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Some Remarks About the Density of Smooth Functions in Weighted Sobolev Spaces

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1. Introduction and statement of main results

In this note we deal with the problem of the density of smooth functions in weighted Sobolev spaces (for general results and references on this topic see, for instance, [14], [10], [2], and the bibliography therein). In order to introduce some definitions, let us fix a bounded open set $\Omega \subseteq \mathbb{R}^n$, a real number $p > 1$, and a function $\lambda : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\lambda(x) > 0 \quad \text{a.e. in } \mathbb{R}^n, \quad \lambda \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{1.1}$$

We define the following function spaces

$$\mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda) = \{u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega) : \|u\|^p_{p,\lambda} = \int_\Omega (|u|^p + |Du|^p)\lambda dx < +\infty\}, \tag{1.2}$$

$$\begin{gathered}\mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda) = \text{the closure of } \mathcal{C}^1(\Omega) \cap \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda) \text{ in } \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)\\ \text{endowed with the norm } \|\cdot\|_{p,\lambda},\end{gathered} \tag{1.3}$$

$$\begin{gathered}\tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega,\lambda) = \text{the completion of } \mathcal{C}^1(\Omega) \cap \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)\\ \text{with respect to the norm } \|\cdot\|_{p,\lambda}.\end{gathered} \tag{1.4}$$

If $u \in \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$, we denote by Du the usual distributional gradient, that exists by definition (1.2). If λ satisfies the additional property

$$\begin{gathered}\text{if } (\varphi_h)_h \subset \mathcal{C}^1(\Omega) \cap \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda), \quad \int_\Omega |\varphi_h|^p \lambda dx \to 0 \text{ and } \int_\Omega |D\varphi_h - \nu|^p \lambda dx \to 0\\ \text{then } \nu(x) = 0 \text{ a.e. in } \Omega,\end{gathered} \tag{1.5}$$

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then if $u \in \tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega,\lambda)$ we can define the gradient ∇u in the following way: if $(\varphi_h)_h \subseteq \mathcal{C}^1(\Omega) \cap \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$ satisfies

$$\int_\Omega |\varphi_h - u|^p \lambda dx \to 0, \qquad \int_\Omega |D\varphi_h - v|^p \lambda dx \to 0,$$

then we set $\nabla u = v$. We remark that condition (1.5) is essential in this context, since in general the gradient of a function in $\tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega,\lambda)$ need not be uniquely defined. An example of this situation is given in [9] (Section 2.1).

An interesting case in which condition (1.5) is satisfied, is when there exists a finite number of points $x_1, \dots, x_k$ in Ω such that $\lambda^{\frac{-1}{p-1}} \in \mathrm{L}^1_{\mathrm{loc}}(\Omega \setminus \{x_1,\dots,x_k\})$, (see [9], Section 2.1). Examples of weights of this kind are considered in the theory of Dirichlet forms (see, for instance, [1], [12]), where condition (1.5) essentially corresponds to the so-called *closability* property of a form. It is interesting to observe that even if $u \in \tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega,\lambda)$ and it has also a distributional gradient, it may occur that $Du \neq \nabla u$ (see Example 2.1 in Section 2). This means that in general $\mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda)$ and $\tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega,\lambda)$ are different spaces and that $\mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$ need not be complete.

If λ satisfies the stronger condition

$$\lambda^{-\frac{1}{p-1}} \in \mathrm{L}^1(\Omega), \tag{1.6}$$

then $\mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$ is a reflexive Banach space and $\tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega,\lambda) = \mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda) \subseteq \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$ (see, for instance, [9], Section 2.1, and [6], Lemma 1.1). Therefore it is a natural problem to investigate when $\mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda) = \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$.

The first positive answer to this question is probably the classical result by Meyers and Serrin ([11]) for the case $\lambda \equiv 1$. Other results in this direction are proven in [14], [2], [10] when $\lambda(x)$ is of the type $\mathrm{dist}(x,\partial\Omega)$ (or, more generally, a positive smooth function of $\mathrm{dist}(x,\partial\Omega)$). Another case in which $\mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda) = \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$ is when λ belongs to the Muckenhoupt class A_p, i.e.,

$$\sup_B \left(\fint_B \lambda dx \right) \left(\fint_B \lambda^{-\frac{1}{p-1}} dx \right)^{p-1} < +\infty, \tag{A_p}$$

where the supremum is taken over all balls B contained in $\mathbb{R}^n$ and $\fint_A f dx$ denotes the average

$$\fint_A f dx = \frac{1}{|A|} \int_A f dx$$

of a function $f \in \mathrm{L}^1(A)$, over a measurable set $A \subset \mathbb{R}^n$ with finite positive Lebesgue measure $|A|$ (see, for instance, [6], Theorem 2.3). Condition (A_p), first introduced in [13], in a context of real analysis, has also been used in many papers about the regularity of the solutions of degenerate elliptic equations (see, for instance, [9]).

The above results suggest to investigate whether conditions (1.1), (1.6) alone are enough to prove the equality $\mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda) = \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$.

If $n = 1$ the result is true and can be obtained as a consequence of some results concerning the relaxation of variational integrals in connection with the so-called *gap* and *Lavrentiev phenomenon* (see [7], [5], and Remark 2.7 in Section 2).

For the case $n > 2$, if Ω is the unit open ball of $\mathbb{R}^n$, in [8] there is an example of a quadratic non isotropic form $a(x) = (a_{i,j}(x))$ satifying the inequality

$$|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq w(x)|\xi|^2, \tag{1.7}$$

for a.e. $x \in \mathbb{R}^n$, for every $\xi \in \mathbb{R}^n$, with w positive measurable function such that $w \in \mathrm{L}^s(\mathbb{R}^n)$, $s > 1$, for which the spaces

$$\mathrm{W}(\Omega, a) = \{u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(\Omega) : \|u\|_a^2 = \int_\Omega u^2 dx + \int_\Omega \sum_{i,j=1}^{n} a_{i,j} D_i u D_j u dx < +\infty\}$$

and

$$\mathrm{H}(\Omega, a) = \text{ the closure of } \mathcal{C}^1(\mathbb{R}^n) \text{ in } \mathrm{W}(\Omega, a) \text{ endowed with the norm } \|u\|_a$$

are different.

In this note we show that, if $n = 2$, then for every $p > 1$ the equality between $\mathrm{W}^{1,\mathrm{p}}(\Omega, \lambda)$ and $\mathrm{H}^{1,\mathrm{p}}(\Omega, \lambda)$ does not hold any more for a suitable weight λ verifying (1.1) and (1.6). More precisely, by adapting a technique of Zhikov ([15]), we give, for every $p > 1$, an example of a weight $\lambda = \lambda(x)$ which is positive and continuous for every $x \neq 0$, such that $\lambda^{-\frac{1}{p-1}} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2)$, for which $\mathrm{H}^{1,\mathrm{p}}(B(0,1), \lambda)$ is strictly contained in $\mathrm{W}^{1,\mathrm{p}}(B(0,1), \lambda)$ (see Example 2.2 in Section 2), where $B(x, r)$ denotes the open ball centered at x, with radius $r > 0$. Moreover, if $p > 2$, it is possible to choose the weight λ regular (see also Remark 2.5 in Section 2).

Finally, we note that the weight λ that we construct in Section 2 permits also to give an example of the so called gap-phenomenon arising in the relaxation of variational integral functionals (see Remark 2.7, in Section 2).

2. Some examples of H≠W

First of all we give an example in the case $p = 2$, $n = 1$, of a weight λ verifying (1.1) and (1.5), for which there exists a function $u \in \tilde{\mathrm{H}}^{1,\mathrm{p}}(\Omega, \lambda)$ whose gradient ∇u is different from the distributional gradient Du.

Example 2.1. Let $\lambda(x) = |x|^{1+\alpha}$, with $\alpha > 0$, $\Omega =]-1, 1[$. Since $\lambda^{-1} \in \mathrm{L}^1(\Omega \setminus B(0, \varepsilon))$ for every $\varepsilon > 0$, condition (1.5) follows immediately. Let $u(x) = \frac{x}{|x|}$ and let $u_h(x) = v(hx)$, where $v \in \mathcal{C}^1(\mathbb{R})$ is a function such that $v \equiv 1$ on $[1, +\infty[$, $v \equiv -1$ on $]-\infty, -1]$. Then $u_h \in \mathcal{C}^1(\overline{\Omega})$ and

$$\int_\Omega |u_h - u|^2 \lambda dx \to 0, \quad \int_\Omega (u_h')^2 \lambda dx \to 0.$$

Therefore, by (1.4), $u \in \tilde{\mathrm{H}}^{1,2}(\Omega, \lambda)$, $\nabla u = 0$, and obviously $u \notin \mathrm{W}^{1,2}(\Omega, \lambda)$.

We now give an example in the case $n = 2$ of a weight λ verifying (1.1), (1.6), for which $\mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda) \neq \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$. To this aim we need to recall some technical results.

First we recall that by Theorem 1 in [3] one can easily obtain that for every convex, bounded, open set A, there exists a positive constant $c_1 = c_1(n, A)$ such that

$$|v(x) - v_A| \leq c_1 \int_A |x-y|^{1-n} |Dv(y)| dy \tag{2.1}$$

for every $x \in \overline{A}$, for every $v \in \mathcal{C}^0(\overline{A}) \cap \mathrm{W}^{1,1}(A)$, where $v_A = -\!\!\!\!\!\!\int_A v dx$. By (2.1) and by a direct application of Hölder's inequality we obtain that, given A as before, $q > 1$, λ verifying (1.1), we have

$$|v(x) - v_A| \leq c_1 \Big(\int_A |Dv(y)|^q \lambda(y) dy \Big)^{\frac{1}{q}} \Big(\int_A \lambda^{-\frac{1}{q-1}}(y) |x-y|^{(1-n)\frac{q}{q-1}} dy \Big)^{1-\frac{1}{q}}, \tag{2.2}$$

for every $v \in \mathcal{C}^0(\overline{A}) \cap \mathrm{W}^{1,1}(A)$, and for every $x \in \overline{A}$, for which

$$\int_A \lambda^{-\frac{1}{q-1}}(y) |x-y|^{(1-n)\frac{q}{q-1}} dy < +\infty. \tag{2.3}$$

If we choose $\lambda \equiv 1$ and $q > n$, from (2.2) it follows at once the classical Morrey's estimate, i.e., there exists a positive constant $c_2 = c_2(n, q, A) > 0$ such that

$$|v(x) - v(y)| \leq c_2 \mathrm{diam}(A)^{1-\frac{n}{q}} \|Dv\|_{\mathrm{L}^{\mathrm{q}}(A)} \tag{2.4}$$

for every $v \in \mathcal{C}^0(\overline{A}) \cap \mathrm{W}^{1,1}(A)$ and for every x, $y \in \overline{A}$.

Finally, we recall that, by Hölder's inequality, for every $1 \leq q < p$, and for every bounded open set A, if

$$\int_A \lambda^{-\frac{q}{p-q}} dx < +\infty,$$

then $\mathrm{W}^{1,\mathrm{p}}(A,\lambda)$ is continuosly embedded in the classical Sobolev space $\mathrm{W}^{1,\mathrm{q}}(A,1) \equiv \mathrm{W}^{1,\mathrm{q}}(A)$. In fact, for every $u \in \mathrm{W}^{1,\mathrm{p}}(A,\lambda)$ one has

$$\Big(\int_A (|u|^q + |Du|^q) dx \Big)^{\frac{1}{q}} \leq c \Big(\int_A (|u|^p + |Du|^p) \lambda dx \Big)^{\frac{1}{p}} \Big(\int_A \lambda^{-\frac{q}{p-q}} dx \Big)^{\frac{1}{q}-\frac{1}{p}}. \tag{2.5}$$

Example 2.2. (i) Case $p > 2$. Let $\Omega = B(0,1) \subseteq \mathbb{R}^2$ and let p, α, $\beta \in \mathbb{R}$, with $p > 2$ and

$$0 < \alpha < \beta < 2(p-1). \tag{2.6}$$

Given $\varepsilon \in]0, \frac{\pi}{4}[$, we denote by $S_\varepsilon = \{(x_1,x_2) \in \Omega : \tan\varepsilon < \frac{x_2}{x_1} < \tan(\frac{\pi}{2} - \varepsilon)\}$, $S_\varepsilon^+ = S_\varepsilon \cap \{x_2 > 0\}$, $S_\varepsilon^- = S_\varepsilon \cap \{x_2 < 0\}$. Let us choose a π-periodic, smooth function $k : \mathbb{R} \to [\alpha, \beta]$ such that $k(\theta) = \alpha$ if $\varepsilon < \theta < \frac{\pi}{2} - \varepsilon$, $k(\theta) = \beta$ if $\frac{\pi}{2} < \theta < \pi$, $k'(0) = 0$.

Then we define the weight $\lambda : \mathbb{R}^2 \to [0, +\infty[$ as

$$\lambda(x) = \begin{cases} |x|^{k(\arccos \frac{x_1}{|x|})} & \text{if } |x| \neq 0, \\ 0 & \text{if } |x| = 0. \end{cases} \tag{2.7}$$

It is clear that $\lambda \in \mathcal{C}^0(\mathbb{R}^2 \setminus \{0\})$ and

$$|x|^\beta \leq \lambda(x) \leq |x|^\alpha \qquad \text{for every } x \in \Omega, \tag{2.8}$$

so that the continuity on $\mathbb{R}^2$ follows. By (2.6) and (2.8) it follows also that

$$\lambda^{-\frac{1}{p-1}} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2).$$

Now we define $u : \Omega \to \mathbb{R}$ as

$$u(x) = \begin{cases} 1 & \text{if } x_1, x_2 > 0, \\ 0 & \text{if } x_1, x_2 < 0, \\ \frac{x_2}{|x|} & \text{if } x_1 < 0 < x_2, \\ \frac{x_1}{|x|} & \text{if } x_2 < 0 < x_1. \end{cases} \tag{2.9}$$

A direct computation shows that $u \in \mathrm{W}^{1,\mathrm{p}}(\Omega, \lambda)$ if

$$\beta > p - 2. \tag{2.10}$$

We claim that for every $p > 2$ there exist α and β verifying (2.6) and (2.10), such that $u \notin \mathrm{H}^{1,\mathrm{p}}(\Omega, \lambda)$.

By contradiction, let us assume that there exists a sequence $(u_h)_h \subseteq \mathcal{C}^1(\Omega) \cap \mathrm{W}^{1,\mathrm{p}}(\Omega, \lambda)$ converging to u in $\mathrm{W}^{1,\mathrm{p}}(\Omega, \lambda)$. Then for every q, $2 < q < p$, such that

$$0 < \alpha < 2(\frac{p}{q} - 1), \tag{2.11}$$

by (2.5) $u_h \to u$ in $\mathrm{W}^{1,\mathrm{q}}(S_\varepsilon)$. By applying inequality (2.4) to $v = u_h$, first with $A = S^+_\varepsilon \cap B(0, R)$, $0 < R < 1$, $y = (0, 0)$, then with $A = S^-_\varepsilon \cap B(0, R)$, $x = (0, 0)$, and taking the sum of the two ones we obtain that there exists a constant $c = c(q, p)$ such that

$$|u_h(x) - u_h(y)| \leq cR^{1-\frac{2}{q}} \|Du_h\|_{\mathrm{L}^\mathrm{q}(S_\varepsilon)} \tag{2.12}$$

for every x, $y \in S_\varepsilon$, for every h, for every $0 < R < 1$. Since we can suppose that, up to a subsequence, $u_h(x) \to u(x)$ for a.e. $x \in \Omega$, by passing to the limit in (2.12) we obtain a contradiction.

(ii) Case $p = 2$. Let Ω, S^+_ε, S^-_ε, k be as in (i), with $\varepsilon \in]0, \frac{\pi}{2}[$, $\alpha = -1$, and $\beta = 1$. Then we define the weight $\lambda : \mathbb{R}^2 \setminus \{0\} \to [0, +\infty[$ as

$$\lambda(x) = \begin{cases} \left(\ln^{-2}\left(\frac{e}{|x|}\right)\right)^{k(\arccos \frac{x_1}{|x|})} & \text{if } 0 < |x| \leq 1 \\ 1 & \text{if } |x| > 1. \end{cases} \tag{2.13}$$

It is clear that $\lambda \in \mathcal{C}^0(\mathbb{R}^2 \setminus \{0\})$ and

$$\ln^{-2}\Big(\frac{e}{|x|}\Big) \le \lambda(x) \le \ln^2\Big(\frac{e}{|x|}\Big)$$

for every $0 < |x| < 1$, from which it follows at once

$$\lambda \text{ and } \lambda^{-1} \in \bigcap_{q\ge 1} \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^2). \tag{2.14}$$

Let u be the function defined by (2.9); then a direct computation shows that $u \in \mathrm{W}^{1,2}(\Omega,\lambda)$. We claim now that $u \notin \mathrm{H}^{1,2}(\Omega,\lambda)$. By contradiction, let us assume that there exists a sequence $(u_h)_h \subset \mathcal{C}^1(\Omega) \cap \mathrm{W}^{1,2}(\Omega,\lambda)$ converging to u in $\mathrm{W}^{1,2}(\Omega,\lambda)$. Then, by (2.2) and (2.3), if we choose $A = S^+_\varepsilon \cap B(0,R)$ $(0 < R < 1)$, $q = 2$, $n = 2$, $x = 0$, $v = u_h$, we obtain that there exists $c = c(S^+_\varepsilon, R) > 0$ such that

$$|u_h(0) - \fint_{S^+_\varepsilon \cap B(0,R)} u_h dy| \le \tag{2.15}$$
$$c\Big(\int_{S^+_\varepsilon \cap B(0,R)} \frac{1}{\lambda(y)|y|^2} dy\Big)^{\frac{1}{2}} \Big(\int_{S^+_\varepsilon \cap B(0,R)} |Du_h|^2 \lambda(y) dy\Big)^{\frac{1}{2}},$$

for every $h \in \mathbf{N}$. Analogously

$$|u_h(0) - \fint_{S^-_\varepsilon \cap B(0,R)} u_h dy| \le \tag{2.16}$$
$$c\Big(\int_{S^-_\varepsilon \cap B(0,R)} \frac{1}{\lambda(y)|y|^2} dy\Big)^{\frac{1}{2}} \Big(\int_{S^-_\varepsilon \cap B(0,R)} |Du_h|^2 \lambda(y) dy\Big)^{\frac{1}{2}},$$

for every $h \in \mathbf{N}$. But

$$\fint_{S^+_\varepsilon \cap B(0,R)} u_h dx \to 1 \qquad \text{and} \qquad \fint_{S^-_\varepsilon \cap B(0,R)} u_h dx \to 0$$

and therefore, by (2.15) and (2.16) we have a contradiction.

(iii) Case $1 < p < 2$. Let Ω, S^+_ε, S^-_ε, k be as in (i), with $\varepsilon \in]0, \frac{\pi}{2}[$, $\alpha = -1$, and $\beta = 0$. Then we define the weight $\lambda : \mathbb{R}^2 \setminus \{0\} \to [0,+\infty[$ as

$$\lambda(x) = \begin{cases} \Big(|x|^{2-p} \ln^{2(1-p)}\Big(\frac{e}{|x|}\Big)\Big)^{k(\arccos \frac{x_1}{|x|})} & \text{if } 0 < |x| \le 1 \\ 1 & \text{if } |x| > 1. \end{cases} \tag{2.17}$$

It is clear that $\lambda \in \mathcal{C}^0(\mathbb{R}^2 \setminus \{0\})$ and, by definition

$$1 \le \lambda(x) \le |x|^{p-2} \ln^{2(p-1)}\Big(\frac{e}{|x|}\Big)$$

for every $0 < |x| < 1$, from which it follows at once that

$$\lambda \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^2) \qquad \text{and} \qquad \lambda^{-\frac{1}{p-1}} \in \mathrm{L}^\infty(\mathbb{R}^2). \tag{2.18}$$

Then, if u is defined by (2.9), a direct computation shows that $u \in \mathrm{W}^{1,\mathrm{p}}(\Omega,\lambda)$. However, by arguing as in (ii) we can prove that $u \notin \mathrm{H}^{1,\mathrm{p}}(\Omega,\lambda)$.

Remark 2.3. Actually, with the same notations of example 2.2, by (2.1) we can get also that $\mathcal{C}^0(\Omega) \cap \mathrm{W}^{1,p}(\Omega,\lambda)$ is not dense in $\mathrm{W}^{1,p}(\Omega,\lambda)$. In fact, by repeating the same arguments used in example 2.2, it follows that u cannot be approximated in $\mathrm{W}^{1,p}(\Omega,\lambda)$ by a sequence $(v_h)_h \subseteq \mathcal{C}^0(\Omega) \cap \mathrm{W}^{1,p}(\Omega,\lambda)$.

Remark 2.4. We want to underline that the weight λ defined by (2.7), (2.13), or (2.17) does not belong to the Muckenhoupt class A_p. In fact by a simple computation it can be proved that

$$\Big(\fint_{B(0,R)} \lambda dx\Big)\Big(\fint_{B(0,R)} \lambda^{-\frac{1}{p-1}} dx\Big)^{p-1} \to +\infty \qquad \text{as } R \to 0^+.$$

Remark 2.5. If $p > 3$, it is easy to see that we can choose $\alpha > 1$ in (2.6), so that $\lambda \in \mathcal{C}^1(\mathbb{R}^2)$. Moreover, we remark that in cases (i), (ii), (iii) of Example 2.2, by (2.8), (2.14), and (2.18), we actually have that $\lambda^{-\frac{1}{p-1}} \in \mathrm{L}^q_{\mathrm{loc}}(\mathbb{R}^2)$ with $q > 1$.

Remark 2.6. Let us set $\tilde{\Omega} = \Omega \times A$, where Ω is the unit open ball of $\mathbb{R}^2$ and A is a bounded open subset of $\mathbf{R}^{\mathbf{m}}$, and let us define the functions $\tilde{u}$ and $\tilde{\lambda}$ as

$$\tilde{u}(x,y) = u(x) \qquad \tilde{\lambda}(x,y) = \lambda(x)$$

for $x \in \Omega$ and $y \in A$. By applying Fubini's theorem one obtains an example of H$\neq$W in dimension $n = 2 + m > 2$.

Remark 2.7. Example 2.2 permits to construct also an example of gap phenomenon in the relaxation of variational integral functionals (see, for instance, [5], [7], [8]). More precisely, given $p > 1$, $n = 2$, $\Omega = B(0,1)$, let us define F, $\overline{F} : \mathrm{W}^{1,1}(\Omega) \to [0,+\infty]$ as

$$F(v) = \int_\Omega |Dv|^p \lambda dx,$$

$$\overline{F}(v) = \inf\{\liminf_{h\to+\infty} F(v_h) : v_h \in \mathcal{C}^1(\Omega), v_h \rightharpoonup v \text{ weakly in } \mathrm{W}^{1,1}(\Omega)\},$$

where λ is given by (2.7), (2.13), or (2.17). These functionals are sequentially lower semicontinuous with respect to the weak convergence in $\mathrm{W}^{1,1}(\Omega)$ (see, for instance, [4]). But if we take $v = u$, u defined in (2.9), as $u \in \mathrm{W}^{1,p}(\Omega,\lambda) \setminus \mathrm{H}^{1,p}(\Omega,\lambda)$, and the spaces $\mathrm{H}^{1,p}(\Omega,\lambda)$, $\mathrm{W}^{1,p}(\Omega,\lambda)$ are the domains of $\overline{F}$ and F respectively, we have that

$$F(u) < +\infty = \overline{F}(u).$$

In particular, for a suitable constant $c > 0$ we have

$$\mathrm{Inf}\Big\{\int_\Omega |Dv|^p \lambda dx + c\int_\Omega |v-u| dx : v \in \mathcal{C}^1(\Omega)\Big\} >$$

$$\mathrm{Inf}\Big\{\int_\Omega |Dv|^p \lambda dx + c\int_\Omega |v-u| dx : v \in \mathrm{W}^{1,1}(\Omega)\Big\}.$$

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