Journal of Applied Analysis
Vol. 6, No. 2 (2000), pp. 259–282

INTEGRALS OF LEGENDRE POLYNOMIALS AND SOLUTION OF SOME PARTIAL DIFFERENTIAL EQUATIONS

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Received March 3, 2000

Abstract. We show a connection between the polynomials whose inflection points coincide with their interior roots (let us write shorter PIPCIR), Legendre polynomials, and Jacobi polynomials, and study some properties of PIPCIRs (Part I). In addition, we give new formulas for some classical orthogonal polynomials. Then we use PIPCIRs to solve some partial differential equations (Part II).

1. Part I. Properties of PIPCIRs

1.1. Relation to classical polynomials.

Since translating all the roots an equal amount or multiplying a polynomial by a constant will not affect the position of the roots relative to any critical or inflection points, we restricted our attention to a polynomial with the first and last roots at $x = \pm 1$, given by

$$Q_n(x) = (1 - x^2)q_{n-2}(x), \qquad n \geq 2. \tag{1}$$

1991 *Mathematics Subject Classification.* Primary: 42C05; Secondary: 26C05, 26C10, 26C99.

Key words and phrases. Orthogonal polynomials, Legendre polynomials, Jacobi polynomials, generating function.

This research was supported by Morris Brown College Research Fund.

ISSN 1425-6908

Let us call a polynomial whose inflection points coincide with their interior roots in a shorter way: PIPCIR. It will be shown that the zeros of these polynomials are all real, distinct, and they lie in the interval $[-1,1]$.

The requirement all inflection points to coincide with all roots of $Q_n(x)$ except ± 1 yields:

$$\begin{aligned} &Q_n''(x) = -n(n-1)q_{n-2}(x), \qquad \text{or} \\ &(1-x^2)Q_n''(x) + n(n-1)Q_n(x) = 0. \end{aligned} \tag{2}$$

If $n = 2$, the function does not have neither inflection points, nor interior roots between 1 and -1, but it equals zero at 1 and -1. We may include $Q_2(x)$ into the family of PIPCIRs.

Let us differentiate the equation (2) with respect to x:

$$-2xQ_n'' + (1-x^2)Q_n''' + n(n-1)Q_n' = 0,$$

and denote $y_{n-1} = Q_n'$:

$$(1-x^2)y_{n-1}'' - 2xy_{n-1}' + n(n-1)y_{n-1} = 0. \tag{3}$$

We have now well-known Legendre's differential equation whose bounded on $[-1,1]$ solutions are known as Legendre polynomials: $y_{n-1} = L_{n-1}(x)$, $n \geq 1$. One can find properties of these polynomials in [1] or [2]. They are normalized so that $L_n(1) = 1$ for all n. If

$$Q_n(x) = -\int_x^1 L_{n-1}(x)dx, \tag{4}$$

then $Q_n'(x) = L_{n-1}(x)$ and $Q_n'' = L_{n-1}'(x)$.

We see that polynomials $Q_n(x)$ defined by (4) satisfy the equation (2), and $Q_n(1) = 0$ for all $n \geq 1$. Moreover, $Q_n(-1) = 0$ for $n \geq 2$, since $\int_{-1}^1 L_{n-1}(x)dx = 0$, because $L_{n-1}(x)$ is orthogonal to $L_0(x) = 1$. Thus,

$$Q_n(1) = Q_n(-1) = 0, \qquad n \geq 2. \tag{5}$$

Using (4), we get an explicit expression for $Q_n(x)$ from the formula for Legendre polynomials ([1, p. 120]):

$$Q_n(x) = \sum_{k=0}^{N} \frac{(-1)^k(2n-2k-3)!!}{(2k)!!(n-2k)!} x^{n-2k}, \quad n \geq 2, \tag{6}$$

$$\text{and} \quad Q_n(0) = \frac{(-1)^{(n-2)/2}(n-3)!!}{n!!}$$

where $N = n/2$ or $(n-1)/2$ according as n is even or odd, or $N = [n/2]$. The PIPCIR $Q_n(x)$ is even for even n and odd for odd n.

Remind that $\quad n!! = n(n-2)(n-4)(n-6)\dots, \quad 0!! = 1, \quad (-1)!! = 1.$

If $n = 1$, we evaluate the integral immediately: $-\int_t^1 1dx = -(1-t) = t-1$. This function cannot be included in the family of PIPCIRs since it is not an odd function and has only one root.

Explicit formula for $q_n(x)$ is the following

$$q_n(x) = \frac{1}{(n+1)(n+2)} \sum_{k=0}^{N} \frac{(-1)^{k+1}(2n-2k+1)!!}{(2k)!!(n-2k)!} x^{n-2k}, \tag{7}$$

where $N = [n/2]$ and

$$q_n(1) = -\frac{1}{2}, \qquad q_n(-1) = \frac{(-1)^{n+1}}{2}. \tag{8}$$

You may see examples of polynomials $Q_n(x)$ and $q_n(x)$:

$$Q_2(x) = \frac{x^2-1}{2}, \qquad Q_3(x) = \frac{x^3-x}{2},$$
$$Q_4(x) = \frac{5x^4-6x^2+1}{8}, \qquad Q_5(x) = \frac{7x^5-10x^3+3x}{8},$$
$$Q_6(x) = \frac{21x^6-35x^4+15x^2-1}{16},$$

$$q_0(x) = -\frac{1}{2}, \qquad q_1(x) = -\frac{x}{2},$$
$$q_2(x) = \frac{-5x^2+1}{8}, \qquad q_3(x) = \frac{-7x^3+3x}{8},$$
$$q_4(x) = \frac{-21x^4+14x^2-1}{16}, \qquad q_5(x) = \frac{-33x^5+30x^3-5x}{16}.$$

If we substitute the formula (1) into the equation (2), we determine that the polynomial $q_n(x)$ of degree n satisfies the differential equation

$$(1-x^2)q_n'' - 4xq_n' + n(n+3)q_n = 0. \tag{9}$$

The bounded solution of this equation is the Jacobi polynomial $P_n^{(1,1)}(x)$, or ultraspherical polynomial $P_n^{(3/2)}(x)$. One can find properties of these polynomials in [2]. In particular, the zeros of these polynomials are all real, distinct, and lie in the interior of the interval $[-1, 1]$. Hence, we have

Theorem 1. *The polynomial* $Q_n(x) = (1-x^2)q_{n-2}(x)$, $n \geq 2$, *has* $(n-2)$ *distinct real zeros in the interior of the interval* $[-1, 1]$ *and two zeros at its ends.*

The equation (2) may be considered as a particular case of the equation for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with $\alpha = -1$, $\beta = -1$ ([2, p. 59]). But polynomials $P_n^{(\alpha,\beta)}(x)$ belong to the family of *classical* orthogonal polynomials only for $\alpha > -1$, $\beta > -1$ (see [2, pp. 28, 57]). That is why these polynomials were not under investigation.

The normalization of Jacobi polynomials is $P_n^{(1,1)}(1) = n+1$ (see [2, p. 57]). Since $q_n(1) = -1/2$ (see (8)), we have:

$$P_n^{(1,1)}(x) = -2(n+1)q_n(x). \tag{10}$$

There are many important properties and recurrence formulas for Legendre and Jacobi polynomials (see [1], [2]). All of them may be transferred into formulas for PIPCIRs. We shall consider some of them.

1.2. Rodrigues formula and corollaries.

Rodrigues formula holds for arbitrary α and β (see [2, p. 66]); for $\alpha = \beta$ we have:

$$(x^2-1)^\alpha P_n^{(\alpha,\alpha)}(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n+\alpha}\right]. \tag{11}$$

If $\alpha = \beta = -1$, it becomes

$$P_n^{(-1,-1)}(x) = \frac{x^2-1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^{n-1}\right], \qquad n > 1.$$

Wishing to find $P_n^{(-1,-1)}(0)$, we determine the coefficient of x^n in the binomial $(x^2-1)^{n-1}$, then evaluate

$$P_n^{(-1,-1)}(0) = \frac{(-1)^{(n-2)/2}(n-1)!!}{2n!!}.$$

Hence, comparing this equality with (6), we have for PIPCIRs

$$P_n^{(-1,-1)}(x) = \frac{n-1}{2}Q_n(x), \qquad n > 1 \tag{12}$$

and

$$Q_n(x) = \frac{x^2-1}{2^{n-1}n!(n-1)}\frac{d^n}{dx^n}\left[(x^2-1)^{n-1}\right], \qquad n > 1. \tag{13}$$

Now formulas (1), (10) and (12) yield

$$(x^2-1)P_n^{(1,1)}(x) = 4P_{n+2}^{(-1,-1)}(x).$$

Combining the formulas (1), (4) and (12), we obtain relation between Legendre polynomials $L_n(x) = P_n^{(0,0)}(x)$ and Jacobi polynomials $P_n^{(1,1)}(x)$ and

$P_n^{(-1,-1)}(x)$:

$$\frac{1-x^2}{2n} P_{n-1}^{(1,1)}(x) = \int_x^1 P_n^{(0,0)}(x)dx$$

$$\frac{2}{n} P_{n+1}^{(-1,-1)}(x) = -\int_x^1 P_n^{(0,0)}(x)dx.$$

For Legendre polynomials ($\alpha = \beta = 0$) we have:

$$L_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2-1)^n\right].$$

Taking into account (4) and the fact that since $x = \pm 1$ are zeros of multiplicity $n-1$ for the function $(x^2-1)^{n-1}$, they are zeros of multiplicity 1 for the $(n-2)$ derivative, we can write (note: $\frac{d^0}{dx^0} f(x) = f(x)$):

$$Q_n(x) = \frac{1}{2^{n-1}(n-1)!} \frac{d^{n-2}}{dx^{n-2}} \left[(x^2-1)^{n-1}\right], \qquad n > 1. \tag{14}$$

Now we have two expressions for $Q_n(x)$; equating them, we obtain the formula

$$(x^2-1)\frac{d^n}{dx^n}\left[(x^2-1)^{n-1}\right] = n(n-1)\frac{d^{n-2}}{dx^{n-2}}\left[(x^2-1)^{n-1}\right]. \tag{15}$$

Theorem 2. *Each function* $(n > 1)$ *in* (15) *is a polynomial of degree n, that has n real distinct roots in the interval* $[-1, 1]$, *two of them are* $x = -1$, *and* $x = 1$. *Other roots coincide with inflection points of this polynomial.*

This statement is an obvious corollary from (13) and (14) and from Theorem 1.

1.3. Orthogonality property.

Theorem 3. *The functions* $Q_n(x)$ *and* $Q_m(x)$ $(n \neq m)$ *are orthogonal with respect to the weight function* $w(x) = 1/(1-x^2)$*:*

$$\int_{-1}^1 \frac{Q_n(x)Q_m(x)}{1-x^2} dx = 0 \qquad (n \neq m) \tag{16}$$

and

$$||Q_n||^2 = \int_{-1}^1 \frac{(Q_n(x))^2}{1-x^2} dx = \frac{2}{n(n-1)(2n-1)}. \tag{17}$$

Since $w(x)$ *is not continuous on* $[-1, 1]$, PIPCIRs *do not belong to classical orthogonal polynomials, but because* $Q_n(-1) = Q_n(1) = 0$, *all integrals* (16), (17) *are proper.*

The functions $q_n(x)$ and $q_m(x)$ $(n \neq m)$ are orthogonal with respect to the weight function $(1-x^2)$ and

$$||q_n||^2 = \int_{-1}^{1} (q_n(x))^2(1-x^2)dx = \frac{2}{(n+2)(n+1)(2n+3)}. \tag{18}$$

These statements are immediate corollary from orthogonality of Legendre polynomials and from the formula

$$\int_{-1}^{1} [L_n(x)]^2 dx = \frac{2}{2n+1}.$$

1.4. Generating functions.

It is known that the function $W(h,x) = (1-2xh+h^2)^{-1/2}$ is the generating function for the Legendre polynomials; that is

$$W(h,x) = (1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} L_n(x)h^n,$$

and this series converges for $|h| < 1$ when $|x| \leq 1$.

The PIPCIRs are integrals of the Legendre polynomials, but the integral of $L_0(x) = 1$ is not included into the family of PIPCIRs. Therefore we denote as

$$U(h,x) = -h\int_{x}^{1} (W(h,t)-1)dt = 1 - xh - \sqrt{1-2xh+h^2}$$

and it can be shown in a standard way that $U(h,x)$ is the generating function for the PIPCIRs:

$$U(h,x) = 1 - xh - \sqrt{1-2xh+h^2} = \sum_{n=2}^{\infty} Q_n(x)h^n. \tag{19}$$

The function $U(h,x)$ satisfies the equation

$$h^2\frac{\partial^2 U}{\partial h^2} + (1-x^2)\frac{\partial^2 U}{\partial x^2} = 0 \tag{20}$$

that can be verified by direct substitution.

The generating function for the family of polynomials $q_n(x)$ is

$$V(h,x) = \frac{U(h,x)}{h^2(1-x^2)} = \frac{1}{h^2(1-x^2)}(1 - xh - \sqrt{1-2xh+h^2})$$
$$= -\frac{1}{1-xh+\sqrt{1-2xh+h^2}} = \sum_{n=0}^{\infty} q_n(x)h^n. \tag{21}$$

The function $V(h,x)$ satisfies the equation

$$\frac{1}{h^2}\frac{\partial}{\partial h}\left(h^4\frac{\partial V}{\partial h}\right)+\frac{1}{1-x^2}\frac{\partial}{\partial x}\left((1-x^2)^2\frac{\partial V}{\partial x}\right)=0$$

that can be verified by direct substitution.

In [2, p. 68], we can find the generating function for Jacobi polynomials $P_n^{(1,1)}(x)$:

$$\sum_{n=0}^{\infty}P_n^{(1,1)}(x)h^n=\frac{4}{\sqrt{1-2xh+h^2}(1-h+\sqrt{1-2xh+h^2})^2}$$

or, p. 82, for ultraspherical polynomials $P_n^{(3/2)}(x)=[(n+2)/2]P_n^{(1,1)}(x)$:

$$\sum_{n=0}^{\infty}P_n^{(3/2)}(x)h^n=\frac{1}{(1-2xh+h^2)^{3/2}}.$$

Using (21) and (10), we can write:

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{n+1}P_n^{(1,1)}(x)h^n=\frac{1}{1-xh+\sqrt{1-2xh+h^2}},$$

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}P_n^{(3/2)}(x)h^n=\frac{1}{1-xh+\sqrt{1-2xh+h^2}}.$$

1.5. Estimation of the functions $Q_n(x)$ and $q_n(x)$.

If we take $x=\cos\theta=(e^{i\theta}+e^{-i\theta})/2$, where $i^2=-1$, we can get in a standard way, using (19), for odd n

$$Q_n(\cos\theta)=\frac{2(2n-3)!!}{(2n)!!}\cos n\theta-2\sum_{k=1}^{(n-1)/2}\frac{(2n-2k-3)!!(2k-3)!!}{(2n-2k)!!(2k)!!}\cos(n-2k)\theta,$$

and for even n:

$$Q_n(\cos\theta)=\frac{2(2n-3)!!}{(2n)!!}\cos n\theta-2\sum_{k=1}^{n/2-1}\frac{(2n-2k-3)!!(2k-3)!!}{(2n-2k)!!(2k)!!}\cos(n-2k)\theta-\left(\frac{(n-3)!!}{n!!}\right)^2.$$

Using this, we can show that for $-1\le x\le 1$ the following estimations hold:

(a) $|Q_n(x)|<\dfrac{4(2n-3)!!}{(2n)!!}$;

(b) $|Q_n(x)|\le|Q_n(0)|=\dfrac{(n-3)!!}{n!!}$ for even n ;

(c) $|q_n(x)|\le\dfrac{1}{2}$.

1.6. Asymptotic property.

The asymptotic behavior of polynomials $P_n^{(\alpha,\beta)}$ for $\alpha > -1/2$, $\alpha - \beta > -2m$ and $\alpha + \beta \geq -1$ is described in [3]. We shall use it for $\alpha = \beta = 1$:

$$P_n^{(1,1)}(\cos\theta) = (n+1)\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^{-1}\left(\frac{\theta}{\sin\theta}\right)^{1/2} \times \left[\sum_{k=0}^{m-1} A_k(\theta)\frac{J_{k+1}((n+3/2)\theta)}{(n+3/2)^{k+1}} + \theta\, O((n+3/2)^{-m})\right]$$

where $J_k(x)$ is the Bessel function of the first kind of order k, the coefficients $A_k(\theta)$ are analytic functions for $0 \leq \theta < \pi$. The O-term is uniform with respect to $0 \leq \theta \leq \pi - \varepsilon$, where ε is an arbitrary positive number.

As a corollary, this gives

$$P_n^{(1,1)}(\cos\theta) = 2(n+1)(\sin\theta)^{-1}\left(\frac{\theta}{\sin\theta}\right)^{1/2} \times \left[\frac{J_1((n+3/2)\theta)}{n+3/2} + A_1(\theta)\frac{J_2((n+3/2)\theta)}{(n+3/2)^2} + \sigma_2\right],$$

where

$$A_1(\theta) = \frac{3(1-\theta\cot\theta)}{8\theta} \quad \text{and} \quad |\sigma_2| \leq \frac{E}{n+3/2}\theta^3, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

and E is constant.

Rewrite the leading term in terms of $x = \cos\theta$

$$P_n^{(1,1)}(x) = \frac{2(n+1)}{\sqrt{1-x^2}}\left(\frac{\arccos x}{\sqrt{1-x^2}}\right)^{1/2} \times \left[\sum_{k=0}^{m-1} A_k(\arccos x)\frac{J_{k+1}((n+3/2)\arccos x)}{(n+3/2)^{k+1}} + \arccos x\, O((n+3/2)^{-m})\right].$$

Using (10), we obtain:

$$q_n(x) = -\frac{\sqrt{\arccos x}}{(1-x^2)^{3/4}}\left[\sum_{k=0}^{m-1} A_k(\arccos x)\frac{J_{k+1}((n+3/2)\arccos x)}{(n+3/2)^{k+1}} + \arccos x\, O((n+3/2)^{-m})\right]$$

and for PIPCIR we have:

$$Q_{n+2}(x) = -(1-x^2)^{1/4}\sqrt{\arccos x}\Bigg[\sum_{k=0}^{m-1} A_k(\arccos x)\frac{J_{k+1}((n+3/2)\arccos x)}{(n+3/2)^{k+1}} + \arccos x\, O((n+3/2)^{-m})\Bigg]$$

or,

$$Q_n(x) = -(1-x^2)^{1/4}\sqrt{\arccos x}\Bigg[\sum_{k=0}^{m-1} A_k(\arccos x)\frac{J_{k+1}((n-1/2)\arccos x)}{(n-1/2)^{k+1}} + \arccos x\, O((n-1/2)^{-m})\Bigg].$$

Corollary.

$$Q_n(x) = -(1-x^2)^{1/4}\sqrt{\arccos x}\Bigg[\frac{J_1((n-1/2)\arccos x)}{n-1/2} + A_1(\theta)\frac{J_2((n-1/2)\theta)}{(n-1/2)^2} + \sigma_2\Bigg]$$

where

$$A_1(\arccos x) = \frac{3(1-\sqrt{1+x^2}\arccos x)}{8\arccos x}$$

$$\text{and} \quad |\sigma_2| \le \frac{E}{n-1/2}(\arccos x)^3, \qquad 0 \le x \le 1.$$

2. Part II. Applications of PIPCIRs

The set of PIPCIRs is a family of orthogonal polynomials with respect to weight $1/(1-x^2)$ (see (16), (17)).

Theorem 4. *If $f(x)$ is continuous on the interval $I: -1 \le x \le 1$, its derivative is piecewise continuous, the curve $y = f'(x)$ is rectifiable, and $f(-1) = f(1) = 0$, then there exists a series of* PIPCIRs *with constant coefficients*

$$B_2Q_2(x) + \ldots + B_nQ_n(x) + \ldots,$$

where

$$B_n = \frac{n(n-1)(2n-1)}{2}\int_{-1}^{1}\frac{f(x)Q_n(x)}{1-x^2}dx = \frac{n(n-1)(2n-1)}{2}\int_{-1}^{1} f(x)q_{n-2}(x)dx \tag{22}$$

which:

(a) *converges everywhere on I,*
(b) *converges to $f(x)$ at each point on I,*
(c) *is such that the series after multiplication by an arbitrary $Q_k(x)$ is termwise integrable on I and converges to the integral of $f(x)Q_k(x)$.*

Proof. This theorem is corollary from the similar statement about Legendre polynomials. We differentiate the given function $f(x)$ and find the series of Legendre polynomials that converges to $f'(x)$:

$$f'(x) = A_0L_0(x) + A_1L_1(x) + \ldots + A_nL_n(x) + \ldots$$

where

$$A_n = \frac{2n+1}{2}\int_{-1}^{1} f'(x)L_n(x)dx.$$

Then we integrate this series termwise from x to 1 and use (4):

$$-\int_x^1 f'(t)dt = f(x) = A_0(x-1) + A_1Q_2(x) + \ldots + A_nQ_{n+1}(x) + \ldots .$$

Since $f(-1) = f(1) = 0$ and $Q_n(-1) = Q_n(1) = 0$ for $n \geq 2$, we must have $A_0 = 0$. The coefficients A_n may be evaluated in terms of polynomials Q_n (use (2)):

$$\begin{aligned} A_n &= \frac{2n+1}{2}\int_{-1}^{1} f'(x)L_n(x)dx = \frac{2n+1}{2}\left(f(x)L_n(x)\Big|_{-1}^{1} - \int_{-1}^{1} f(x)L_n'(x)dx \right) \\ &= -\frac{2n+1}{2}\int_{-1}^{1} f(x)Q''_{n+1}(x)dx = \frac{(2n+1)n(n+1)}{2}\int_{-1}^{1} \frac{f(x)Q_{n+1}(x)}{1-x^2}dx \end{aligned}$$

and we set for $n \geq 2$:

$$\begin{aligned} B_n = A_{n-1} &= \frac{n(n-1)(2n-1)}{2}\int_{-1}^{1} \frac{f(x)Q_n(x)}{1-x^2}\,dx \\ &= \frac{n(n-1)(2n-1)}{2}\int_{-1}^{1} f(x)q_{n-2}(x)dx. \end{aligned}$$

The last integral shows that we do not have an improper integral. □

Requirement $f(-1) = f(1) = 0$ makes certain restrictions for application of this series. But this obstacle can be overcome. For an arbitrary continuous function $f(x)$ we define

$$g(x) = f(x) - \frac{f(1)+f(-1)}{2} - \frac{f(1)-f(-1)}{2}x.$$

Since $g(1) = g(-1) = 0$, we can find a representation

$$g(x) = \sum_{n=2}^{\infty} B_n Q_n(x)$$

where

$$B_n = \frac{n(n-1)(2n-1)}{2} \int_{-1}^{1} g(x) q_{n-2}(x) dx.$$

Thus

$$f(x) = \frac{f(1) + f(-1)}{2} + \frac{f(1) - f(-1)}{2} x + \sum_{n=2}^{\infty} B_n Q_n(x).$$

Examples.

$$f_1(x) = \begin{cases} x+2 & \text{if } -1 \le x \le 0 \\ 2-x & \text{if } 0 \le x \le 1 \end{cases} \approx 1 - 1.5Q_2(x) + 0.875Q_4(x)$$

(see Fig. 1)

$$f_2(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0 \\ x & \text{if } 0 \le x \le 1 \end{cases} \approx \frac{1+x}{2} + 0.75Q_2(x) - 0.4375Q_4(x)$$

(see Fig. 2)

$$f_3(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 0 \\ 1-x^2 & \text{if } 0 \le x \le 1 \end{cases} \approx -1.75Q_2(x) - 0.625Q_3(x) + 0.4375Q_4(x)$$

(see Fig. 3)

$$\sin(\pi x) \approx -4.7765Q_3(x) + 1.82981Q_5(x)$$ (see Fig. 4)

$$\cos(\pi x) \approx -1 - 3Q_2(x) + 3.63872Q_4(x)$$ (see Fig. 5)

$$e^x \approx \cosh(1) + x\sinh(1) + 1.10364Q_2(x) + 0.357814Q_3(x)$$ (see Fig. 6).

Figures 4, 5, 6 show graphs of PIPCIR expansion in comparison with Taylor polynomial expansion of the same degree.

The set of PIPCIRs is a family of orthogonal polynomials with respect to weight $1/(1-x^2)$ (see (16), (17)). Sometimes it is convenient to normalize polynomials, so that its norm would be 1. Then we shall have family of orthonormal polynomials:

$$\widehat{Q}_n(x) = \sqrt{\frac{n(n-1)(2n-1)}{2}} Q_n(x), \tag{23}$$

$$||\widehat{Q}_n|| = \int_{-1}^{1} \frac{(\widehat{Q}_n(x))^2}{1-x^2} dx = 1.$$

Theorem 5. *Let $f(x)$ be continuous on the interval $I = [-1, 1]$, $f(-1) = f(1) = 0$, and its expansion in series of normalized* PIPCIRs

$$f(x) = \sum_{n=2}^{\infty} \widehat{B}_n \widehat{Q}_n(x) \tag{24}$$

converges uniformly on I. Then

$$\int_{-1}^{1} \frac{(f(x))^2}{1-x^2} dx = \sum_{n=2}^{\infty} \widehat{B}_n^2.$$

Proof. Multiply the series (24) by $f(x)/(1-x^2)$ and integrate over I:

$$\int_{-1}^{1} \frac{(f(x))^2}{1-x^2} dx = \sum_{n=2}^{\infty} \widehat{B}_n \int_{-1}^{1} \frac{f(x)\widehat{Q}_n(x)}{1-x^2} dx = \sum_{n=2}^{\infty} \widehat{B}_n^2.$$

□

2.1. Intervals different from $[-1, 1]$.

If a function $f(x)$ is continuous on the interval $[a, b]$, we may represent it as a sum of series of PIPCIRs by substitution a new variable $x = (1/2)[(b-a)t + b + a]$. When x varies from a to b, we have t varying from -1 to 1. As a result, we shall have:

$$f(x) = \sum_{n=2}^{\infty} B_n Q_n \left(\frac{2x - b - a}{b - a} \right),$$

where

$$\begin{aligned} B_n =& \frac{n(n-1)(2n-1)}{2} \int_{-1}^{1} f\left(\frac{(b-a)t + b + a}{2} \right) q_{n-2}(t) dt \\ =& \frac{n(n-1)(2n-1)}{b-a} \int_{a}^{b} f(x) q_{n-2} \left(\frac{2x - b - a}{b - a} \right) dx. \end{aligned}$$

If an interval is $[-b, b]$ then

$$f(x) = \sum_{n=0}^{\infty} B_n Q_n \left(\frac{x}{b} \right), \qquad B_n = \frac{n(n-1)(2n-1)}{2b} \int_{-b}^{b} f(x) q_{n-2} \left(\frac{x}{b} \right) dx.$$

If a function is continuous on the interval $[0, 1]$ (or $[0, b]$), we can extend it on the interval $[-1, 0]$ (or $[-b, 0]$), making it even. Then we can find series

$$\begin{aligned} f(x) =& \sum_{k=0}^{\infty} B_{2k} Q_{2k}(x) \qquad \text{where} \\ B_{2k} =& 2k(2k-1)(4k-1) \int_{0}^{1} f(x) q_{2k-2}(x) dx \end{aligned}$$

(make corresponding correction for the interval $[0, b]$).

If $f(0) = 0$, we have choice to extend the function $f(x)$ on the interval $[-1, 0]$ by making an extension odd or even (the extended function must be continuous). If we choose the odd extension then

$$f(x) = \sum_{k=0}^{\infty} B_{2k+1} Q_{2k+1}(x) \qquad \text{where}$$

$$B_{2k+1} = 2k(2k+1)(4k+1) \int_0^1 f(x) q_{2k-1}(x) dx.$$

2.2. The distance formula and corollary.

Let R denote the distance from the origin O of the fixed point M_0. Let r be the spherical coordinate denoting the distance from the origin of a variable point M. Let D be the distance M_0M. We shall show that

$$D = R - r\cos\theta - R\sum_{n=2}^{\infty} \left(\frac{r}{R}\right)^n Q_n(\cos\theta) \qquad \text{for } r < R$$

and

$$D = r - R\cos\theta - r\sum_{n=2}^{\infty} \left(\frac{R}{r}\right)^n Q_n(\cos\theta) \qquad \text{for } r > R,$$

where $x = \cos\theta$, and θ being the angle of intersection of vectors $\boldsymbol{OM_0}$ and $\boldsymbol{OM}$, and $Q_n(x)$ is the PIPCIR of degree n.

From the triangle OM_0M we can find (using Law of Cosines):

$$D = \sqrt{r^2 + R^2 - 2Rr\cos\theta}. \tag{25}$$

Using the generating function and its series representation (19),

$$U(h, x) = 1 - xh - \sqrt{1 - 2xh + h^2} = \sum_{n=2}^{\infty} h^n Q_n(x),$$

we shall get, setting $h = r/R$ for $r < R$ and $x = \cos\theta$:

$$\begin{aligned} U\left(\frac{r}{R}, \cos\theta\right) &= 1 - \frac{r}{R}\cos\theta - \sqrt{1 - 2\frac{r}{R}\cos\theta + \frac{r^2}{R^2}} \\ &= 1 - \frac{r}{R}\cos\theta - \frac{D}{R} = \sum_{n=2}^{\infty} \left(\frac{r}{R}\right)^n Q_n(\cos\theta). \end{aligned}$$

Hence we obtain:

$$D = R - r\cos\theta - R\sum_{n=2}^{\infty} \left(\frac{r}{R}\right)^n Q_n(\cos\theta) \qquad \text{for } r < R.$$

If $r > R$, we set $h = R/r$ and obtain the second equality.

This formula is of special interest, if we recall that reciprocal of distance D is represented as a series of Legendre polynomials ([1, p. 207]):

$$\frac{1}{D} = \frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n L_n(\cos\theta) \qquad \text{for } r < R,$$

and

$$\frac{1}{D} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{R}{r}\right)^n L_n(\cos\theta) \qquad \text{for } r > R.$$

Using this, we may find an interesting connection between PIPCIRs and Legendre polynomials:

$$\left(R - r\cos\theta - R\sum_{n=2}^{\infty}\left(\frac{r}{R}\right)^n Q_n(\cos\theta)\right)\left(\frac{1}{R}\sum_{n=0}^{\infty}\left(\frac{r}{R}\right)^n L_n(\cos\theta)\right) = 1.$$

If we temporary set: $Q_1(x) = x$, $\cos\theta = x$, $r/R = t$, then

$$R\left(1 - \sum_{n=1}^{\infty} t^n Q_n(x)\right)\left(\frac{1}{R}\sum_{n=0}^{\infty} t^n L_n(x)\right) = 1.$$

Multiply series and equate coefficients of t^n $(n > 0)$ to 0:

$$\sum_{k=1}^{n} Q_k(x) L_{n-k}(x) = L_n(x).$$

Here are some particular cases:

$$\begin{aligned}
L_1(x) &= Q_1(x) = x\\
L_2(x) &= Q_1(x)L_1(x) + Q_2(x)\\
L_3(x) &= Q_1(x)L_2(x) + Q_2(x)L_1(x) + Q_3(x)\\
L_4(x) &= Q_1(x)L_3(x) + Q_2(x)L_2(x) + Q_3(x)L_3(x) + Q_4(x).
\end{aligned}$$

Using this system of equations, we may express Legendre polynomials as a sum of products of PIPCIRs, or PIPCIRs as a sum of products of Legendre polynomials.

For a small value of n we can do it immediately:

$$\begin{aligned}
L_1(x) &= Q_1(x)\\
L_2(x) &= Q_1^2(x) + Q_2(x)\\
L_3(x) &= Q_1^3(x) + 2Q_1(x)Q_2(x) + Q_3(x)\\
L_4(x) &= Q_1^4(x) + 3Q_1^2(x)Q_2(x) + 2Q_1(x)Q_3(x) + Q_2^2(x) + Q_4(x)
\end{aligned}$$

or

$$\begin{aligned}
Q_1(x) &= L_1(x)\\
Q_2(x) &= L_2(x) - L_1^2(x)\\
Q_3(x) &= L_3(x) - 2L_1(x)L_2(x) + L_1^3(x)\\
Q_4(x) &= L_4(x) - 2L_1(x)L_3(x) + 3L_1^2(x)L_2(x) - L_2^2(x) - L_1^4(x)
\end{aligned}$$

and so on.

Recall the relation between a PIPCIR and a Jacobi polynomial:

$$Q_n(x) = (1-x^2)q_{n-2}(x) = -\frac{(1-x^2)}{2(n-1)}P_{n-2}^{(1,1)}(x).$$

Using this, we can obtain new formulas for Jacobi polynomials:

$$\frac{1-x^2}{2}\sum_{k=0}^{n}\frac{1}{k+1}P_k^{(1,1)}(x)L_{n-k}(x) = xL_{n+1}(x) - L_{n+2}(x),$$

and

$$\begin{aligned}
P_0^{(1,1)}(x) &= \frac{-2}{1-x^2}\left(L_2(x) - L_1^2(x)\right)\\
P_1^{(1,1)}(x) &= \frac{-4}{1-x^2}\left(L_3(x) - 2L_1(x)L_2(x) + L_1^3(x)\right)\\
P_2^{(1,1)}(x) &= \frac{-6}{1-x^2}\left(L_4(x) - 2L_1(x)L_3(x) + 3L_1^2(x)L_2(x) - L_2^2(x) - L_1^4(x)\right),
\end{aligned}$$

and so on.

2.3. Solution of some partial differential equations.

Recall that PIPCIRs are solutions of the equation (2). If we set $x = \cos\theta$ then these polynomials will satisfy equations

$$\frac{d^2Q_n}{d\theta^2} - \cot\theta\frac{dQ_n}{d\theta} + n(n-1)Q_n = 0. \tag{26}$$

Theorem 6. *The equations*

(a) $(1-x^2)y'' - c^2y = 0, \qquad y(-1) = y(1) = 0$

(b) $\dfrac{d^2y}{d\theta^2} - \cot\theta\dfrac{dy}{d\theta} - c^2y = 0, \qquad y(0) = y(\pi) = 0$

have only trivial solution on the interval $[-1,1]$ *for any* c.

The proof is standard.

If we change the conditions to $y(0) = y(1) = 0$ or $y'(0) = y(1) = 0$, the conclusion will be the same.

Example 1. Consider the equation

$$(1-x^2)\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \quad 0 \le x \le 1, \quad t \ge 0, \tag{27}$$

with initial condition $w(x,0) = f(x)$ for all $0 \le x \le 1$ and boundary conditions $w(0,t) = 0$, $w(1,t) = 0$ for $t \ge 0$. This requires that $f(0) = 0$, $f(1) = 0$.

We shall seek a solution of this problem by separation of variables. First we shall find a solution of (27) of the form $w(x,t) = F(x)G(t)$. Standard operations will give the following equations for F and G:

$$(1-x^2)F'' + \lambda F = 0 \quad \text{and} \quad G' + \lambda k G = 0. \tag{28}$$

We rewrite boundary conditions $w(0,t) = 0$, $w(1,t) = 0$ in terms of the functions F and G: $F(0)G(t) = 0$ and $F(1)G(t) = 0$ for all t. This means that $F(0) = 0$ and $F(1) = 0$.

Since $f(0) = 0$, we may extend the function $f(x)$ to entire interval $[-1,1]$ making it odd and the extended function is continuous. Then we shall look for solution using PIPCIRs of odd order only. As it was stated in Theorem 6, the first of the equations (28) has a nontrivial solution only if $\lambda > 0$. If we set $\lambda = 2n(2n+1)$, $n > 0$, we find a solution of this equation $F(x) = Q_{2n+1}(x)$. After that we determine a particular solution of the second equation (28) with $\lambda = 2n(2n+1)$:

$$G(t) = e^{-2n(2n+1)kt}.$$

Let

$$w(x,t) = \sum_{n=1}^{\infty} B_{2n+1} Q_{2n+1}(x) e^{-2n(2n+1)kt}. \tag{29}$$

For any coefficients B_{2n+1} this function satisfies the equation (27) and boundary condition $w(0,t) = 0$, $w(1,t) = 0$. For $t = 0$ we must have

$$f(x) = \sum_{n=1}^{\infty} B_{2n+1} Q_{2n+1}(x),$$

and we find coefficients B_{2n+1}, using formulas (22):

$$\begin{aligned} B_{2n+1} =& 2n(2n+1)(4n+1)\int_0^1 \frac{f(x)Q_{2n+1}(x)}{1-x^2}dx \\ =& 2n(2n+1)(4n+1)\int_0^1 f(x)q_{2n-1}(x). \end{aligned}$$

The function (29) with defined coefficients B_n is the solution of the problem (27).

Example 2. If we have the equation

$$(1-x^2)\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial w}{\partial t}, \quad 0 \le x \le 1, \quad t \ge 0, \tag{30}$$

with initial condition $w(x,0) = f(x)$ and boundary conditions $\frac{\partial w}{\partial x}(0,t) = 0$, $w(1,t) = 0$ (that requires $f(1) = 0$), then after the same procedure, we may extend the function $f(x)$ from the interval $[0,1]$ to the interval $[-1,1]$ making it even. Since a function $Q_n(x)$ for even n satisfies both boundary conditions $Q_n(1) = 0$ and $\frac{\partial Q_n}{\partial x}(0) = 0$, the function

$$w(x,t) = \sum_{n=1}^{\infty} B_{2n} Q_{2n}(x) e^{-2n(2n-1)kt} \tag{31}$$

satisfies this conditions for any coefficients B_{2n}. For $t = 0$ we must have

$$f(x) = \sum_{n=1}^{\infty} B_{2n} Q_{2n}(x)$$

and we find coefficients B_{2n} using formulas (22):

$$\begin{aligned} B_{2n} =& 2n(2n-1)(4n-1)\int_0^1 \frac{f(x)Q_{2n}(x)}{1-x^2}dx \\ =& 2n(2n-1)(4n-1)\int_0^1 f(x)q_{2n-2}(x)dx. \end{aligned}$$

The function (31) with defined coefficients B_n is a solution of the problem (30).

Example 3. Consider the equation

$$(1-x^2)\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial^2 w}{\partial t^2}, \quad k > 0, \quad 0 \le x \le 1, \quad t \ge 0, \tag{32}$$

with initial conditions $w(x,0) = f(x)$, $\frac{\partial w}{\partial t}(x,0) = g(x)$ for all $0 \le x \le 1$ and boundary conditions $w(0,t) = 0$, $w(1,t) = 0$ for $t \ge 0$. This requires that $f(0) = 0$, $g(0) = 0$, $f(1) = 0$, $g(1) = 0$.

As in Example 1, we shall find first a solution of (32) in the form $w(x,t) = F(x)G(t)$, separate variables, and we shall get two equations:

$$(1-x^2)F'' + \lambda F = 0 \quad \text{and} \quad G'' + \lambda k G = 0. \tag{33}$$

Since $f(0) = 0$ and $g(0) = 0$, we may extend the functions $f(x)$ and $g(x)$ to entire interval $[-1,1]$ making them odd and the extended functions are continuous. Then we shall look for a solution using PIPCIRs of odd order only. As it was stated in Theorem 6, the first of the equations (33) has nontrivial solution only if $\lambda > 0$. If we set $\lambda = 2n(2n+1)$, $n > 0$, we find a solution of this equation $F(x) = Q_{2n+1}(x)$. After that we determine a particular solution of the second equation (33) with $\lambda = 2n(2n+1)$:

$$\begin{aligned} G_n(t) =& A_n \sin(\sqrt{\lambda k}\, t) + B_n \cos(\sqrt{\lambda k}\, t) \\ =& A_n \sin(\sqrt{2n(2n+1)k}\, t) + B_n \cos(\sqrt{2n(2n+1)k}\, t). \end{aligned}$$

Let

$$\begin{aligned} w(x,t) = \sum_{n=1}^{\infty} Q_{2n+1}(x) \Big[& A_{2n+1} \sin(\sqrt{\lambda k}\, t) \\ & + B_{2n+1} \cos(\sqrt{\lambda k}\, t) \Big]. \end{aligned} \tag{34}$$

Then

$$\frac{\partial w}{\partial t}(x,t) = \sum_{n=1}^{\infty} Q_{2n+1}(x) \left[A_{2n+1}\sqrt{\lambda k} \cos(\sqrt{\lambda k}\, t) - B_{2n+1}\sqrt{\lambda k} \sin(\sqrt{\lambda k}\, t) \right].$$

For any coefficients A_{2n+1}, B_{2n+1} the function $w(x,t)$ satisfies the equation (32) and boundary condition $w(0,t) = 0$, $w(1,t) = 0$. For $t = 0$ we must have

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} B_{2n+1} Q_{2n+1}(x), \\ g(x) &= \sum_{n=1}^{\infty} A_{2n+1} \sqrt{\lambda k}\, Q_{2n+1}(x), \end{aligned}$$

and we find coefficients B_n and A_n using formulas (22):

$$\begin{aligned} B_{2n+1} &= 2n(2n+1)(4n+1) \int_0^1 \frac{f(x) Q_{2n+1}(x)}{1-x^2} dx \\ &= 2n(2n+1)(4n+1) \int_0^1 f(x) q_{2n-1}(x) \\ A_{2n+1} &= \frac{2n(2n+1)(4n+1)}{\sqrt{\lambda k}} \int_0^1 \frac{g(x) Q_{2n+1}(x)}{1-x^2} dx \\ &= \frac{2n(2n+1)(4n+1)}{\sqrt{\lambda k}} \int_0^1 g(x) q_{2n-1}(x). \end{aligned}$$

The function (34) with defined coefficients A_n, B_n is the solution of the problem (32).

If we have the equation

$$(1-x^2)\frac{\partial^2 w}{\partial x^2} = \frac{1}{k}\frac{\partial^2 w}{\partial t^2}, \quad 0 \le x \le 1, \quad t \ge 0,$$

with initial conditions $w(x,0) = f(x)$, $\dfrac{\partial w}{\partial t}(x,0) = g(x)$ for all $0 \le x \le 1$, and boundary conditions $\dfrac{\partial w}{\partial x}(0,t) = 0$, $w(1,t) = 0$, then after the same procedure, we may extend the function $f(x)$ from the interval $[0,1]$ to the interval $[-1,1]$ making it even, and we shall find the answer in the same way as in Example 2.

Example 4. Let $w(r,\varphi,\theta)$ be a function defined on spherical solid of radius R with center at the origin, (r,φ,θ) are spherical coordinates of a point where r is the distance from origin, θ is colatitude from the positive z-axis (cone angle), φ is the angle of sweep about the z-axis. We assume that the function w depends only on two variables, r and θ, and satisfies the partial differential equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial w}{\partial\theta}\right) = 0 \tag{35}$$

and boundary condition $w(R,\theta) = f(\theta)$, where $f(\theta)$, $0 \le \theta \le \pi$, is continuous function. We shall seek a solution of this problem by separation of variables. First we shall find a solution of (35) of the form $w(r,\theta) = F(r)G(\theta)$, and standard operations will give the following equations for F and G:

$$\sin\theta\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{dG}{d\theta}\right) + \lambda G = 0, \qquad \frac{d}{dr}\left(r^2\frac{dF}{dr}\right) - \lambda F = 0.$$

These equations can be rewritten as

$$\frac{d^2G}{d\theta^2} - \cot\theta\frac{dG}{d\theta} + \lambda G = 0, \qquad r^2F'' + 2rF' - \lambda F = 0. \tag{36}$$

By Theorem 6, we conclude that λ must be positive, and we set $\lambda = n(n-1)$. The PIPCIR $Q_n(\cos\theta)$ is a particular solution of the first equation (36). A general solution of the second equation (36) is equal to the function

$$F(r) = A_n r^n + \frac{B_n}{r^{n+1}}, \tag{37}$$

where A_n and B_n are arbitrary constants. The second term on the right in equation (37) becomes infinite at $r = 0$ and is thus unsuitable. Hence we let $B_n = 0$.

Now we can construct the function

$$w_n(r,\theta) = F(r)G(\theta) = \frac{A_n}{R^n}r^nQ_n(\cos\theta)$$

and then

$$w(r,\theta) = \sum_{n=0}^{\infty} A_n\left(\frac{r}{R}\right)^n Q_n(\cos\theta).$$

This function satisfies the equation (35) and $w(r,0) = w(r,\pi) = 0$ for any coefficients A_n. If the function $f(\theta)$ does not satisfy the condition $f(0) = f(\pi) = 0$, then we set

$$g(\theta) = f(\theta) - \frac{f(0)+f(\pi)}{2} - \frac{f(0)-f(\pi)}{2}\cos\theta.$$

Since $g(0) = g(\pi) = 0$, we can find a representation

$$g(\theta) = \sum_{n=2}^{\infty} A_n Q_n(\cos\theta)$$

where

$$A_n = \frac{n(n-1)(2n-1)}{2} \int_0^{\pi} g(\theta) q_{n-2}(\cos\theta)\sin\theta \, d\theta.$$

Thus

$$f(\theta) = \frac{f(0)+f(\pi)}{2} + \frac{f(0)-f(\pi)}{2}\cos\theta + \sum_{n=2}^{\infty} A_n Q_n(\cos\theta).$$

The function

$$\widehat{w}(x,t) = \frac{f(0)+f(\pi)}{2} + \frac{f(0)-f(\pi)}{2}\cos\theta + \sum_{n=0}^{\infty} A_n \left(\frac{r}{R}\right)^n Q_n(\cos\theta)$$

with defined coefficients A_n is a solution of the problem (35).

Note. The function $S(r,\theta) = D/r$, where D is a distance (25), is one of the particular solutions of the equation (35) with the boundary condition $f(\theta) = 2\sin(\theta/2)$.

Example 5. In the situation described in Example 4 we consider spherical shell solid with the radius of the inner surface R_1 and the radius of the outer surface R_2, $R_1 < R_2$. The common center of these surfaces is at the origin. We shall determine the function $w(R,\theta)$ that satisfies the equation (35) and given boundary conditions

$$w(R_1,\theta) = f_1(\theta), \qquad w(R_2,\theta) = f_2(\theta),$$

where $f_1(\theta)$ and $f_1(\theta)$ are continuous functions, $0 \le \theta \le \pi$. As in Example 4, we find the equations (36) and the function (37), but now we cannot reject the second term of it. So, we construct a function

$$w_n(r,\theta) = F(r)G(\theta) = \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) Q_n(\cos\theta)$$

and

$$w(r,\theta) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) Q_n(\cos\theta).$$

For $r = R_1$ and $r = R_2$ we must have:

$$w(R_1,\theta) = f_1(\theta) = \frac{f_1(0)+f_1(\pi)}{2} + \frac{f_1(0)-f_1(\pi)}{2}\cos\theta + \sum_{n=0}^{\infty}\left(A_n R_1^n + \frac{B_n}{R_1^{n+1}}\right)Q_n(\cos\theta)$$

$$w(R_2,\theta) = f_2(\theta) = \frac{f_2(0)+f_2(\pi)}{2} + \frac{f_2(0)-f_2(\pi)}{2}\cos\theta + \sum_{n=0}^{\infty}\left(A_n R_2^n + \frac{B_n}{R_2^{n+1}}\right)Q_n(\cos\theta)$$

Hence, if

$$g_1(\theta) = f_1(\theta) - \frac{f_1(0)+f_1(\pi)}{2} - \frac{f_1(0)-f_1(\pi)}{2}\cos\theta$$

and

$$g_2(\theta) = f_2(\theta) - \frac{f_2(0)+f_2(\pi)}{2} - \frac{f_2(0)-f_2(\pi)}{2}\cos\theta,$$

then

$$A_n R_1^n + \frac{B_n}{R_1^{n+1}} = \frac{n(n-1)(2n-1)}{2}\int_0^{\pi} g_1(\theta)q_{n-2}(cos\theta)\sin\theta\, d\theta,$$

$$A_n R_2^n + \frac{B_n}{R_2^{n+1}} = \frac{n(n-1)(2n-1)}{2}\int_0^{\pi} g_2(\theta)q_{n-2}(\cos\theta)\sin\theta\, d\theta.$$

To define coefficients A_n and B_n, we have to solve the linear system of two equations. It has the unique solution for $R_1 \neq R_2$.

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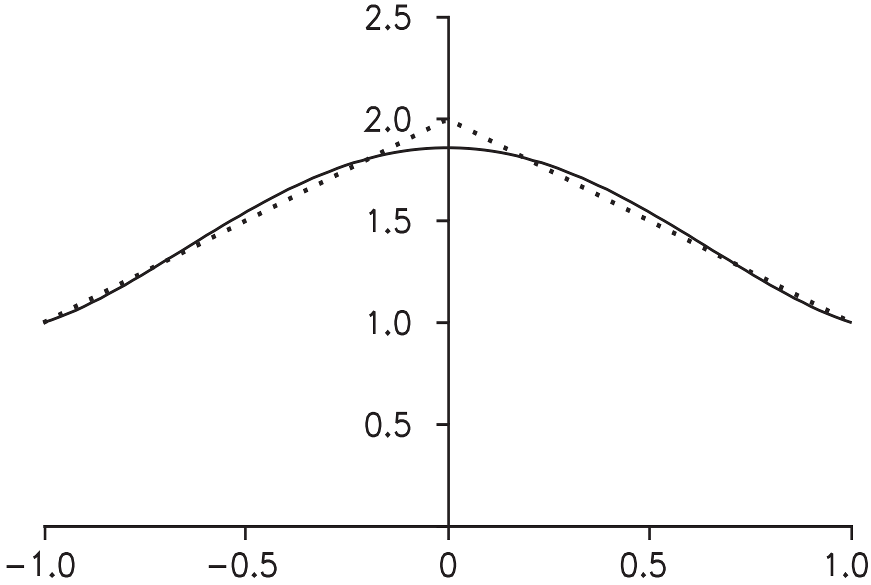

Fig. 1. Graphs of $f_1(x)$ and $s_1(x) = 1 - 1.5Q_2(x) + 0.875Q_4(x)$

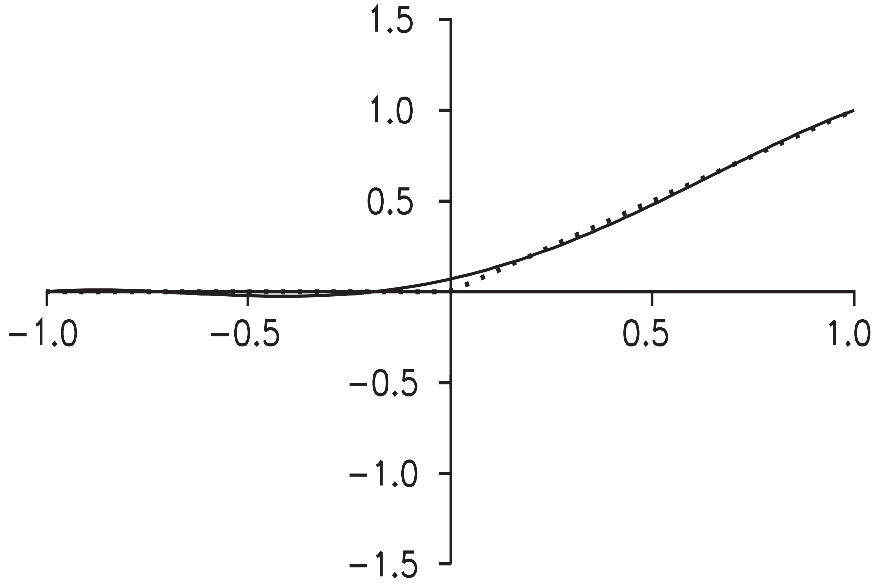

Fig. 2. Graphs of $f_2(x)$ and $s_2(x) = (1 + x)/2 + 0.75Q_2(x) - 0.4375Q_4(x)$

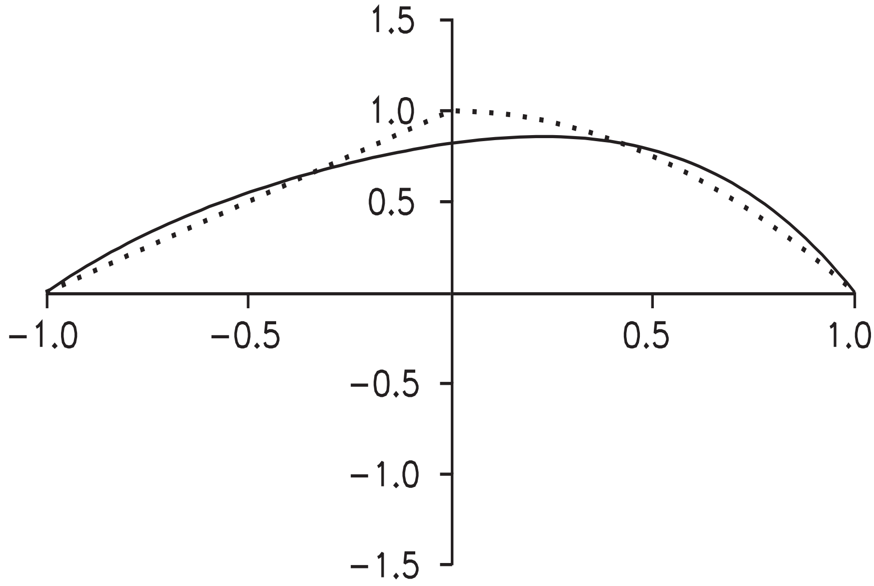

Fig. 3. Graphs of $f_3(x)$ and $s_3(x) = -1.75Q_2(x) - 0.625Q_3(x) + 0.4375Q_4(x)$

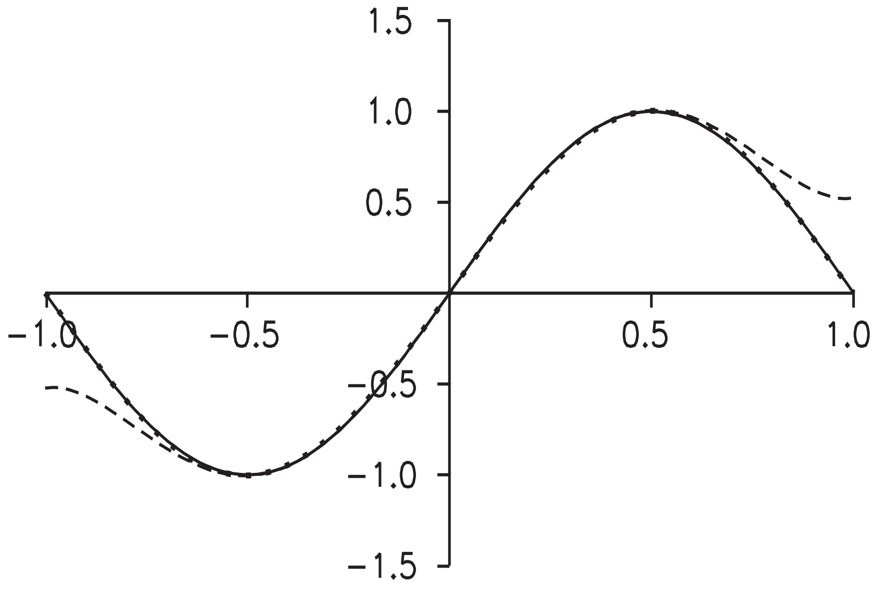

Fig. 4. Graphs of $f(x) = \sin(\pi x)$, $s(x) = -4.7765Q_3(x) + 1.82981Q_5(x)$ and $T(x) = \pi x - (1/6)\pi^3 x^3 + (1/120)\pi^5 x^5$ (dashed curve represents $T(x)$)

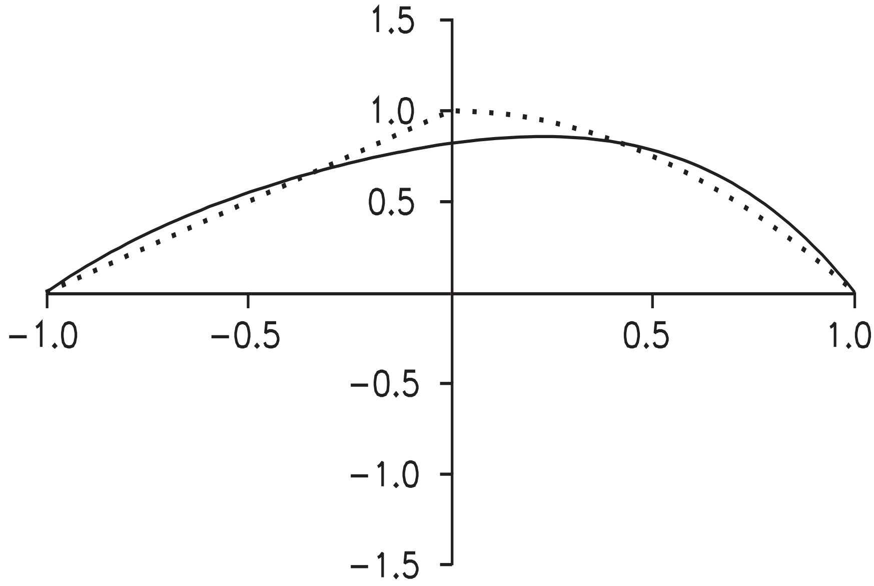

Fig. 5. Graphs of $f(x) = \cos(\pi x)$, $s(x) = -1 - 3Q_2(x) + 3.63872Q_4(x)$ and $T(x) = 1 - (1/2)\pi^2 x^2 + (1/24)\pi^4 x^4$ (dashed curve represents $T(x)$)

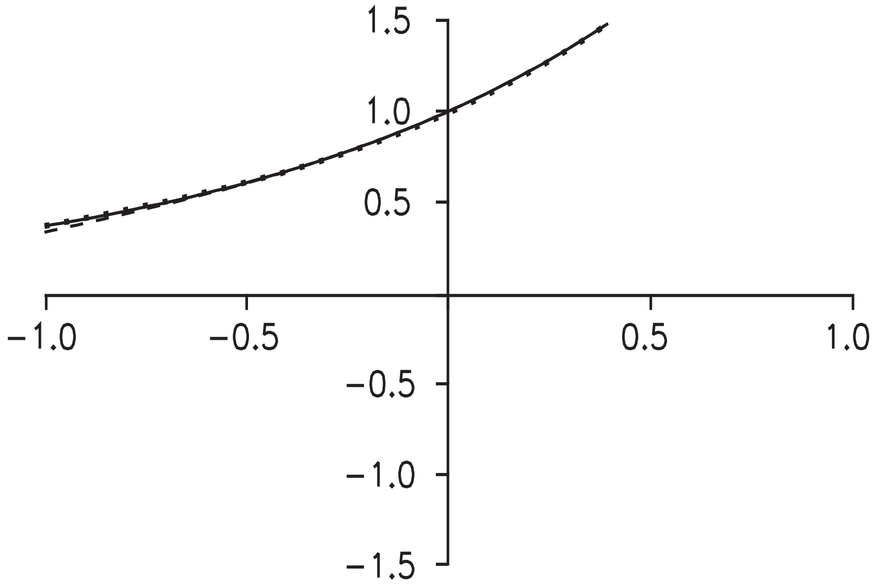

Fig. 6. Graphs of $f(x) = e^x$, $s(x) = \cosh(1) + x\sin(1) + 1.10364Q_2(x) + 0.357814Q_3(x)$ and $T(x) = 1 + x + (1/2)x^2 + (1/6)x^3$ (dashed curve represents $T(x)$)