{c_{n-1}\Delta u_{n-1}\Delta u_{h_n-1}}{u_{h_n-1}u_{h_n}},$$

and hence,

$$\Delta W_n = -q_n \frac{G(x_{\tau_n})H(x_{\tau_n})}{u_{h_n}} - \frac{c_{n-1}\Delta u_{n-1}\Delta u_{h_n-1}}{u_{h_n-1}u_{h_n}}, \quad n \in I_N(h). \tag{2.12}$$

From (2.2) and (2.9), we conclude

$$\Delta u_{h_n-1} \ge \Delta u_{n-1}, \qquad n \in I_N(h)$$

which implies

$$\frac{c_{n-1}\Delta u_{n-1}\Delta u_{h_n-1}}{u_{h_n-1}u_{h_n}} \ge \frac{W_n^2}{W_n + c_{n-1}}, \qquad n \in I_N(h). \tag{2.13}$$

Substituting (2.11) and (2.13) into (2.12), we get

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n G(d_1 C_{\tau_n-1}) \frac{H(x_{\tau_n})}{u_{h_n}}, \quad n \in I_N(h). \tag{2.14}$$

Next, choose N so large that $\sigma_n > N_3$ for all $n > N$. From (2.7), we see that $u_n > x_n$ or $u_{\sigma_n} > x_{\sigma_n}$, for all $n > N$. Thus, again, (2.7) implies

$$u_{n-1} \le x_{n-1} + p_n u_{\sigma_n}, \qquad n \in I_N(h). \tag{2.15}$$

From (2.5) and since $\Delta u_n > 0$, (2.15) is reduced to

$$u_{n-1}(1-p_n) \le x_{n-1},$$

which implies that

$$u_{h_n}(1-p_{\tau_n+1}) \le u_{\tau_n}(1-p_{\tau_n+1}) \le x_{\tau_n}, \qquad n \in I_N(h).$$

Then (2.14) has the following form

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n G(d_1 C_{\tau_n - 1}) \frac{H(u_{h_n}(1-p_{\tau_n+1}))}{u_{h_n}}, \qquad n \in I_N(h). \quad (2.16)$$

Now, by the increasing nature of u_n, one can find a positive number k^* such that $u_n > k^*$, $n \in I_N(h)$. Consequently, in view of (2.10), the inequality (2.16) becomes

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n G(d_1 C_{\tau_n - 1}) \frac{H(k^*(1-p_{\tau_n+1}))}{d_1 C_{h_n - 1}}, \quad \text{for all} \quad n \in I_N(h)$$

which by Lemma 1.2 of [2], implies that equation (2.6) is nonoscillatory. This contradiction completes the proof. □

Theorem 2.2. Suppose that the conditions *(2.1)−−(2.5)* are satisfied, and

$$H(x)\operatorname{sgn} x \ge |x|^\lambda \quad \text{for} \quad x \ne 0 \quad \text{and} \quad \lambda > 0. \quad (2.17)$$

If for every $d \ge 1$, $k > 0$ and all large N ($N \ge 0$), the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^2 u_n = 0, \qquad \text{for all} \quad n \in I_N(h) \quad (2.18)$$

where

$$Q_n^2 = \begin{cases} k^{\lambda-1}(1-p_{\tau_n+1})^\lambda G(dC_{\tau_n-1}), & \lambda \ge 1 \\ (1-p_{\tau_n+1})^\lambda G(dC_{\tau_n-1})(dC_{h_n-1})^{\lambda-1}, & \lambda < 1 \end{cases}$$

is oscillatory, then equation *(1.1)* is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). As in the proof of Theorem 2.1, we proceed to obtain (2.16). But (2.17) implies that

$$\frac{H(u_{h_n}(1-p_{\tau_n+1}))}{u_{h_n}} \ge (1-p_{\tau_n+1})^\lambda u_{h_n}^{\lambda-1}, \qquad n \in I_N(h). \quad (2.19)$$

In view of (2.10), (2.19) and the fact that $u_n > k^* > 0$ for $n > N$, the inequality (2.16) has one of the following forms

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -(k^*)^{\lambda-1} q_n G(d_1 C_{\tau_n-1})(1-p_{\tau_n+1})^\lambda, \qquad \text{for} \quad \lambda \ge 1$$

or

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n G(d_1 C_{\tau_n - 1})(1 - p_{\tau_n + 1})^{\lambda}(d_1 C_{h_n - 1})^{\lambda - 1}, \quad \text{for} \quad \lambda < 1,$$

where $n \in I_N(h)$ and k^* is a constant defined as in the proof of Theorem 2.1. By applying Lemma 1.2 of [2], we arrive at a contradiction. This completes the proof. □

As an application to our results, we consider the equation

$$\Delta^2 x_{n-1} + q_n e^{-|x_n|}|x_n|^{\nu} \operatorname{sgn} x_n = 0, \qquad \lambda > 0, \tag{2.20}$$

the corresponding linear equation according to Theorem 2.1 is,

$$\Delta^2 u_{n-1} + q_n \frac{e^{-d(n-N+1)}}{d(n-N)} k^{\nu} u_n = 0, \qquad d \ge 1,\ k > 0 \quad \text{and} \quad n > N+1. \tag{2.21}$$

Using [11] and Theorem 2.1 we get that (2.20) is oscillatory when

$$\sum_{n=N_1}^{\infty} q_n \frac{e^{-d(n-N+1)}}{d(n-N)} = \infty, \qquad N_1 > N+1.$$

This condition is satisfied, e.g., when $q_n = e^{n^2}$. As far as the authors know, none of the existing criteria can examine the oscillation of (2.20).

Next, for the sake of completeness, we consider equation (1.1) with p_n does not satisfy (2.4) eventually. Sometimes, we use the notation $y(n)$ instead of the indexed form y_n. Suppose that

$$\sigma_n \quad \text{is increasing and } \sigma_n \ge n-1 \quad \text{for} \quad n > 0, \tag{2.22}$$

$$p_n^* = \frac{1}{p(\sigma^{-1}(n))}\left[1 - \frac{1}{p(\sigma^{-1}(\sigma^{-1}(n)-1))}\right], \quad p_n^* \ge 0, \qquad \text{for all} \quad n > 0, \tag{2.23}$$

$$h_n^* = \min\{n,\, (\sigma^{-1} \circ (\tau - 1))(n)\}, \quad \Delta h_n^* \ge 0 \quad \text{for all} \quad n > 0 \tag{2.24}$$

and

$$c_n \ge c_{h_n^*}, \quad \text{for all} \quad n > 0. \tag{2.25}$$

Theorem 2.3. Suppose that the conditions *(2.1)* and *(2.22)–(2.25)* are satisfied. If for every $d \ge 1$, $k > 0$ and all large N ($N \ge 0$), the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^3 u_n = 0, \qquad \text{for all} \quad n \in I_N(h^*) \tag{2.26}$$

where

$$Q_n^3 = \frac{G(dC_{\tau_n - 1})}{dC_{h_n^* - 1}} H(kp^*_{\tau_n}),$$

is oscillatory, then equation *(1.1)* is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Proceeding as in the proof of Theorem 2.1 with h_n is replaced by h_n^*, one can easily see that

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n \frac{G(x_{\tau_n})H(x_{\tau_n})}{u_{h_n^*}}, \qquad n > N \ge 0. \tag{2.27}$$

Using (2.7), we obtain

$$x(n-1) = u(n-1) - p(n)x(\sigma(n))$$

or

$$x(\sigma^{-1}(n) - 1) = u(\sigma^{-1}(n) - 1) - p(\sigma^{-1}(n))x(n), \qquad n > N$$

where σ^{-1} is the inverse function of σ. Thus, for $n > N$, we have

$$\begin{aligned} x_n &= \frac{u(\sigma^{-1}(n) - 1) - x(\sigma^{-1}(n) - 1)}{p(\sigma^{-1}(n))} \\ &= \frac{1}{p(\sigma^{-1}(n))} \Bigg[u(\sigma^{-1}(n) - 1) - \\ &\quad \frac{u(\sigma^{-1}(\sigma^{-1}(n) - 1) - 1) - x(\sigma^{-1}(\sigma^{-1}(n) - 1) - 1)}{p(\sigma^{-1}(\sigma^{-1}(n) - 1))} \Bigg] \\ &\ge \frac{1}{p(\sigma^{-1}(n))} \left[u(\sigma^{-1}(n) - 1) - \frac{u(\sigma^{-1}(\sigma^{-1}(n) - 1) - 1)}{p(\sigma^{-1}(\sigma^{-1}(n) - 1))} \right] \\ &\ge \frac{1}{p(\sigma^{-1}(n))} \left[1 - \frac{1}{p(\sigma^{-1}(\sigma^{-1}(n) - 1))} \right] u(\sigma^{-1}(n) - 1). \end{aligned}$$

Then

$$x_n \ge p_n^* u(\sigma^{-1}(n) - 1), \qquad n \in I_N(h^*)$$

which implies that

$$x_{\tau_n} \ge p^*_{\tau_n} u_{h_n^*}, \qquad n \in I_N(h^*). \tag{2.28}$$

From (2.11), (2.27) and (2.28) we obtain

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n G(d_1 C_{\tau_n - 1}) \frac{H(p^*_{\tau_n} u_{h_n^*})}{u_{h_n^*}}, \quad n \in I_N(h^*). \tag{2.29}$$

Since $\Delta u_{n-1} > 0$ for $n > N$, there exists a positive real number k^* such that $u_n > k^*$ for all $n > N$. Using (2.10), we get

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n G(d_1 C_{\tau_n - 1}) \frac{H(p^*_{\tau_n} k^*)}{d_1 C_{h_n^* - 1}}, \qquad n \in I_N(h^*).$$

By [2, Lemma 1.2], equation (2.26) is nonoscillatory which contradicts the given hypothesis. This completes the proof. □

Theorem 2.4. Suppose that the conditions *(2.1)*, *(2.17)* and *(2.22)–(2.25)* are satisfied. If for every $d \geq 1$, $k > 0$ and all large N ($N \geq 0$), the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^4 u_n = 0, \qquad \text{for all} \quad n \in I_N(h^*) \tag{2.30}$$

where

$$Q_n^4 = \begin{cases} k^{\lambda-1}(p^*_{\tau_n})^\lambda G(dC_{\tau_n-1}), & \lambda \geq 1 \\ (p^*_{\tau_n})^\lambda G(dC_{\tau_n-1})(dC_{h_n^*-1})^{\lambda-1}, & \lambda < 1 \end{cases}$$

is oscillatory, then equation *(1.1)* is oscillatory.

Proof. Using similar arguments as in the proof of Theorem 2.3, we can proceed and obtain (2.29). Now,

$$\frac{H(p_n^* u_{h_n^*})}{u_{h_n^*}} \geq (p_n^*)^\lambda u_{h_n^*}^{\lambda-1}. \tag{2.31}$$

In view of (2.10), (2.31) and the fact that $u_n > k^* > 0$ for $n > N$, the inequality (2.29) is reduced to one of the following forms

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \leq -(k^*)^{\lambda-1} q_n G(d_1 C_{\tau_n-1})(p^*_{\tau_n})^\lambda, \quad n \in I_N(h^*) \quad \text{for} \quad \lambda \geq 1$$

or

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \leq -(d_1 C_{h_n^*-1})^{\lambda-1} q_n G(d_1 C_{\tau_n-1})(p^*_{\tau_n})^\lambda, \qquad \text{for} \quad \lambda < 1,$$

where $n \in I_N(h^*)$. In either case, an application of [2, Lemma 1.2] implies a contradiction with the assumption that equation (2.30) is oscillatory, which completes the proof. □

In the previous results, our linearization process yields second order linear difference equations of the form

$$\Delta(c_{n-1}\Delta y_{n-1}) + \overline{k} a_n y_n = 0, \qquad n \geq 1 \quad \text{and} \quad \overline{k} > 0, \tag{2.32}$$

where $\{a_n\}$ is eventually nonnegative sequence of real numbers. We interest in the oscillation of equation (2.32) for any constant $\overline{k} > 0$. This type of oscillation is called strong oscillation, and is equivalent to saying that equation (2.32) is oscillatory regardless the value of $\overline{k}$(provided that $\overline{k} > 0$). As an example of results concerning with the strong oscillation of (2.32), we extract the following two results from [3] and [11].

Theorem 2.5 ([3])**.** For $c_n \equiv 1$, equation *(2.32)* is strongly oscillatory if

$$\limsup_{n\to\infty} \; n\sum_{i=n}^{\infty} a_i = \infty.$$

Theorem 2.6 ([11])**.** Equation *(2.32)* is strongly oscillatory if *(2.1)* is satisfied and

$$\sum_{i=0}^{\infty} a_i = \infty.$$

Applying these criteria to our results, one can drive many oscillation criteria regarding equation (1.1). As an example, the following corollary is extracted from Theorem 2.1.

Corollary 2.1. Suppose that the conditions *(2.1)–(2.5)* are satisfied. If either

(I) $$\sum_{i=N_1}^{\infty} q_i Q_i^1 = \infty, \quad N_1 \text{ is sufficiently large,}$$

or

(II) $$c_n \equiv 1 \quad \text{and} \quad \limsup_{n\to\infty} \; n\sum_{i=n}^{\infty} q_i Q_i^1 = \infty,$$

where Q_n^1 is defined as in Theorem 2.1, then equation *(1.1)* is oscillatory.

Remark 2.1. Theorems 2.1–2.4 are the discrete analogues of the continuous results which have been established by [4] for the differential equation

$$(c(t)x'(t) + p(t)x(\sigma(t)))' + q(t)f(x(\tau(t))) = 0,$$

and its special case

$$(c(t)x'(t))' + q(t)f(x(\tau(t))) = 0,$$

particularly when $h_n = h_n^* = n$.

3. Forced equations

In this section we drive some sufficient conditions of comparison type for the oscillation of equation (1.2). We need the following two assumptions:

There exists a sequence $\{R_n\}$ of real numbers such that

$$\Delta(c_{n-1}\Delta R_n) = r_n \text{ eventually, and } \{R_n\} \text{ is oscillatory.} \tag{3.1}$$

And

$$P_n = 1 - p_n - \frac{1}{k^*}\left|p_n R_{\tau_n^*+1} - R_n\right| \geq 0,$$
$$\text{for every } k^* > 0 \text{ and all large } n. \quad (3.2)$$

Theorem 3.1. Suppose that *(2.1)–(2.3)*, *(2.5)* and *(3.1)–(3.2)* are satisfied. If for every $d \geq 1$, $k > 0$ and all large N ($N \geq 0$), the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^5 u_n = 0, \quad \text{for all} \quad n \in I_N(h), \tag{3.3}$$

where

$$Q_n^5 = \frac{G(dC_{\tau_n-1} + R_{\tau_n+1})H(kP_{\tau_n+1})}{dC_{h_n-1}},$$

is oscillatory, then equation *(1.2)* is oscillatory.

Proof. Suppose, for the sake of contradiction, that equation (1.2) is nonoscillatory. Without loss of generality, we assume that (1.2) has an eventually positive solution $\{x_n\}$. Let the sequence $\{y_n\}$ be, eventually, defined as follows

$$x_{n-1} + p_n x_{\sigma_n} = y_{n-1} + R_n. \tag{3.4}$$

Substituting into (1.2),

$$\Delta(c_{n-1}\Delta y_{n-1}) + q_n f(x_{\tau_n}) = 0. \tag{3.5}$$

Then

$$\Delta(c_{n-1}\Delta y_{n-1}) \leq 0. \tag{3.6}$$

This inequality implies the nonoscillation of Δy_{n-1} as well as y_n. Thus y_n is either eventually positive or eventually negative. From (3.4), we get

$$x_{n-1} \leq y_{n-1} + R_n, \quad \text{eventually}.$$

Therefore, y_n is eventually positive. Otherwise we get that $0 < x_{n-1} \leq R_n$ eventually which contradicts the oscillation of R_n . Now as in the proof of Theorem 2.1, using (2.1) and (3.6), we obtain

$$\Delta y_{n-1} > 0, \ n > N \qquad \text{for some } N > 0, \tag{3.7}$$

and

$$y_n \leq d_1 C_{n-1}, \qquad n > N. \tag{3.8}$$

From (2.5), (3.4) and the increasing nature of y_n, it is clear that

$$\begin{aligned} x_{n-1} + p_n x_{\sigma_n} &\leq x_{n-1} + p_n(y_{\sigma_n} + R_{\sigma_n+1}) \\ &\leq x_{n-1} + p_n y_{n-1} + p_n R_{\sigma_n+1}, \end{aligned}$$

or

$$y_{n-1} + R_n \leq x_{n-1} + p_n y_{n-1} + p_n R_{\sigma_n+1}, \qquad n > N.$$

Rearranging,

$$(1-p_n)y_{n-1} \le x_{n-1} + p_n R_{\sigma_n+1} - R_n, \qquad n > N. \tag{3.9}$$

Let $\{v_n\}$ be defined by

$$p_n R_{\sigma_n+1} - R_n = v_n y_{n-1}, \qquad n > N.$$

Substituting into (3.9),

$$(1-p_n-v_n)y_{n-1} \le x_{n-1}, \qquad n > N. \tag{3.10}$$

Using (3.7), one can find a constant $k_1 > 0$ such that $y_{n-1} > k_1$ for all $n > N$. The definition of v_n implies that

$$\begin{aligned} |p_n R_{\sigma_n+1} - R_n| &= |v_n|\, y_{n-1} \\ &\ge k_1 |v_n| \ge k_1 v_n \end{aligned}$$

then

$$v_n \le \frac{1}{k_1} |p_n R_{\sigma_n+1} - R_n|, \qquad n > N.$$

It follows from the above inequality and (3.10) that

$$P_n y_{n-1} \le x_{n-1}$$

thus

$$P_{\tau_n+1} y_{h_n} \le P_{\tau_n+1} y_{\tau_n} \le x_{\tau_n}, \qquad n \in I_N(h). \tag{3.11}$$

Now, define

$$W_n = \frac{\Delta(c_{n-1}\Delta y_{n-1})}{y_{h_n-1}}, \qquad n \in I_N(h).$$

As in the proof of Theorem 2.1, it is easy to drive the following inequality from (3.5), (3.11) and the inequality $x_n \le d_1 C_{\tau_n-1} + R_{\tau_n+1}$,

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n \frac{G(d_1 C_{\tau_n-1} + R_{\tau_n+1}) H(P_{\tau_n+1} y_{h_n})}{y_{h_n}}, \qquad n \in I_N(h), \tag{3.12}$$

but

$$k_1 < y_n \le d_1 C_{n-1}, \quad \text{for all} \quad n \in I_N(h).$$

Then, in view of the increasing nature of H, (3.12) implies

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \le -q_n \frac{G(d_1 C_{\tau_n-1} + R_{\tau_n+1}) H(P_{\tau_n+1} k_1)}{d_1 C_{h_n-1}}, \qquad n \in I_N(h).$$

Applying [2, Lemma 1.2], the above inequality implies that equation (3.3) is nonoscillatory which contradicts our assumption. This completes the proof. □

Theorem 3.2. Suppose that the conditions *(2.1)–(2.3)*, *(2.5)*, *(2.17)* and *(3.1)–(3.2)* are satisfied. If for every $d \geq 1$, $k > 1$ and all large N $(N \geq 0)$, the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^6 u_n = 0, \quad \text{for all} \quad n \in I_N(h), \tag{3.13}$$

where

$$Q_n^6 = \begin{cases} k^{\lambda-1}(P_{\tau_n+1})^\lambda G(dC_{\tau_n-1} + R_{\tau_n+1}), & \lambda \geq 1, \\ (P_{\tau_n+1})^\lambda G(dC_{\tau_n-1} + R_{\tau_n+1})(dC_{h_n-1})^{\lambda-1}, & \lambda < 1 \end{cases}$$

is oscillatory, then equation *(1.2)* is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.1, we obtain (3.11). Then the proof can be completed similarly as the proof of Theorem 2.2. We omit the details to avoid repetition. □

Remark 3.1. From the proof of Theorem 3.1, we have

$$0 < x_n \leq d_1 C_{n-1} + R_n, \quad \text{eventually}$$

which implies

$$\liminf_{n\to\infty} \frac{R_n}{d_1 C_{n-1}} \geq 0 \quad \text{for every } d_1 \geq 1.$$

So if

$$\liminf_{n\to\infty} \frac{R_n}{d_1 C_{n-1}} = -\infty \tag{3.14}$$

then we obtain a contradiction. Similarly, if $x_n < 0$ eventually, the condition

$$\limsup_{n\to\infty} \frac{R_n}{d_1 C_{n-1}} = \infty, \tag{3.15}$$

implies a contradiction, too. Hence, the above analysis leads to the following result which generalizes and improves Theorems 3.4 and 3.1 of [5] and [6] respectively.

Corollary 3.1. Suppose that the conditions *(3.14)* and *(3.15)* are satisfied. Then equation *(1.2)* is oscillatory.

In the following two results, we consider equation (1.2) with $p_n \equiv 0$, i.e.,

$$\Delta(c_{n-1}\Delta x_{n-1}) + q_n f(x_{\tau_n}) = r_n. \tag{3.16}$$

Theorem 3.3. Suppose that the conditions *(2.1)–(2.3)*, *(2.5)* and *(3.1)* are satisfied. If for every $d \geq 1$, $k > 0$ and all large N ($N \geq 0$), the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^7 u_n = 0, \quad \text{for all} \quad n \in I_N(h), \tag{3.17}$$

where

$$Q_n^7 = \frac{G(dC_{\tau_n-1} + R_{\tau_n+1})}{dC_{h_n-1}} H(kR^+_{\tau_n+1}/dC_{\tau_n-1}),$$

is oscillatory, then equation *(3.16)* is oscillatory.

Proof. Suppose that (3.16) is nonoscillatory. Then, as usual, (3.16) has a solution x_n which can be assumed to be eventually positive. The proof is similar to that of Theorem 3.1. But, we obtain here another estimate for x_{τ_n} in terms of y_{τ_n} which is defined by (3.4) (with $p_n \equiv 0$). For this purpose, we define a sequence $\{v_n\}$ as follows

$$x_n = v_n y_n, \qquad n > N \text{ for some } N > 0,$$

then (3.4) (with $p_n \equiv 0$) yields

$$v_n y_n = y_n + R_{n+1} > R_{n+1}$$

which in view of (3.8) implies

$$v_n > \frac{R_{n+1}}{d_1 C_{n-1}}, \qquad n \in I_N(h),$$

therefore

$$x_n > \frac{R_{n+1}}{d_1 C_{n-1}} y_n, \qquad n \in I_N(h),$$

from this inequality, we get

$$x_{\tau_n} > \frac{R^+_{\tau_n+1}}{d_1 C_{\tau_n-1}} y_{\tau_n} \geq \frac{R^+_{\tau_n+1}}{d_1 C_{\tau_n-1}} y_{h_n}, \qquad n \in I_N(h).$$

Using the above estimate of x_{τ_n}, similarly as in the proof of Theorem 3.1, one can prove that

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \leq -q_n \frac{G(d_1 C_{\tau_n-1} + R_{\tau_n+1}) H(\frac{R^+_{\tau_n+1}}{d_1 C_{\tau_n-1}} y_{h_n})}{y_{h_n}}, \qquad n \in I_N(h). \tag{3.18}$$

But $k_1 < y_{h_n} < d_1 C_{h_n-1}$, $n \in I_N(h)$. Then

$$\Delta W_n + \frac{W_n^2}{W_n + c_{n-1}} \leq -q_n \frac{G(d_1 C_{\tau_n-1} + R_{\tau_n+1}) H(\frac{R^+_{\tau_n+1}}{d_1 C_{\tau_n-1}} k_1)}{d_1 C_{h_n-1}}, \qquad n \in I_N(h).$$

which, in view of [2, Lemma 1.2], implies that equation (3.17) is nonoscillatory. This contradiction implies the proof. $\square$

When condition (2.17) holds, one can easily prove the following result which is of the same type as Theorems 2.2, 2.4 and 3.2.

Theorem 3.4. Suppose that the conditions *(2.1)–(2.3)*, *(2.5)*, *(2.17)* and *(3.1)* are satisfied. If for every $d \geq 1$, $k > 0$ and all large N $(N \geq 0)$, the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n Q_n^8 u_n = 0, \quad \text{for all} \quad n \in I_N(h),$$

where

$$Q_n^8 = \begin{cases} k^{\lambda-1} G(dC_{\tau_n-1} + R_{\tau_n+1})(R_{\tau_n+1}^+/dC_{\tau_n-1})^\lambda, & \lambda \geq 1, \\ G(dC_{\tau_n-1} + R_{\tau_n+1})(R_{\tau_n+1}^+/dC_{\tau_n-1})^\lambda (dC_{h_n-1})^{\lambda-1}, & \lambda < 1 \end{cases}$$

is oscillatory, then equation *(3.13)* is oscillatory.

As in the preceding section, our results can be combined with any known strong oscillation criteria to obtain several new oscillation criteria regarding equation (1.2). The following result is derived from Theorem 3.1 using Theorem 2.5 and Theorem 2.6. The result is considered as the nonhomogeneous version of Corollary 2.1.

Corollary 3.2. Suppose that the conditions *(2.1)–(2.3)*, *(2.5)* and *(3.1)–(3.2)* are satisfied. If either

(I) $$\sum_{i=N_1}^{\infty} q_i Q_i^5 = \infty, \quad N_1 \text{ is sufficiently large,}$$

or

(II) $$c_n \equiv 1 \quad \text{and} \quad \limsup_{n\to\infty}\ n \sum_{i=n}^{\infty} q_i Q_i^5 = \infty,$$

where Q_n^5 is defined as in Theorem 3.1, then equation *(1.2)* is oscillatory.

A prototype of equation (3.16), namely

$$\Delta(c_{n-1}\Delta x_{n-1}) + q_n x_n = r_n \tag{3.19}$$

has been studied by [10] and recently by [5]. Most of the results obtained there depend essentially on the oscillatory character of the associated homogeneous equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + q_n x_n = 0.$$

So that, those results of [5], [10] should be compared with our results regarding equation (3.19) particularly the following two results which are immediate consequences of Theorems 3.2 and 3.4 respectively.

Corollary 3.3. In addition to the conditions *(2.1)* and *(3.1)–(3.2)* (with $p_n \equiv 0$), suppose that the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + q_n(1 - |R_{n+1}|/k)u_n = 0, \quad \text{for all} \quad k > 0$$

is oscillatory, then equation *(3.19)* is oscillatory.

Corollary 3.4. Suppose that the conditions *(2.1)* and *(3.1)* are satisfied. If for every $d \geq 1$ and all large N such that $n - 1 > N$, the equation

$$\Delta(c_{n-1}\Delta u_{n-1}) + \frac{R^+_{n+1}}{dC_{n-1}} q_n u_n = 0,$$

is oscillatory, then equation *(3.19)* is oscillatory.

General remarks:

1. When $R_n \equiv 0$, it is clear that Theorems 3.1 and 3.2 are reduced to Theorems 2.1 and 2.2, respectively. But Theorems 3.3 and 3.4 do not satisfy this property.
2. It is observed that any of our results does not require whether $\tau_n < n$ or not. This, of course, gives our results the ability of testing many types of equations, i.e., delay or advance or mixed type, by the same test.
3. It is of special importance to obtain similar results as ours when p_n assumes eventually negative values and/or q_n is oscillatory.

References

[1] Agarwal, R.P., Manuel, M.M.S. and Thandapani, E., *Oscillatory and nonoscillatory behavior of second order neutral delay difference equations II*, Appl. Math. Lett. **10** (1997), 103–109.

[2] Chen, S. and Erbe, L.H., *Riccati techniques and discrete oscillations*, J. Math. Anal. Appl. **142** (1989), 468–487.

[3] Cheng, S.S., Yan, T.C. and Li, H.J., *Oscillation criteria for second order difference equation*, Funkcial. Ekvac. **34** (1991), 223–239.

[4] Grace, S.R., *Oscillation criteria of comparison type for nonlinear functional differential equations*, Math. Nachr. **173** (1995), 177–192.

[5] Grace, S.R. and El-Morshedy, H.A., *Oscillation and nonoscillation theorems for certain second order difference equations with forcing term*, J. Math. Anal. Appl. **216** (1997), 614–625.

[6] Grace, S.R. and Lalli, B.S., *Oscillation theorems for forced neutral difference equations*, J. Math. Anal. Appl. **187** (1994), 91–106.

[7] Hooker, J.W. and Patula, W.T., *A second–order nonlinear difference equation: Oscillation and asymptotic behaviour*, J. Math. Anal. Appl. **91** (1983), 9–29.

[8] Kwong, M.K., Hooker, J.W. and Patula, W. T., *Riccati type transformations for second–order linear difference equations II*, J. Math. Anal. Appl. **107** (1985), 182–196.

[9] Mahfoud, W.E., *Oscillation and asymptotic behavior of solutions of Nth order nonlinear delay differential equations*, J. Differential Equations **24** (1977), 75–98.

[10] Patula, W.T., *Growth, oscillation and comparison theorems for second order linear difference equations*, SIAM J. Math. Anal. **10** (1979), 1272–1279.

[11] Szmanda, B., *Oscillation of solutions of second order difference equations*, Portugal. Math. **37**(3–4) (1978), 251–254.

[12] Szmanda, B., *Oscillation criteria for second order non-linear difference equations*, Ann. Polon. Math. **XLIII** (1983), 225–235.

[13] Thandapani, E., Györi, I. and Lalli, B.S., *An application of discrete inequality to second order nonlinear oscillation*, J. Math. Anal. Appl. **186** (1994), 200–208.

[14] Zhang, B.G., *Oscillation and asymptotic behaviour of second order difference equations*, J. Math. Anal. Appl. **173** (1993), 58–68.

[15] Zhang, B.G. and Chen, G.D., *Oscillation of certain second order nonlinear difference equations*, J. Math. Anal. Appl. **199** (1996), 827–841.

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