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SECOND-ORDER CHARACTERIZATIONS OF CONVEX AND PSEUDOCONVEX FUNCTIONS

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Abstract. The present paper gives characterizations of radially u.s.c. convex and pseudoconvex functions $f\colon X \to \mathbb{R}$ defined on a convex subset X of a real linear space $\mathbf{E}$ in terms of first and second-order upper Dini-directional derivatives. Observing that the property f radially u.s.c. does not require a topological structure of $\mathbf{E}$, we draw the possibility to state our results for arbitrary real linear spaces. For convex functions we extend a theorem of Huang, Ng [10]. For pseudoconvex functions we generalize results of Diewert, Avriel, Zang [6] and Crouzeix [4]. While some known results on pseudoconvex functions are stated in global concepts (e.g. Komlosi [11]), we succeeded in realizing the task to confine to local concepts only.

1. Introduction

In this paper $\mathbf{E}$ denotes a real linear space and $f\colon X \to \mathbb{R}$ is a finite-valued real function defined on the set $X \subset \mathbf{E}$. Here $\mathbb{R}$ is the set of the reals and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. Let $\bar{f}\colon \mathbf{E} \to \overline{\mathbb{R}}$ be the extension of f such that $\bar{f}(x) = +\infty$ for $x \in \mathbf{E} \setminus X$.

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The *upper Dini-directional derivative* of f at $x \in X = \mathrm{dom}\,\bar{\mathrm{f}}$ in direction $u \in \mathbf{E}$ is defined as an element of $\overline{\mathbb{R}}$ by

$$f'_+(x;u) = \limsup_{t \downarrow 0} t^{-1}(\bar{f}(x+tu) - f(x)).$$

We introduce the *second-order upper Dini-directional derivative* of f at $x \in X$ in direction $u \in \mathbf{E}$ for which $f'_+(x;u)$ is finite by

$$f''_+(x;u) = \limsup_{t \downarrow 0} 2t^{-2}(\bar{f}(x+tu) - f(x) - t f'_+(x;u)).$$

In the case of an infinite $f'_+(x;u)$ the derivative $f''_+(x;u)$ will not be considered.

We recall that a function $f\colon X \to \mathbb{R}$ is radially upper semicontinuous (radially u.s.c. for short) on the convex set X if the function $\varphi(t) = f(x + t(y-x))$, $t \in [0,\,1]$, is upper semicontinuous for every $x, y \in X$.

While the upper semicontinuity is a topological concept, let us turn attention that the property of f being radially u.s.c. does not require a topological structure on $\mathbf{E}$. In fact, we use the function $\varphi\colon [0,\,1] \to \mathbb{R}$ applying only the topology on $\mathbb{R}$. Similarly, the Dini-directional derivatives are defined through restrictions of f on linearly parameterized rays and also do not require a topology on $\mathbf{E}$. Neither do the definitions of convexity and pseudoconvexity. Hence, arbitrary real linear space is the most natural environment for the considered concepts, which is in consistency with our point of view.

Further we use to say "local concepts", which in the framework of lack of topology on $\mathbf{E}$ needs some explanation. Actually, speaking for "local concepts" (such as Dini derivatives) we face the situation, when some segment or ray is given, which is linearly parameterized by some parameter t. A concept related to this segment or ray is called "local", if it is determined by an arbitrary small interval of variation of t.

It is well-known that a twice continuously differentiable function f of n variables defined on an open convex set X is convex if and only if its Hessian $\nabla^2 f(x)$ is positively semidefinite for all $x \in X$. Various generalizations are due to Chaney [1], Cominetti, Correa [3], Huang, Ng [10], Yang, Jeyakumar [14], and Yang [15]. The present paper is related to these results. It gives characterization of radially u.s.c. convex and pseudoconvex functions in Dini derivatives (and partially in other local concepts). For convex functions we extend a theorem of Huang, Ng [10]. For pseudoconvex functions we generalize results of Diewert, Avriel, Zang [6] and Crouzeix [4].

2. Second-order characterization of convex functions

Let $X \subset \mathbf{E}$ be a convex set. The function $f\colon X \to \mathbb{R}$ is said to be *strongly convex* if there exists a constant $\kappa > 0$ such that for all $x,\ y \in X,\ \lambda \in [0,1]$ it holds

$$f\left(\lambda y + (1-\lambda)x\right) \le \lambda f(y) + (1-\lambda) f(x) - \kappa\lambda(1-\lambda)\|y-x\|^2.$$

Consider the following conditions:

(C_1)	$f'_+(x;u) + f'_+(x;-u) \ge 0$	if the expression in the left hand side has sense;
(C_2)	$f'_+(x;u) + f'_+(x;-u) = 0$	implies $f''_+(x;u) \ge 0$;
(C_3)	$f'_+(x;u) + f'_+(x;-u) = 0$	implies $f''_+(x;u) \ge 2\kappa\|u\|^2$;
(C_4)	$f'_+(x;u) + f'_+(x;-u) = 0$	implies $f''_+(x;u) > 0$.

We add the following comments to these conditions. Concerning (C_1) recall that the upper Dini-directional derivatives $f'_+(x;u)$ and $f'_+(x;-u)$ are elements of $\overline{\mathbb{R}}$ and if the left hand side of (C_1) is the sum $(+\infty)+(-\infty)$ or $(-\infty)+(+\infty)$ we consider this expression as having no sense. All other sums involving infinities have sense: $(+\infty)+(+\infty) = +\infty$, $(-\infty)+(-\infty) = -\infty$ and for a finite $(+\infty)+a = a+(+\infty) = +\infty$ and $(-\infty)+a = a+(-\infty) = -\infty$. Concerning (C_2), (C_3), (C_4) we see with the regard to the definition of the summation in $\overline{\mathbb{R}}$ the equality $f'_+(x;u) + f'_+(x;-u) = 0$ could have place only if $f'_+(x;u)$ and $f'_+(x;-u)$ are finite, whence the second-order upper Dini-directional derivative $f''_+(x;u)$ does exist.

The following theorem gives characterizations of different kinds of convex functions.

Theorem 2.1. *Let $f\colon X \to \mathbb{R}$ be radially u.s.c. on the convex set $X \subset \mathbf{E}$. Then*

- (i) *f is convex on X if and only if Conditions (C_1) and (C_2) hold for each $x \in X$ and $u \in \mathbf{E}$;*
- (ii) *f is strongly convex on X with a constant $\kappa > 0$ if and only if Conditions (C_1) and (C_3) hold for each $x \in X$ and $u \in \mathbf{E}$;*
- (iii) *If Conditions (C_1) and (C_4) hold for each $x \in X$ and $u \in \mathbf{E}\setminus\{0\}$, then f is strictly convex on X.*

Proof. (i) Suppose that f is convex. Then the extended function $\bar{f}$ is also convex, whence for $x \in X$ it holds

$$\bar{f}(x+tu) + \bar{f}(x-tu) \ge 2f(x).$$

Consequently, if the expression $f'_+(x;u) + f'_+(x;-u)$ has sense we have

$$0 \le \limsup_{t\downarrow 0} t^{-1}\big(\bar{f}(x+tu) - f(x) + \bar{f}(x-tu) - f(x)\big) \le f'_+(x;u) + f'_+(x;-u),$$

which proves (C_1). We check now Condition (C_2). If $f'_+(x;u)$ is finite, then for all sufficiently small $t > 0$ it holds $x + tu \in X$ and $f'_+(x;u)$ coincides with the usual directional derivative $f'(x;u)$. From the inequality $f(x + tu) - f(x) - tf'(x;u) \geq 0$ it follows that $f''_+(x;u) \geq 0$. If $f'_+(x;u)$ is infinite, then $f'_+(x;u) + f'_+(x;-u)$ cannot be 0.

Conversely, assume that Conditions (C_1) and (C_2) hold for each $x \in X$ and $u \in \mathbf{E}$. Let us fix $x,\ y \in X$ and $\varepsilon > 0$. Define the function $\psi\colon [0,1] \to \mathbb{R}$ putting

$$\psi(t) = f(x + t(y-x)) - \varepsilon t(1-t) - f(x)(1-t) - f(y)t, \quad 0 \leq t \leq 1.$$

Obviously ψ is finite and $\psi(0) = \psi(1) = 0$. Since also ψ is u.s.c., then by the generalized Weierstrass Theorem, it attains its largest value over $[0,1]$ at some point ξ. We show that $\xi \in \{0,1\}$. Assume in the contrary that $0 < \xi < 1$. An easy calculation accounting the maximality property of ξ gives

$$\begin{aligned}
\psi'_+(\xi;1) =& f'_+(x+\xi(y-x);y-x) - \varepsilon(1-2\xi) + f(x) - f(y) \leq 0,\\
\psi'_+(\xi;-1) =& f'_+(x+\xi(y-x);x-y) + \varepsilon(1-2\xi) - f(x) + f(y) \leq 0.
\end{aligned}$$

These inequalities show that if $f'_+(x+\xi(y-x);y-x)$ or $f'_+(x+\xi(y-x);x-y)$ take eventually infinite value, these values could be only $-\infty$, whence $f'_+(x+\xi(y-x);y-x) + f'_+(x+\xi(y-x);x-y)$ has sense and with regard to Condition (C_1) we have

$$\psi'_+(\xi;1) + \psi'_+(\xi;-1) = f'_+(x+\xi(y-x);y-x) + f'_+(x+\xi(y-x);x-y) \geq 0.$$

The obtained inequality together with the inequalities $\psi'_+(\xi;1) \leq 0$ and $\psi'_+(\xi;-1) \leq 0$ gives $\psi'_+(\xi;1) = \psi'_+(\xi;-1) = 0$. Consequently,

$$f'_+(x+\xi(y-x);y-x) + f'_+(x+\xi(y-x);x-y) = 0.$$

From $\psi'_+(\xi;1) = 0$ and ξ maximal point for ψ we have

$$\psi''_+(\xi;1) = \limsup_{t\downarrow 0} 2t^{-2}(\psi(\xi+t) - \psi(\xi)) \leq 0.$$

On the other hand we infer from Condition (C_2) that

$$\psi''_+(\xi;1) = f''_+(x+\xi(y-x);y-x) + 2\varepsilon > 0,$$

which is a contradiction.

Thus, ξ does not attain its maximum on $(0,1)$, which implies that

$$\psi(t) \leq \max\{\psi(0),\psi(1)\} = 0 \quad \text{for all} \quad t \in [0,1].$$

This inequality can be rewritten in the form

$$f((1-t)x + ty) \leq \varepsilon t(1-t) + (1-t)f(x) + tf(y)$$

and passing to a limit with $\varepsilon \to 0$ we conclude that f is convex.

(*ii*) To adapt the proof of strongly convex case to the convex one, we must change only in the proof the function ψ into

$$\psi_1(t) = f(x + t(y - x)) - (\varepsilon - \kappa\|y - x\|^2)t(1 - t) - f(x)(1 - t) - f(y)t.$$

(*iii*) We apply reasoning similar to that of (*i*) to the function

$$\psi_2(t) = f(x + t(y - x)) - f(x)(1 - t) - f(y)t.$$

□

The following example shows that Condition (C_1) cannot be dropped and Condition (C_2) truncated only to $f''_+(x; u) \geq 0$ for all $x \in X$ and $u \in \mathbf{E}$ as in the classical case.

Example 2.1. The function $f(x) = -|x|$, where $|\cdot|$ is the usual absolute value of a real number, satisfies the equality $f''_+(x; u) = 0$ for all x, $u \in \mathbb{R}$. It is continuous, but not convex. Obviously, $f'_+(0; 1) + f'_+(0; -1) = -2$.

The following example shows that the assumption that f is radially u.s.c. cannot be dropped in Theorem 2.1.

Example 2.2. The function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational;} \\ 0, & \text{otherwise,} \end{cases}$$

satisfies Conditions (C_1) and (C_2), but f is not convex. This function is also not u.s.c.

Theorem 2.1 (*i*) is an extension of Theorem 2 in Huang, Ng [10]. In their work the corresponding theorem is proved in terms of the second-order derivative of Ben-Tal and Zowe. In comparison to the present paper the considerations in Huang, Ng [10] are restricted only to a regular locally Lipschitz function on an open convex set. Condition (C_1) in fact does not appear there, since in case of a regular locally Lipschitz function it is trivially satisfied. When the function is regular locally Lipschitz and the set X is open the second-order Dini derivative coincides with the derivative used in Theorem 2 of Huang, Ng [10]. Characterization of convexity through second-order directional derivatives give also Chaney [1], Cominetti, Correa [3], Yang, Jeyakumar [14], and Yang [15], where again functions defined on open sets from restricted classes are considered satisfying ad hoc condition (C_1).

Related necessary and sufficient conditions for convexity of a continuous function f in terms of other type lower generalized directional derivatives are derived in Ginchev, Ivanov [8].

3. Characterizations of pseudoconvex functions

In this section we use the following well-known definitions.

The function f defined on the convex set $X \subset \mathbf{E}$ is said to be *quasiconvex* on X if

$$f((1-t)x+ty) \le \max\{f(x), f(y)\} \text{ for all } t \in [0,1].$$

The function f defined on the convex set $X \subset \mathbf{E}$ is said to be *upper Dini (strictly) pseudoconvex* on X if the following implications holds:

$$x,\ y \in X,\ f(y) < f(x) \quad \text{implies} \quad f'_+(x; y-x) < 0$$
$$(x,\ y \in X,\ f(y) \le f(x),\ x \ne y \quad \text{implies} \quad f'_+(x; y-x) < 0).$$

For brevity we omit the words "upper Dini" saying simply "pseudoconvex" or "strictly pseudoconvex" function.

The point $x \in X$ is said to be a *stationary point* of the function $f\colon X \to \mathbb{R}$ with respect to the upper Dini-directional derivative if $f'_+(x; u) \ge 0$ for all $u \in \mathbf{E}$.

The following proposition due to Komlosi [11] is a generalization of a well-known characterization of pseudoconvex differentiable functions derived in Crouzeix, Ferland [5].

Proposition 3.1. *Let f be a radially u.s.c. quasiconvex function defined on the open convex set $X \subset \mathbb{R}^n$. If f attains a global minimum at x whenever x is an stationary point of f, then f is pseudoconvex on X.*

The main result of this section is Theorem 3.1, which gives a characterization of pseudoconvex functions. In the contrary to Proposition 3.1 and similarly to the convex case from Theorem 2.1 our point of view is to consider an arbitrary and not necessarily open set $X \subset \mathbf{E}$ (recall that even more, we do not assume a topological structure on $\mathbf{E}$) and to state our result in local and not global concepts. The price we pay is that in the sufficiency parts of Theorem 3.1 (i_1) and (i_2) we confine to the class of functions satisfying Condition (C_1), which does not contain all the pseudoconvex functions (such a restriction does not appear in Proposition 3.1).

Example 3.1. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x, & x < 0, \\ x, & x \ge 0, \end{cases}$$

is pseudoconvex and it does not satisfy Condition (C_1) at $x = 0$.

We consider the following conditions:

(C_5)	$f'_+(x;u)=0$	implies	$\exists\,\delta>0\colon\ \forall\, t\in(0,\delta)\colon$ $f(x)\le \bar f(x+tu)$;
(C_6)	$f'_+(x;u)=0$	implies	$\exists\,\delta>0\colon\ \forall\, t\in(0,\delta)\colon$ $f(x)< \bar f(x+tu)$;
(C_7)	$f'_+(x;u)=0$	implies	$f''_+(x;u)\ge 0$;
(C_8)	$f'_+(x;u)=f''_+(x;u)=0$	implies	$\exists\,\delta>0\colon\ \forall\, t\in(0,\delta)\colon$ $f(x)\le \bar f(x+tu)$;
(C_9)	$f'_+(x;u)=0$	implies	$f''_+(x;u)> 0$.

Theorem 3.1. *Let f be a radially u.s.c. function defined on the convex set $X\subset \mathbf{E}$.*

(First-order conditions)

(i_1) *If f is pseudoconvex, then Conditions (C_5) is satisfied for all $x\in X$ and $u\in\mathbf{E}$. Conversely, if Conditions (C_1) and (C_5) are satisfied for all $x\in X$ and $u\in\mathbf{E}$, then f is pseudoconvex.*

(ii_1) *If Conditions (C_1) and (C_6) are satisfied for all $x\in X$ and $u\in\mathbf{E}\backslash\{0\}$, then f is strictly pseudoconvex.*

(Second-order conditions)

(i_2) *If f is pseudoconvex, then Conditions (C_7) and (C_8) are satisfied for all $x\in X$ and $u\in\mathbf{E}$. Conversely, if Conditions (C_1), (C_7) and (C_8) are satisfied for all $x\in X$ and $u\in\mathbf{E}$, then f is pseudoconvex.*

(ii_2) *If Conditions (C_1) and (C_9) are satisfied for all $x\in X$ and $u\in\mathbf{E}\backslash\{0\}$, then f is strictly pseudoconvex.*

Proof. We prove only the second-order conditions, since the proof of the first-order conditions can be obtained as an obvious simplification of that of the second-order ones.

(i_2) Suppose that f is pseudoconvex on X. Let us fix $x\in X$ and $u\in\mathbf{E}$ such that $f'_+(x;u)=0$. Then $f(x)\le\bar f(x+tu)$ for all $t\ge 0$. Indeed, assume the contrary that $\bar f(x+tu)<f(x)$ for some $t>0$. By the definition of $\bar f$, $x+tu\in X$. Therefore, the positive homogeneity of $f'_+(x;\cdot)$ contradicts the assumption $f'_+(x;u)=0$. Hence Condition (C_8) has place. Now Condition (C_7) is implied by

$$f''_+(x;u)=\limsup_{t\downarrow 0} 2t^{-2}(\bar f(x+tu)-f(x)).$$

Conversely, assume that additional Condition (C_1) is satisfied together with Conditions (C_7) and (C_8) for all $x\in X$ and $u\in\mathbf{E}$. Our purpose is to show that f is pseudoconvex.

We show first the quasiconvexity of f. Indeed, if this is not the case, there for some $x,\ y\in X$ and for the function $\varphi(t)=f(x+t(y-x))$, $0\le t\le 1$, we would have

$$\begin{aligned}\varphi(\bar\xi) =& f((1-\bar\xi)x+\bar\xi y) = \max_{0\le t\le 1}\varphi(t)\\ &> \max\{\varphi(0),\varphi(1)\} = \max\{f(x),f(y)\}\end{aligned} \tag{3.1}$$

(the maximum is attained, which is guaranteed by the generalized Weierstrass Theorem accounting that f is radially u.s.c.). From the maximality of $\bar\xi$ we get

$$f'_+(x+\bar\xi(y-x);y-x)\le 0,\ f'_+(x+\bar\xi(y-x);x-y)\le 0.$$

These inequalities and Condition (C_1) give

$$f'_+(x+\bar\xi(y-x);y-x) = f'_+(x+\bar\xi(y-x);x-y) = 0. \tag{3.2}$$

Now the maximality property of $\bar\xi$ and Condition (C_7) imply

$$f''_+(x+\bar\xi(y-x);y-x) = f''_+(x+\bar\xi(y-x);x-y) = 0.$$

Further the maximality of $\bar\xi$ and Condition (C_8) imply that φ is constant in a neghbourhood of $\bar\xi$. This property is valid for each maximizer $\xi\in(0,1)$, whence φ is constant on the whole interval $[0,1]$. The latter is a consequence of the following reasoning. Let

$$\Xi = \bigcup\{(\alpha,\beta)\mid 0\le\alpha<\bar\xi<\beta\le 1,\ \varphi \text{ is constant on } (\alpha,\beta)\}.$$

Then Ξ is an open interval (ξ_*,ξ^*) and $\varphi(\xi)=\varphi(\bar\xi)$ for $\xi_*<\xi<\xi^*$. We prove that also $\varphi(\xi_*)=\varphi(\xi^*)=\varphi(\bar\xi)$. From the maximality of $\bar\xi$ and the upper semicontinuity of φ we have

$$\varphi(\xi^*)\ge\limsup_{\xi\uparrow\xi^*}\varphi(\xi)=\varphi(\bar\xi)\ge\varphi(\xi^*),$$

where $\xi\uparrow\xi^*$ denotes $\xi\to\xi^*,\ \xi<\xi^*$. Similarly $\varphi(\xi_*)=\varphi(\bar\xi)$. This implies that $\xi_*=0$ and $\xi^*=1$. Indeed, ξ^* is a maximizer of φ. If $\xi^*<1$, then from the proved property there exist $\alpha,\ \beta$ such that $\xi^*\in(\alpha,\beta)\subset[0,1]$ and φ is constant on (α,β). Now φ is constant on the interval $[\xi_*,\beta)$, which contradicts the choice of ξ^*. Hence $\xi^*=1$. Similarly we prove $\xi_*=0$. Thus, φ is constant on $[\xi_*,\xi^*]=[0,1]$. The property φ is constant on $[0,1]$ contradicts however the strict inequality (3.1), which proves the quasiconvexity of f.

We show that f is pseudoconvex on X. Assume the contrary: there exist $x,\ y\in X$ such that $f(y)<f(x)$ and $f'_+(x;y-x)\ge 0$. Thanks to quasiconvexity $f'_+(x;y-x)=0$. It follows from Condition (C_7) and the quasiconvexity that $f''_+(x;y-x)=0$. Condition (C_8) and the quasiconvexity imply an existence of $\delta>0$ such that

$$f(x)=f(x+t(y-x))\quad\text{for all}\quad t\in[0,\delta).$$

In order to get a contradiction denote by $x^* = (1-t^*)x + t^* y$, where

$$t^* = \sup\{t \in [0,1] \mid \varphi \text{ is constant on } [0,t)\}.$$

From the upper semicontinuity and quasiconvexity of φ we have

$$\varphi(t^*) \geq \limsup_{t \uparrow t^*} \varphi(t) = \varphi(0) \geq \varphi(t^*).$$

Hence $f(x^*) = f(x)$. Assume that $t^* < 1$. From the quasiconvexity $f'_+(x^*; y-x^*) \leq 0$. On the other hand, by the choice of x^*, $f'_+(x^*; x^*-y) = 0$, whence applying Condition (C_1) we get that $f'_+(x^*; y-x^*) = 0$. Now Condition (C_7) and the quasiconvexity imply that $f''_+(x^*; y-x^*) = 0$. From the quasiconvexity and Condition (C_8) we conclude that $f(x^*) = f(x^* + t(y-x^*))$ for $0 < t < \delta$ and some $\delta > 0$. This contradicts the choice of x^*. The obtained contradiction implies that $t^* = 1$, which is impossible, since $f(y) < f(x)$. Hence, f is pseudoconvex.

(ii_2) The proof of this claim is simpler and it almost repeats the general parts of case (i_2). To show the quasiconvexity we see that the maximality of $\bar{\xi}$ and equality (3.2) contradict Condition (C_9). In the proof of the strict pseudoconvexity equality $f'_+(x; y-x) = 0$ and the quasiconvexity contradict Condition (C_9) again. □

Theorem 3.1 is a generalization of some well-known results of Diewert, Avriel, Zang [6, Theorems 10, 11 and Corollaries 10.1, 11.1], Crouzeix [4, Propositions 3, 4 and Theorem 2] for a directionally differentiable function or a twice continuously differentiable function on an open set X.

Remark 3.1. Some classes of functions, which satisfy condition (C_1) are the Gâteaux-differentiable, quasidifferentiable in the sense of Pshenichnyi [13], or regular locally Lipschitz in the sense of Clarke [2] functions. Another functions, which fulfill this condition are the upper Dini subdifferentiable functions, that is the functions for which the upper Dini subdifferential

$$\partial f(x) = \{\xi \in \mathbf{E}^* \mid \langle \xi, u \rangle \leq f'_+(x; u) \ \forall\, u \in \mathbf{E}\},$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between $\mathbf{E}$ and its dual space E^*, is nonempty for all $x \in X$.

Looking at Conditions (C_5)–(C_9) we see that (C_5), (C_6) and (C_8) are bit different. Although they are local, they involve not only the values of the derivatives, but also the values of the function f over a finite interval. Further we comment this situation, but we give first an example showing that the sufficient part of Theorem 3.1 (i_2) fails to be true without Condition (C_8).

Example 3.2. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2-1)^3$ satisfies everywhere Conditions (C_1) and (C_7), but it is not pseudoconvex. This function does not fulfill Condition (C_8) at the points $x = 1$ and $x = -1$.

Indeed, each differentiable function satisfies Condition (C_1), since then $f'_+(x;-u) = -f'_+(x;u)$. Here we can check this condition also directly from $f'_+(x;u) = 6x(x^2-1)^2u$. The second-order derivative is $f''_+(x;u) = 6(x^2-1)^2u^2 + 24x^2(x^2-1)u^2$. For $u = 0$ Condition (C_7) is trivially satisfied. Let $u \neq 0$ and $f'_+(x;u) = 0$. Then $x \in \{-1, 0, 1\}$. For $x = 0$ we have $f''_+(x;u) = 6u^2 > 0$ and $x = \pm 1$ gives $f''_+(x;u) = 0$. Therefore (C_7) holds for each $x \in \mathbb{R}$ and $u \in \mathbb{R}$. The function f is however not pseudoconvex, since for $x = 1$, $y = 0$ we have $f(y) = f(0) = -1 < 0 = f(1) = f(x)$, but $f'_+(x; y-x) = 0$.

Commenting that the conclusion in Condition (C_8) is not given in terms of directional derivatives, we could say the following. Obviously, conditions in terms of directional derivatives can be reformulated as conditions over small intervals. For instance, Condition (C_7) can be given in the form:

$f'_+(x;u) = 0$ *implies that for each* $\varepsilon > 0$ *and all* $\delta > 0$ *there exists* $t \in (0,\delta)$ *such that* $\bar{f}(x+tu) > f(x) - 0.5\,t^2\varepsilon$.

It seems that the conclusion in Condition (C_8) cannot be given in terms of directional derivatives. However sometimes it is advantageous to remain by the directional derivatives, since then known calculus rules can be applied. Having this in mind, we come easily to the following corollary.

Corollary 3.1. *If* $f\colon X \to \mathbb{R}$ *is pseudoconvex, then Condition* (C_7) *is satisfied and*

$$f'_+(x;u) = f''_+(x;u) = 0 \quad \textit{implies} \quad f'''_+(x;u) \geq 0. \tag{3.3}$$

Conversely, assuming (C_1) *and* (C_7) *are satisfied and that*

$$f'_+(x;u) = f''_+(x;u) = 0 \quad \textit{implies} \quad f'''_+(x;u) > 0, \tag{3.4}$$

then f *is strictly pseudoconvex.*

In general the applied above upper third-order Dini-directional derivative is defined as element of $\overline{\mathbb{R}}$ iff $f''_+(x;u)$ is finite and

$$f'''_+(x;u) = \limsup_{t\downarrow 0} \frac{6}{t^3}\Big(\bar{f}(x+tu) - f(x) - t f'_+(x;u) - \frac{1}{2}t^2 f''_+(x;u)\Big).$$

Corollary 3.1 is an obvious consequence of Theorem 3.1, taking into account that (C_8) implies (3.3) and (3.4) implies that there exists $\delta > 0$ such that $\bar{f}(x+tu) > f(x)$ for all $t \in (0,\delta)$.

Obviously, assertions like Corollary 3.1 can be formulated in terms of higher-order upper Dini-directional derivatives. Such derivatives can be introduced in manner similar to the ones considered in [7] and [8].

4. Final remarks

Remark 4.1. Theorem 3.1 and Corollary 3.1 remain true if Condition (C_1) is replaced by the following one:

(C_1') $\quad f'_+(x;u) \le 0$ and $f'_+(x;-u) \le 0$ implies $f'_+(x;u) = f'_+(x;-u) = 0$.

In fact in the proof of Theorem 3.1 we have used not exactly (C_1), but the weaker Condition (C_1'). Now we gain the possibility to recognize a wider class of pseudoconvex functions. For instance, the pseudoconvex function from Example 3.1 does not satisfy (C_1), but satisfies (C_1'). Still, the following example shows that there exist pseudoconvex functions, which do not satisfy even this weaker condition.

Example 4.1. The function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \min\{x, x^2\}$ is pseudoconvex and it does not satisfy Condition (C_1').

Each convex function satisfies (C_1) and in such a sense this condition is natural for the convex case. It is some link between Sections 2 and 3, but for pseudoconvex functions (C_1) is not necessarily satisfied. The simplification to (C_1') comes from the proof of Theorem 3.1. It remains still an open question, whether arbitrary pseudoconvex functions can be characterized (with similarly looking necessary and sufficient conditions) in terms of directional derivatives.

Remark 4.2. Theorem 2.1 and Theorem 3.1 remain true if Condition (C_1) in them is replaced respectively by the given below Conditions (C_1^c) and (C_1^p) (the superscript c stands for convex and p for pseudoconvex):

(C_1^c) $\quad f'_+(x;u) \le a$ and $f'_+(x;-u) \le -a$ for some $a \in \mathbb{R}$ implies $f'_+(x;u) + f'_+(x;-u) \ge 0$;

(C_1^p) $\quad f'_+(x;u) \le 0$ and $f'_+(x;-u) \le 0$ implies $f'_+(x;u) + f'_+(x;-u) \ge 0$.

Obviously Condition (C_1^c) is weaker than (C_1) and Condition (C_1^p) is equivalent to (C_1'). We involve these conditions for purely esthetic reasons, observe that they are in some sense similar to each other.

Let us underline again that the present paper gives second-order characterizations of nonsmooth convex and pseudoconvex functions in terms of the upper Dini-directional derivatives. For twice continuously differentiable functions characterizations of different types of generalized convexity in terms of Hessians are well-known, and Giorgi, Thierfelder [9] is a good survey on the subject. The case of pseudoconvex functions is studied in Komlosi [12] who introduces the concept of a quasi-Hessian. A generalization of these results could be a task for another paper.

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References

[1] Chaney, R. W., *Second-order directional derivatives for nonsmooth functions*, J. Math. Anal. Appl. **128** (1987), 495–511.

[2] Clarke, F. H., *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, 1983.

[3] Cominetti, R., Correa, R., *A generalized second-order derivative in nonsmooth optimization*, SIAM J. Control Optim. **28** (1990), 235–245.

[4] Crouzeix, J. P., *Characterizations of generalized convexity and generalized monotonicity, a survey*, in: "Generalized Convexity, Generalized Monotonicity: Recent Results" (Luminy, 1996), Nonconvex Optim. Appl. **27**, Kluwer Academic Publisher, Dordrecht, 1998, 237–256.

[5] Crouzeix, J. P., Ferland, J. A., *Criteria for quasiconvexity and pseudoconvexity: relationships and comparisons*, Math. Programming **23** (1982), 193–205.

[6] Diewert, W. E., Avriel, M., Zang, I., *Nine kinds of quasiconcavity and concavity*, J. Econom. Theory **25** (1981), 397–420.

[7] Ginchev, I., *Higher order optimality conditions in nonsmooth optimization*, Optimization **51**(1) (2002), 42–72.

[8] Ginchev, I., Ivanov, V. I., *Higher order directional derivatives for nonsmooth functions*, C. R. Acad. Bulgare Sci. **54**(11) (2001), 33–38.

[9] Giorgi, G., Thierfelder, J., *Constrained quadratic forms and generalized convexity of C^2-functions revisited*, in: "Generalized Convexity and Optimization for Economic and Financial Decisions", G. Giorgi, F. Rossi, eds., Pitagora Editrice, Bologna, 1999, 179–219.

[10] Huang, L. R., Ng, K. F., *On lower bounds of the second-order directional derivatives of Ben-Tal, Zowe, and Chaney*, Math. Oper. Res. **22** (1997), 747–753.

[11] Komlosi, S., *Some properties of nondifferentiable pseudoconvex functions*, Math. Programming **26** (1983), 232–237.

[12] Komlosi, S., *On pseudoconvex functions*, Acta. Sci. Math. (Szeged) **57** (1993), 569–586.

[13] Pshenichnyi, B. N., *Necessary Conditions for an Extremum*, 2nd ed., Nauka, Moscow, 1982, (English translation of the 1st ed. Pure and Applied Mathematics **4**, Marcel Dekker, Inc., New York, 1971).

[14] Yang, X. Q., Jeyakumar, V., *Generalized second-order directional derivatives and optimization with $C^{1,1}$ functions*, Optimization **26** (1992), 165–185.

[15] Yang, X. Q., *Generalized second-order characterizations of convex functions*, J. Optim. Theory Appl. **82**(1) (1994), 173–180.

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