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Quasi-convex Functions and Quasi-monotone Operators

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Dedicated to R. T. Rockafellar on his 60th Birthday

The notions of a quasi-monotone operator and of a cyclically quasi-monotone operator are introduced, and relations between such operators and quasi-convex functions are established.

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1. Introduction

Let E be a Hausdorff locally convex space, X a convex set in it, and E^* the dual space. We say an operator $f : X \to E^*$ is *quasi-monotone* if

$$\min\{ \langle y - x, f(x)\rangle, \langle x - y, f(y)\rangle \} \leq 0 \tag{1.1}$$

for all $x, y \in X$, and *cyclically quasi-monotone* if

$$\min\{ \langle x_{i+1} - x_i, f(x_i)\rangle : i = 0, \dots, k \} \leq 0 \tag{1.2}$$

for all integer k and all cycles $x_0, x_1, \dots, x_k, x_{k+1} = x_0$ in X.
Such operators are closely related to the so-called demand functions in mathematical economics [4]. Denote $X = \mathrm{int}\mathbb{R}^n_+$ and suppose that there exists the single solution $d(p)$ for the extremal problem

$$u(q) \to \max, \quad pq \leq 1,$$

where an increasing utility function u on $\mathbb{R}^n_+$ is assumed to be given, and a price vector $p \in X$ is considered as a parameter. The demand function d, showing demand for n products

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as a vector function of their prices, satisfies the strong axiom of revealed preference due to Houthakker: no cycle $p^0, p^1, \dots, p^k, p^{k+1} = p^0$ can exist in X such that

$$(p^{i+1} - p^i)d(p^i) \leq 0, \ i = 0, 1, \dots, k,$$

and $d(p^i) \neq d(p^j)$ for at least one pair of indices $i, j = 0, \dots, k$ with $i \neq j$. The same condition for $k = 1$ only is known as the weak axiom of revealed preference due to Samuelson. It is easily seen that each axiom implies the corresponding quasi-monotonicity property for the operator $f = -d : X \to \mathbb{R}^n$, the strong axiom implies (1.2), and the weak axiom implies (1.1).

The simplest examples of quasi-monotone (resp. cyclically quasi-monotone) operators are monotone (resp. cyclically monotone) ones. Their definitions are obtained by replacing minima with sums in (1.1) and (1.2) respectively.

The notion of a (not necessarily single-valued) monotone operator was first proposed by Kachurovskii [1] (see also [2, 3]), and properties and applications of such operators were studied later by many authors. Cyclically monotone operators were introduced and investigated by Rockafellar [6–9].

Given a smooth function u on a convex domain, characterizations of its convexity in terms of monotonicity or cyclical monotonicity of grad$u$ are known as follows:

$$u \text{ is convex } \Leftrightarrow \ \text{grad}u \text{ is monotone } \Leftrightarrow \ \text{grad}u \text{ is cyclically monotone.}$$

The goal of the present paper is to get similar characterization theorems connecting quasi-convex functions with quasi-monotone and cyclically quasi-monotone operators.

2. Characterization theorems

Let X be a convex set in a real vector space. Recall that a function $u : X \to \mathbb{R}^1$ is said to be *quasi-convex* if its sublevel sets

$$\{ x : u(x) \leq \alpha \}, \quad \alpha \in \mathbb{R}^1,$$

are convex or, which is the same thing, if

$$u((1-t)x + ty) \leq \max\{u(x), u(y)\}$$

whenever $x, y \in X$, $0 < t < 1$.

It follows from this definition that u is quasi-convex if and only if all the functions $\varphi(u, x, y; \cdot)$, $x, y \in X$, on the segment $[0, 1]$ are so, where

$$\varphi(u, x, y; t) : = u((1-t)x + ty). \tag{2.1}$$

Theorem 2.1. *Let X be a convex open set in a Hausdorff locally convex space E, and suppose that a function $u : X \to \mathbb{R}^1$ has two properties as follows:*

(i) *u is Gâteaux differentiable on X, i.e. for every $x \in X$ there exists an element* grad$u(x) \in E^*$ *such that*

$$\lim_{t \to 0} \frac{u(x + th) - u(x)}{t} = \langle h, \ \text{grad}u(x) \rangle \quad \text{for all } h \in E; \tag{2.2}$$

(ii) *for every* $x, y \in X$ *the function* $\varphi(u, x, y; \cdot)$ *given by* (2.1) *is absolutely continuous on* $[0, 1]$.

The following assertions are then equivalent:

(a) u *is quasi-convex,*

(b) *the operator* $\operatorname{grad} u$ *is cyclically quasi-monotone,*

(c) *the operator* $\operatorname{grad} u$ *is quasi-monotone.*

Remark 2.2. Assumptions (i) and (ii) are clearly satisfied if u is C^1. For the finite-dimensional case of the theorem, see also [5, Proposition 3.1].

Before to pass on to a more general characterization theorem, let us formulate two assumptions, (D1) and (D2), on a function $u : X \to \mathbb{R}^1$, where X is a convex set in a vector space. The assumptions are expressed in terms of the functions $\varphi(u, x, y; \cdot)$ (see (2.1)) as follows:

(D1) for every $x, y \in X$ and every t, $0 \le t < 1$, there exists the right derivative

$$\mathrm{D}\varphi(u, x, y; t) := \lim_{\Delta t \downarrow 0} \frac{\varphi(u, x, y; t + \Delta t) - \varphi(u, x, y; t)}{\Delta t};$$

(D2) for every $x, y \in X$, $\varphi(u, x, y; \cdot)$ is absolutely continuous on $[0, 1]$.

It follows from (D1) and (D2) that

$$\int_0^1 \mathrm{D}\varphi(u, x, y; t)\, dt = u(y) - u(x) \quad \text{whenever } x, y \in X. \tag{2.3}$$

Note that if (D1) holds, then for every $x, y \in X$ the directional derivative is defined as follows:

$$u'(x, y - x) := \mathrm{D}\varphi(u, x, y; 0) = \lim_{t \downarrow 0} \frac{u(x + t(y - x)) - u(x)}{t}. \tag{2.4}$$

Theorem 2.3. *Let* X *be a convex subset in a vector space and* $u : X \to \mathbb{R}^1$ *satisfy* (D1) *and* (D2). *The following assertions are then equivalent:*

(a) u *is quasi-convex;*

(b) *for every integer* k *and every cycle* $x_0, x_1, \ldots, x_k, x_{k+1} = x_0$ *in* X, *the inequality*

$$\min\{u'(x_i, x_{i+1} - x_i) : i = 0, \ldots, k\} \le 0$$

holds;

(c) *for every* $x, y \in X$, *the inequality*

$$\min\{u'(x, y - x), u'(y, x - y)\} \le 0$$

holds.

Remark 2.4. Similarly to the case of cyclically monotone operators, the notion of cyclical quasi-monotonicity can be generalized in the natural way on multivalued operators. A well-known result of Rockafellar [6,9] asserts that subdifferentials of proper semi-continuous convex functions are characterized as maximal cyclically monotone operators. Here a multivalued cyclically monotone (resp. cyclically quasi-monotone) operator f is called maximal if no cyclically monotone (resp. cyclically quasi-monotone) operator g exists with $\mathrm{gr} f \subset \mathrm{gr} g$. This Rockafellar's characterization theorem cannot be generalized on quasi-convex functions, because for any (single-valued) cyclically quasi-monotone operator f, the operator $g(x) := \{\alpha f(x) : \alpha \geq 0\}$ is cyclically quasi-monotone as well, and $\mathrm{gr} f \subset \mathrm{gr} g$ provided $f \not\equiv 0$. It follows that for every non-constant smooth quasi-convex function u the single-valued cyclically quasi-monotone operator $f = \mathrm{grad} u$ is not maximal.

3. Proofs

Proof of Theorem 2.1 Observe that if u is Gâteaux differentiable on X, then (D1) holds and $u'(x, y - x) = \langle y - x, \mathrm{grad} u(x)\rangle$ for all $x, y \in X$. Theorem 2.1 is then a direct consequence of Theorem 2.3.

Proof of Theorem 2.3 (a)⇒(b). If (b) fails, then for some cycle $x_0, x_1, \dots, x_{k+1} = x_0$ in X,

$$u'(x_i, x_{i+1} - x_i) > 0, \quad i = 0, 1, \dots, k.$$

We have $\mathrm{D}\varphi(u, x_i, x_{i+1}; 0) > 0$, so

$$\varphi(u, x_i, x_{i+1}; 0) < \varphi(u, x_i, x_{i+1}; t) \quad \text{for small } t > 0, \tag{3.1}$$

and as $\varphi(u, x_i, x_{i+1}; \cdot)$ is quasi-convex on $[0, 1]$, it follows from (3.1) that

$$\varphi(u, x_i, x_{i+1}; 0) < \varphi(u, x_i, x_{i+1}; 1),$$

i.e. $u(x_i) < u(x_{i+1})$. We obtain a contradictory chain of inequalities

$$u(x_0) < u(x_1) < \ldots < u(x_k) < u(x_0),$$

and the contradiction means that (b) is true.
(b)⇒(c). Obvious.
(c)⇒(a). Suppose u is not quasi-convex. There exist then $x, y \in X$ and t_0, $0 < t_0 < 1$, such that

$$u((1 - t_0)x + t_0 y) > \max\{u(x), u(y)\}. \tag{3.2}$$

We claim that there exist t_1 and t_2, $0 < t_1 < t_0 < t_2 < 1$, such that

$$\mathrm{D}\varphi(u, x, y; t_1) > 0 \tag{3.3}$$

and

$$\mathrm{D}\varphi(u, y, x; 1 - t_2) > 0. \tag{3.4}$$

Indeed, if $\mathrm{D}\varphi(u, x, y; t) \leq 0$ for all t, $0 < t < t_0$, then, taking into account (2.3) and the identity

$$\varphi(u, x, (1 - t_0)x + t_0 y; t) = \varphi(u, x, y; t_0 t),$$

we obtain

$$\begin{aligned} u((1-t_0)x+t_0y) &= u(x) + \int_0^1 \mathrm{D}\varphi(u,x,(1-t_0)x+t_0y;t)\,dt \\ &= u(x) + \int_0^1 \mathrm{D}\varphi(u,x,y;t_0t)\,dt \\ &= u(x) + \frac{1}{t_0}\int_0^{t_0} \mathrm{D}\varphi(u,x,y;\tau)\,d\tau \le u(x), \end{aligned}$$

which contradicts (3.2).
Similarly, if $\mathrm{D}\varphi(u,y,x;t) \le 0$ for all t, $0<t<1-t_0$, then

$$\begin{aligned} u((1-t_0)x+t_0y) &= u(y) + \int_0^1 \mathrm{D}\varphi(u,y,(1-t_0)x+t_0y;t)\,dt \\ &= u(y) + \int_0^1 \mathrm{D}\varphi(u,y,x;(1-t_0)t)\,dt \\ &= u(y) + \frac{1}{1-t_0}\int_0^{1-t_0} \mathrm{D}\varphi(u,y,x;\tau)\,d\tau \le u(y), \end{aligned}$$

which again contradicts (3.2). The claim is thus proved.
Set now

$$x_k := (1-t_k)x + t_ky, \quad k=1,2,$$

and, by using (3.3) (3.4) and the identities

$$\begin{aligned} \varphi(u,x_1,x_2;t) &= \varphi(u,x,y;\ t_1+t(t_2-t_1)), \\ \varphi(u,x_2,x_1;t) &= \varphi(u,y,x;\ 1-t_2+t(t_2-t_1)), \end{aligned}$$

one obtains

$$\begin{aligned} u'(x_1,x_2-x_1) &= \mathrm{D}\varphi(u,x_1,x_2;0) = (t_2-t_1)\ \mathrm{D}\varphi(u,x,y;t_1) > 0, \\ u'(x_2,x_1-x_2) &= \mathrm{D}\varphi(u,x_2,x_1;0) = (t_2-t_1)\ \mathrm{D}\varphi(u,y,x;1-t_2) > 0, \end{aligned}$$

hence

$$\min\{u'(x_1,x_2-x_1), u'(x_2,x_1-x_2)\} > 0,$$

a contradiction with (c).

Remark : After submitting the present paper, in November 1994, I have seen a manuscript by Aussel, Corvellec and Lassonde [10], where some related results on connections between (non-differentiable) quasi-convex functions and multivalued quasi-monotone operators in Banach spaces are proved in a different way.

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