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BOUNDED SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS IN UNBOUNDED DOMAINS

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Abstract. We prove here the existence of a bounded, radial solution in unbounded domain of the nonlinear elliptic problem

$$\begin{array}{l}\Delta u = f(\|x\|, u) \quad \text{for} \quad \|x\| > 1,\ x \in \mathbb{R}^n \\ u(x) = 0 \ \ \text{for} \ \ \|x\| = 1\end{array}$$

under some asymptotic and sign condition on f. Under stronger assumptions it is proved that this solution must be of constant sign. The existence of radial solutions, vanishing at ∞, of some semilinear equation is also established here.

1. Introduction

The paper is concerned with the existence of bounded solutions of

$$\begin{array}{l}\Delta u = f(\|x\|, u) \quad \text{for} \quad \|x\| > 1, \quad x \in \mathbb{R}^n \\ u(x) = 0 \quad \text{for} \quad \|x\| = 1\end{array} \tag{1}$$

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where nonlinearity $f : [1,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the following sign-condition: there exists a positive constant M such that

$$f(r,u)u \geq 0 \quad \text{for} \quad |u| \geq M, r \in [1,\infty)$$

and asymptotic condition:

$$\lim_{r\to\infty} \sup_{|u|\leq M} |f(r,u)| = 0 \quad \text{for} \quad \text{each} \quad M > 0$$

Boundary value problems on unbounded domains present specific difficulties. In variational approach the methods, working well in the case of bounded domain, break down because of lack of compactness, therefore other methods have to be used (cf. [1], [9], [18]). One way to observe this is the fact that the compact embedding $H_0^1 \hookrightarrow L^2$ is no longer valid when we consider problems in unbounded domains.

A good survey of known results on the existence of positive solutions is [6]. Only solutions convergent to 0 at infinity are considered there. Similar results have been obtained in [8], [13]. The existence of radial solutions is proved in [3], [5]. In [12], [17] the authors consider bounded solutions of some specific elliptic equations. In this paper we consider general nonlinearity under above-mentioned assumptions. We use perturbation method together with some fixed point theorem (which follows from Leray-Schauder degree theory).

Fix $n \geq 3$. Substitution $r = \|x\|$ leads to the following boundary value problem involving ordinary differential equation:

$$\begin{aligned} &v'' + \frac{n-1}{r}v' = f(r,v) \\ &v(1) = 0 \\ &v \quad \text{—} \quad \text{bounded} \quad \text{on} \ [1,\infty) \end{aligned} \tag{2}$$

Similar problems for ordinary differential equations have been considered in [14]. Solutions vanishing at infinity are also considered there. We can reformulate (2) in the following fashion:

$$Lv = N(v) \tag{3}$$

where L is the linear differential operator defined by the left-hand side of (1) (with the boundary condition taken into account) on the subspace of $BC([1,\infty))$ (the space of bounded and continuous functions) and $N(v)(r) = f(r,v(r))$ is a Nemitski operator. For $n \geq 3$ L has one dimensional kernel spanned by the function: $v_1(r) = 1 - r^{2-n}$ so the problem is at resonance. The theory of coincidence-degree developed in [4] and [11] does not work, because the image of the operator L is not closed so it is not a Fredholm operator. Instead we use here perturbation technique — all the eigenvalues of L are in $(-\infty, 0]$. Therefore for $\lambda > 0$ the operator $L - \lambda I$ is injective so the problem:

$$Lv - \lambda v = N(v) \tag{4}$$

may be replaced by

$$v = (L - \lambda I)^{-1} N(v) \tag{5}$$

and some fixed point theorem can be applied.

2. Auxiliary theorems

We shall apply the Schaefer fixed point theorem (see [10, Corollary 4.4.12, p. 71]):

Theorem 1. Let X be a Banach space and let $A : X \mapsto X$ be a completely continuous operator, such that the set

$$\{x \in X : x = \mu Ax, \quad \text{for} \quad \text{some} \ \ \mu \in [0, 1)\}$$

is bounded. Then A has a fixed point.

The equation (4) can be rewritten as:

$$\begin{aligned} & v'' + \frac{n-1}{r} v' - \lambda v = f(r, v) \\ & v(1) = 0 \\ & v \quad \text{—} \quad \text{bounded} \quad \text{on} \quad [1, \infty) \end{aligned} \tag{6}$$

while the equation (5) can be expressed as

$$v(r) = \int_1^\infty G_\lambda(r, s) f(s, v(s)) \, ds \tag{7}$$

where $v \in BC([1, \infty))$ and G_λ is a Green function for (4).
In order to apply Theorem 1 we shall use the following compactness criterion (for more general version see [15]):

Theorem 2. Define an integral operator A on $BC([1, \infty))$ by

$$Av(r) = \int_1^\infty G(r, s) f(s, v(s)) \;\; ds.$$

Assume that

(a) functions $G : [1, \infty) \times [1, \infty) \to \mathbb{R}$ and $f : \mathbb{R} \times [1, \infty) \to \mathbb{R}$ are continuous,

(b) there exist positive constants v, N such that

$$|G(r, s)| \le N e^{-v|r-s|}, \tag{8}$$

(c) there exists a function $b \in BC([1,\infty))$ such that, for each $M > 0$,

$$\lim_{r\to\infty} \sup_{|v|\le M} |f(r,v) - b(r)| = 0.$$

Then A maps $BC([1,\infty))$ into $BC([1,\infty))$ and is completely continuous.

3. Existence result

Theorem 3. Let $n \ge 3$. Assume that $f : [1,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies:

(i) there exists $M > 0$ such that for all $|v| \ge M$ and $r \in [1,\infty)$

$$f(r,v)v \ge 0, \tag{9}$$

(ii) for each $M > 0$

$$\lim_{r\to\infty} \sup_{|v|\le M} |f(r,v)| = 0. \tag{10}$$

Then boundary value problem (2) has at least one solution.

Proof. The proof of this theorem will be divided into three steps:

1 — the complete continuity of integral operator defined by right-hand side of (7) (with λ fixed)

2 — a priori bound for solutions of (7) or equivalently (6) independent of parameter μ from Theorem 1 (λ still fixed) and in consequence the existence of solution for (6)

3 — extraction from the sequence of solutions for (6) v_{λ_n} ($\lambda_n \to 0^+$) a subsequence convergent to a solution v_0 for (2).

1st step

First we shall show the complete continuity of A_λ defined by the right hand side of (7). In order to find the Green function G_λ for (6) we consider another equation

$$z'' + z\left(\frac{k}{r^2} - \lambda\right) = 0 \tag{11}$$

(where $k = -(n-1)(n-3)/4 \le 0$) obtained from (6) by substitution $v(r)r^{(n-1)/2} = z(r)$. From the linear theory (see [2]) one obtains two solutions $\phi_{\lambda,1}$, $\phi_{\lambda,2}$ of the equation (11) such that:

$$\begin{aligned} &\left|\phi_{\lambda,1}(r)e^{\sqrt{\lambda}r} - 1\right| \to 0 \quad \text{as} \;\; r \to \infty \\ &\left|\phi_{\lambda,2}(r)e^{-\sqrt{\lambda}r} - 1\right| \to 0 \quad \text{as} \;\; r \to \infty. \end{aligned} \tag{12}$$

Therefore any linear combination of $\phi_{\lambda,1}$, $\phi_{\lambda,2}$ satisfying

$$z(1) = 0 \quad \text{and} \quad z(r)r^{(1-n)/2} \;\text{— bounded} \quad \text{for} \;\; r \in [1,\infty)$$

must be identically 0. The same holds for homogeneous problem (6). So we are ready to construct the Green function for (6). Taking (if necessary) $\phi_{\lambda,2}$ as a combination of the above solutions we can assume that $\phi_{\lambda,2}(1) = 0$. Moreover

$$\begin{aligned} |\phi_{\lambda,1}(r)| &\le \hat{N}_\lambda e^{-\sqrt{\lambda}r} \\ |\phi_{\lambda,2}(r)| &\le \hat{N}_\lambda e^{\sqrt{\lambda}r} \end{aligned} \tag{13}$$

for some positive constant $\hat{N}_\lambda$. Thus the functions: $\varphi_{\lambda,1}(r) = r^{(1-n)/2}\phi_{\lambda,1}(r)$, $\varphi_{\lambda,2}(r) = r^{(1-n)/2}\phi_{\lambda,2}(r)$ are solutions for the homogeneous problem (6). Moreover, by (13) there exist positive constants $\bar{N}_\lambda, \bar{\mu}_\lambda$ such that

$$\begin{aligned} |\varphi_{\lambda,1}(r)| &\le \bar{N}_\lambda e^{-\bar{\mu}_\lambda r} \\ |\varphi_{\lambda,2}(r)| &\le \bar{N}_\lambda e^{\bar{\mu}_\lambda r} \end{aligned} \tag{14}$$

$$\varphi_{\lambda,2}(1) = 0. \tag{15}$$

Multiplying $\varphi_{\lambda,1}$ by some constant, we can assume that

$$\varphi_{\lambda,1}(1)\varphi'_{\lambda,2}(1) = 1. \tag{16}$$

(none of them cannot be zero, since otherwise it would be a non-zero solution to homogenous problem (6)). The existence of solutions $\varphi_{\lambda,1}$, $\varphi_{\lambda,2}$ with the above properties can be also derived after (6) is transformed to the Bessel equation and then solutions are given by the modified Bessel functions of order $(N/2)-1$ of the first and second kind (cf. [7], [16], in the case of $N = 3$ we can explicitly write formulas for $\varphi_{\lambda,1}$ and $\varphi_{\lambda,2}$). The Green function for the problem (6) has the following form:

$$G_\lambda(r,s) = \begin{cases} -s^{n-1}\varphi_{\lambda,1}(s)\varphi_{\lambda,2}(r) & \text{for } s > r \\ -s^{n-1}\varphi_{\lambda,2}(s)\varphi_{\lambda,1}(r) & \text{for } s \le r \end{cases} \tag{17}$$

since it satisfies the following conditions:

1° For any $s \in [1,\infty)$, $G(\cdot,s)$ satisfies the homogenous equation, i.e.

$$\frac{\partial^2 G}{\partial r^2}(r,s) + \frac{n-1}{r}\frac{\partial G}{\partial r}(r,s) - \lambda G(r,s) = 0$$

for any $r \ne s$,

2° $\displaystyle\lim_{s\to r^-}\frac{\partial G}{\partial r}(r,s) - \lim_{s\to r^+}\frac{\partial G}{\partial r}(r,s) = 1$ for any $s \in (1,\infty)$,

3° $G(\cdot,s)$ satisfies the boundary condition for any $s \in \mathbb{R}$ (i.e. is bounded on $[1,\infty)$ and $G(1,s) = 0$ for every $s \in [1,\infty)$).

The first and the third follow immediately from the properties of $\varphi_{\lambda,1}$, $\varphi_{\lambda,2}$. The second one is ascertained by the fact that Wronskian of homogeneous equation satisfies

$$W'(r) = \frac{1-n}{r}W(r)$$

so $W(r) = ar^{1-n}$. Considering (15) and (16) we obtain

$$W(1) = \varphi_{\lambda,1}(1)\varphi'_{\lambda,2}(1) - \varphi_{\lambda,2}(1)\varphi'_{\lambda,1}(1) = 1$$

so $a = 1$. The conditions (i)–(iii) ensure the equivalence of the problems (6) and (7).

From (14) it easily follows that there exist some positive constants N_λ, μ_λ such that

$$|G(r,s)| \leq N_\lambda e^{-v_\lambda |r-s|}. \tag{18}$$

So the assumptions of Theorem 2 on the function G_λ are satisfied. Furthermore (10) implies (c) from Theorem 2 with $b \equiv 0$ so we have obtained the complete continuity of the operator A_λ.

2nd step

Now, with λ fixed, we shall prove that solutions for the family of problems

$$v(r) = \int_1^\infty G_\lambda(r,s)\mu f(s, v(s))\, ds \tag{19}$$

or

$$\begin{array}{l} v'' + \dfrac{n-1}{r} v' - \lambda v = \mu f(r, v) \\ v(1) = 0 \\ v \quad \text{— bounded} \end{array} \tag{20}$$

are a priori bounded independently of μ in $BC([1,\infty))$. Suppose that for some μ there exists a solution v (for convenience we omit the dependence of v on μ) for the problem (20) such that $\|v\|_\infty := \sup_{r\in[1,\infty)} |v(r)| > M$, with M from assumption (ii). Then there exists r_0 such that $v(r_0) = M$. Now we shall consider two cases:

1° v is strictly increasing for $r > r_0$ or

2° there exists $r_1 \geq r_0$ such that $v(r_1) \geq M$, $v'(r_1) = 0$ and $v''(r_1) \leq 0$.

In the first case for all r such that $v(r) \geq M$

$$v''(r) + \frac{n-1}{r} v'(r) = \lambda v(r) + \mu f(r, v(r)) \geq M\lambda \tag{21}$$

and multiplying by r^{n-1} and integrating on the interval $[r_0, r]$ we get

$$\begin{array}{rcl} v'(r) & \geq & v'(r_0)(\dfrac{r_0}{r})^{n-1} + M\lambda \dfrac{1}{n}(r - \dfrac{r_0^n}{r^{n-1}}) \\ v'(r) & \geq & M\lambda \dfrac{1}{n}(r - r_0) \end{array}$$

so $v'(r) \to \infty$ as $r \to \infty$, which contradicts boundedness of v.

In the second case we have

$$f(r_1, v(r_1)) = v''(r_1) + \frac{n-1}{r} v'(r_1) - \lambda v(r_1) < 0 \tag{22}$$

which contradicts (9).
So we have proved that $v(r) \leq M$. In a similar fashion we prove that $v(r) \geq -M$, thus obtaining a priori bound for $\|v\|_\infty$.
As we have announced using Theorem 1 we obtain the existence of the solution for (7).

3rd step
Now we shall prove that the sequence v_{λ_m} has a subsequence that converges (uniformly on compacts) as $\lambda_m \to 0^+$ to the solution v_0 for (2). From the first part it follows that solutions for (6) are bounded independently of λ_m by the constant M, so the sequence v_{λ_m} is bounded. Fix $k > 1$. After substituting $z_{\lambda_m}(r) = v_{\lambda_m}(r)r^{(n-1)/2}$ we obtain equivalent equation to (6) i.e.

$$z''_{\lambda_m} + z_{\lambda_m}(\frac{k}{r^2} - \lambda_m) = r^{(n-1)/2} f(r, z_{\lambda_m} r^{(1-n)/2}). \tag{23}$$

Since f is bounded on $[1,p] \times [-M, M]$ from (23) we get an estimate for $\sup_{r\in[1,p]} \left|z''_{\lambda_m}(r)\right|$. Then for $r \in [1, \frac{p+1}{2}]$ we have the following Taylor formula

$$z_{\lambda_m}(p) = z_{\lambda_m}(r) + z'_{\lambda_m}(r)(p-r) + z''_{\lambda_m}(r + \theta(p-r))\frac{(p-r)^2}{2} \tag{24}$$

and for $r \in [(p+1)/2, p]$ we have

$$z_{\lambda_m}(1) = z_{\lambda_m}(r) + z'_{\lambda_m}(r)(1-r) + z''_{\lambda_m}(r + \theta(1-r))\frac{(1-r)^2}{2}. \tag{25}$$

Since in both cases the coefficient at $z'_{\lambda_m}(r)$ is bounded away from zero, we also obtain bound on $\sup_{r\in[1,p]} |z'(r)|$. In consequence from Ascoli-Arzelá theorem we obtain that the sequences: z_{λ_m} and z'_{λ_m} have convergent subsequences in $C([1,p])$. Since p was arbitrary by diagonal procedure we can extract a subsequence $z_{\lambda_{m_n}}$ convergent with its derivative (uniformly on compacts) to some z_0. Then from the equation (23) we get uniform convergence of $z''_{\lambda_{m_n}}$ on compacts. Therefore z_0 is a solution of (2) with $\lambda = 0$. Then $v_0(r) = z_0(r)r^{(1-n)/2}$ is a solution for (2) for all $r \in [1, \infty)$. Moreover it is bounded since the sequence $v_{\lambda_{m_n}}$ was bounded independently of λ_{m_n} by the constant M. □

Corollary 1. From the integral equation (7) it follows any solution v_λ ($\lambda > 0$) satisfies

$$\lim_{t\to\infty} |v_\lambda(t)| = 0$$

thus we have established the existence of solution for semilinear boundary value problem:

$$\begin{array}{l} \Delta u - \lambda u = g(\|x\|)h(u) \quad \text{for} \quad \|x\| > 1 \\ u(x) = 0 \quad \text{for} \quad \|x\| = 1 \\ \lim_{\|x\|\to\infty} u(\|x\|) = 0. \end{array} \tag{26}$$

Note however that the solution of problem (2) does not have to converge to zero at infinity.

Corollary 2. If we assume that the nonlinearity f satisfies the assumptions of Theorem 3 with (9) replaced by the following condition: there exist positive constants M_1, M_2 such that

$$f(r,u) \geq 0 \quad \text{for} \quad u \geq M_1 \quad \text{and} \quad r \in [1,\infty)$$

and

$$f(r,u) \leq 0 \quad \text{for} \quad u \leq -M_2 \quad \text{and} \quad r \in [1,\infty)$$

then from the proof of Theorem 3 there exists a non-zero solution u of (2) such that $-M_2 \leq u(t) \leq M_1$ for all $r \in [1,\infty)$.

Remark 1. If f satisfies the above condition with $M_1 = 0$ or $M_2 = 0$ then there exists a negative or positive solution of (2), respectively.

Example 1. Let $n \geq 3$. The following boundary value problem:

$$\begin{array}{l} \Delta u = g(\|x\|)h(u) \quad \text{for} \quad \|x\| > 1 \\ u(x) = 0 \quad \text{for} \quad \|x\| = 1 \\ u \quad \text{---} \quad \text{bounded} \end{array} \tag{27}$$

has a non-zero solution provided that g is a non-negative function satisfying

$$\lim_{t\to\infty} |g(t)| = 0$$

and h is a polynomial of odd degree l such that $h(0) \neq 0$ and coefficient at l-th power is positive.

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