Journal of Applied Analysis
Vol. 12, No. 2 (2006), pp. 293–299

EXTENSION THEOREM FOR A FUNCTIONAL EQUATION

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Received June 23, 2004 and, in revised form, September 27, 2006

Abstract. We are going to prove an extension theorem on a functional equation for special means studied by Domsta and Matkowski.

1. Introduction

Let $I \subset \mathbb{R}_+$ be a non-empty, open interval and we denote by $\mathcal{CM}(I)$ the class of continuous, strictly monotone functions on I. If $\varphi, \psi \in \mathcal{CM}(I)$, $J \subset I$ and there are $a, b \in \mathbb{R}$, $a \neq 0$, so that $\psi(x) = a\varphi(x) + b$ for all $x \in J \subset I$, then we say that φ is equivalent to ψ on J. In notation $\varphi \overset{J}{\sim} \psi$ or $\varphi(x) \sim \psi(x)$, $x \in J$. Let $\varphi \in \mathcal{CM}(I)$, then we can form the following quantity

$$M_\varphi(x,y) := \varphi^{-1}\left(\frac{x\varphi(x) + y\varphi(y)}{x+y}\right), \qquad x, y \in I,$$

where φ^{-1} is the inverse function of φ. It is easy to prove that $M_\varphi\colon I^2 \to I$ is a strict and symmetric mean on I (see [3], [7]).

2000 *Mathematics Subject Classification.* Primary 26D10.
Key words and phrases. Means, functional equation, extension theorem.
The research was supported by OTKA Grant No. T043080.

ISSN 1425-6908

Originally J. Matkowski raised the following problem at the 40.ISFE in Gronow. For which functions $\varphi, \psi \in \mathcal{CM}(I)$ does the functional equation

$$M_\varphi(x,y) + M_\psi(x,y) = x + y\,, \qquad x, y \in I\,, \tag{1}$$

hold? Domsta's and Matkowski's result [2] refers to this as follows. If additionally φ or ψ is four times continuously differentiable, then $\varphi(x) \sim 1/x$ and $\psi(x) \sim 1/x$, $x \in I$.

Obviously it is not the complete solution of the original problem, because one may expect not to impose any other regularity conditions on the unknown functions, indeed the problem mentioned above is similar to the Matkowski-Sutô problem which was solved in [7] by Daróczy and Páles without any regularity condition. Then instead of M_φ and M_ψ in (1), quasi-arithmetic means are present (see [4], [5], [7]).

In this work we would like to take a small step forward in the solution of the original problem. So we are going to prove, that if the unknown functions $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of (1) and there exists a non-empty, open interval $J \subset I$ so that the unknown functions are equivalent to the function $x \to 1/x$ on J, then this holds on the whole interval I. It is known, that this type of extension theorem played a major role in solving the Matkowski-Sutô problem (see [4], [5], [6], [7], [8]). With our theorem we generalize Domsta's and Matkowski's result, because in this way it is enough to suppose that at least one of the unknown functions is four times continuously differentiable on a subinterval of I (see Theorem 2).

2. Main result

We need the following lemma to prove our main theorem.

Lemma 1. *If $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of* (1) *and $\varphi \overset{I}{\sim} \Phi$, $\psi \overset{I}{\sim} \Psi$, then Φ and Ψ are solutions of* (1)*, too.*

Proof. By the assumptions there exist $a, b, c, d \in \mathbb{R}$, $ac \neq 0$, so that

$$\Phi(x) = a\varphi(x) + b \qquad \text{and} \qquad \Psi(x) = c\psi(x) + d$$

for all $x, y \in I$. Then by equation (1)

$$\varphi^{-1}\left(\frac{\dfrac{a(x\varphi(x) + y\varphi(y)) + b(x+y)}{x+y} - b}{a}\right)$$

$$+\,\psi^{-1}\left(\frac{\dfrac{c(x\psi(x)+y\psi(y))+d(x+y)}{x+y}-d}{c}\right)$$
$$=x+y$$

for all $x, y \in\ I$. It means, Φ and Ψ also fulfil (1). □

Theorem 1. *Let $I \subset \mathbb{R}_+$ be a non-empty, open interval and $\psi, \varphi \in \mathcal{CM}(I)$ satisfy*

$$\varphi^{-1}\left(\frac{x\varphi(x)+y\varphi(y)}{x+y}\right)+\psi^{-1}\left(\frac{x\psi(x)+y\psi(y)}{x+y}\right)=x+y \tag{2}$$

for all $x, y \in I$. If there exists an interval $J \subset I$ of positive length, so that

$$\varphi(x)\sim\frac{1}{x} \quad \text{and} \quad \psi(x)\sim\frac{1}{x}\,, \qquad \text{on } x \in J\,, \tag{3}$$

then both φ and ψ are in the equivalence class of $\frac{1}{x}$, $x \in I$.

Proof. According to the lemma we can suppose that

$$\varphi(x)=\frac{1}{x} \quad \text{and} \quad \psi(x)=\frac{1}{x}\,, \qquad x \in J.$$

Let $K \subset I$ be the interval of maximal length containing J so that

$$\varphi(x)=\psi(x)=\frac{1}{x}\,, \qquad x \in K. \tag{4}$$

We are going to show that $K = I$. Since φ and ψ are continuous, K is closed in I. Suppose to the contrary that $K \neq I$, then either $\inf K$ or $\sup K$ is an interior point of I. Say, $c := \inf K$ is an interior point of I. Then

$$I :=]a, b[\,, \qquad K := [c, d\}\,, \qquad a < c < d \leq b\,,$$

where $\}$:= [or]. We can assume {=] and $d < b$ without losing the generality to our further examination.

First we prove, that there exist δ_1, δ_2 positive real numbers $\delta_2 \leq d - c$, so that:

(i) For $x \in]c - \delta_1, c]$ and $y \in [d - \delta_2, d]$ we have

$$\frac{x\varphi(x)+y\varphi(y)}{x+y}=\frac{x\varphi(x)+1}{x+y}\in[1/d, 1/c]\,,$$

and

$$\frac{x\psi(x)+y\psi(y)}{x+y}=\frac{x\psi(x)+1}{x+y}\in[1/d, 1/c].$$

$$\text{(ii)} \qquad \frac{(c-\delta_1)\varphi(c-\delta_1)+1}{c-\delta_1+d-\delta_2} = \frac{1}{c}.$$

For this let us introduce

$$\delta_2(\lambda) = d - \big(\lambda c + (1-\lambda)d\big) = \lambda(d-c)\,, \quad \lambda \in [0,1]\,.$$

Then, as φ is strictly decreasing,

$$\varphi(d) = \frac{1}{d} < \frac{\varphi(c)c + \varphi(d-\delta_2(\frac{1}{2}))(d-\delta_2(\frac{1}{2}))}{c+d-\delta_2(\frac{1}{2})} < \frac{1}{c} = \varphi(c)$$

and further because of the continuity of φ there exists $\delta_1 > 0$ such that

$$\frac{1}{d} < \frac{\varphi(x)x + \varphi(d-\delta_2(\frac{1}{2}))(d-\delta_2(\frac{1}{2}))}{x+d-\delta_2(\frac{1}{2})} < \frac{1}{c} \qquad \text{for all } x \in [c-\delta_1, c]\,.$$

For next steps we assume that $\varphi(c-\delta_1)(c-\delta_1) \neq 1$, which is possible by the definition of c, $(c = \inf K)$. Again by continuity we can state

$$\frac{1}{d} \le \frac{\varphi(x)x+\varphi(y)y}{x+y} \le \frac{1}{c}\,, \text{ for all } x \in [c-\delta_1, c] \text{ and } y \in [d-\delta_2(\tfrac{1}{2}), d]\,.$$

Thus (i) is fulfilled by φ with $\delta_2 = \delta_2(\frac{1}{2})$. Similarly goes the proof for ψ. Now, if $\delta_2(\frac{1}{2})$ replaced for δ_2 does not fulfil (ii), then by monotonicity of φ

$$\frac{(c-\delta_1)\varphi(c-\delta_1) + (d-\delta_2(\frac{1}{2}))\varphi(d-\delta_2(\frac{1}{2}))}{c-\delta_1+d-\delta_2(\frac{1}{2})} < \frac{1}{c} \tag{5}$$

and

$$\frac{\varphi(c-\delta_1)(c-\delta_1)+\varphi(c)c}{c-\delta_1+c} > \frac{1}{c}. \tag{6}$$

Since $c = d - \delta_2(1)$, applying the Bolzano theorem to the following function

$$Q(\lambda) := \frac{(c-\delta_1)\varphi(c-\delta_1) + (d-\delta_2(\lambda))\varphi(d-\delta_2(\lambda))}{c-\delta_1+d-\delta_2(\lambda)}\,,$$

because of the inequalities (5), (6) there exists $\lambda_0 \in]1/2, 1[$ such that (ii) is true with $\delta_2 = \delta_2(\lambda_0)$. Obviously, then (i) also remains true.

After this we are going to prove that $\varphi(x) = \psi(x) = 1/x$ on an interval L with $\inf L < \inf K = c$. For, let $x \in [c-\delta_1, c]$ and $y \in [d-\delta_2, d] \subset K$, then (2), (ii), and the property of K imply

$$\frac{x+y}{x\varphi(x)+1} + \frac{x+y}{x\psi(x)+1} = x+y\,.$$

From this and (4) we obtain

$$\frac{1}{x^2} = \varphi(x)\psi(x) \qquad \text{for all } x \in [c-\delta_1, d]. \tag{7}$$

Then (using (7)) two cases are possible:

$$\frac{1}{c-\delta_1} > \varphi(c-\delta_1) \qquad \text{and} \qquad \frac{1}{c-\delta_1} < \psi(c-\delta_1), \tag{8}$$

or

$$\frac{1}{c-\delta_1} < \varphi(c-\delta_1) \qquad \text{and} \qquad \frac{1}{c-\delta_1} > \psi(c-\delta_1). \tag{9}$$

We can deal with them at the same time but (2) is symmetric in φ and in ψ. Therefore we shall use (8) only. Moreover, we can assume without losses that

$$\varphi(c-\delta_1) \neq \frac{1}{c-\delta_1} \qquad \text{and} \qquad \psi(c-\delta_1) \neq \frac{1}{c-\delta_1}$$

(changing if it necessary the value of δ_1).

Using (2) and (7) we obtain

$$\begin{aligned} &(x\varphi(x)+y\varphi(y))\left(x+y-M_\varphi(x,y)\right)^2\varphi\left(x+y-M_\varphi(x,y)\right) \\ &= xy\varphi(x)\varphi(y)(x+y) \end{aligned} \tag{10}$$

for all $x, y \in [c-\delta_1, d]$.

From (10) for all $x \in [c-\delta_1, c]$ and $y \in [d-\delta_2, d]$ we have

$$(x\varphi(x)+1)\left(x+y-\frac{x+y}{x\varphi(x)+1}\right)^2\varphi\left(x+y-\frac{x+y}{x\varphi(x)+1}\right) = x\varphi(x)(x+y)\,.$$

Simplifying this we have

$$x\varphi(x)\left[(x+y)\frac{x\varphi(x)}{x\varphi(x)+1}\varphi\left(\frac{x+y}{x\varphi(x)+1}x\varphi(x)\right)-1\right] = 0\,,$$

since $x\varphi(x) \neq 0$

$$\varphi\left(\frac{x+y}{x\varphi(x)+1}x\varphi(x)\right) = \frac{x\varphi(x)+1}{x\varphi(x)(x+y)}\,, \tag{11}$$

for every $x \in [c-\delta_1, c]$ and $y \in [d-\delta_2, d]$. In (11) performing the substitutions $x = c-\delta_1, y \in [d-\delta_2, d]$ we obtain

$$\begin{aligned}&\varphi\left(\frac{c-\delta_1+y}{(c-\delta_1)\varphi(c-\delta_1)+1}(c-\delta_1)\varphi(c-\delta_1)\right) \\ &= \frac{(c-\delta_1)\varphi(c-\delta_1)+1}{(c-\delta_1)\varphi(c-\delta_1)(c-\delta_1+y)} \\ &\left(= \frac{1}{-}\frac{c-\delta_1+y}{(c-\delta_1)\varphi(c-\delta_1)+1}(c-\delta_1)\varphi(c-\delta_1)\right).\end{aligned} \tag{12}$$

This and (8) imply that $\varphi(x) = \psi(x) = 1/x$ holds on such an interval L, that $\inf L < \inf K$ (according to inequality (8), $(c-\delta_1)\varphi(c-\delta_1) < 1$). This contradicts to the maximality of K. □

With the previous theorem we can immediately generalize Domsta's and Matkowski's result ([2, Theorem 3]).

Theorem 2. *Let $I \subset \mathbb{R}_+$ be a non-empty, open interval. If $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of* (1) *and there is a non-empty, open interval $J \subset I$ so that φ or ψ is four times continuously differentiable on J, then*

$$\varphi(x) \sim \frac{1}{x} \qquad \text{and} \qquad \psi(x) \sim \frac{1}{x}$$

on $x \in I$.

Proof. We obtain it immediately from the Domsta's and Matkowski's result and from the previous theorem. □

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