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Existence of Solutions for Unilateral Problems With Multivalued Operators

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Dedicated to R. T. Rockafellar on his 60th Birthday

We prove some new results about the existence of solutions of variational inequalities with quasilinear operators having the generalized pseudo-monotone property. We also consider the case where data are Radon measures or L^1 elements.

Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, $1 < p < n$, $\psi \in L^\infty(\Omega)$. The problem we are going to consider, when f lies in the dual space of $H_0^{1,p}(\Omega)$, has the form

$$\begin{cases} \text{find} \quad u \in H^{1,p}(\Omega), \quad u \geq \psi \\ \text{find } \zeta \in \mathcal{A}(u) \quad \text{satisfying} \\ \langle \zeta, v-u \rangle \geq \langle f, v-u \rangle \quad \text{for any} \quad v \in H_0^{1,p}(\Omega), \quad v \geq \psi, \end{cases} \tag{0}$$

where $\mathcal{A}$ is a multivalued operator defined on $H_0^{1,p}(\Omega)$ with values in its dual $H^{-1,p'}(\Omega)$ ($p' = p/(p-1)$). More precisely we take a multivalued map a defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ with values in $\mathbb{R}^n$, maximal monotone with respect to the last variable. Then $\zeta \in \mathcal{A}(u)$ if and only if $\zeta = -\operatorname{div} g$ and g is a measurable selection of the map

$$x \in \Omega \mapsto a(x, u(x), Du(x)) \subset \mathbb{R}^n.$$

When a is single-valued and strictly monotone with respect to the last variable, some particular case of this problem was developed for example by Pascali and Sburlan in [8], ch. VI.
When $f \in L^1(\Omega)$, after a suitable formulation of the inequality in (0), various existence results, which consider particular cases of single valued and strictly monotone operators, were obtained for example by L. Boccardo and T. Gallouet in [2] and L. Boccardo and

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G.R. Cirmi in [3]. Another kind of existence result for single valued operators was provided by J.M. Rakotoson in [9] for the case of f Radon measure.

In this work we first give, in section 2, an existence theorem with f in $H^{-1,p'}(\Omega)$ and a multivalued operator $\mathcal{A}$ of the above kind. The form of the problem (0) is adopted for example by R.T. Rockafellar in [10] and by G. Dal Maso - A. Defranceschi in [5].

In section 3 we deal with the cases where f is a bounded Radon measure or an L^1 function. In the first case we give an existence theorem for the problem stated in analogy with the one in [9]. Then when we take $f \in L^1(\Omega)$, we prove another existence theorem after stating the problem in analogy with the one in [2].

1. Formulation of the problem

Notation and Hypotheses

Let Ω be a bounded open set of $\mathbb{R}^n$, $n \geq 2$, we denote by $H_0^{1,p}(\Omega)$, for $1 < p < \infty$, the usual Sobolev space, by $H^{-1,p'}(\Omega)$ its dual; by $\langle .,. \rangle$ the scalar product on $\mathbb{R}^n$ or the duality between $H_0^{1,p}(\Omega)$ and $H^{-1,p'}(\Omega)$. The symbol $\| \|_{L^p}$ will denote the norm in $L^p(\Omega)^n$ or in $L^p(\Omega)$ and $x_h \rightharpoonup x$ will mean that the sequence $(x_h)_{h\in N}$ of a certain dual of a Banach space, converges to x in the weak topology.

Let $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$ be $\mathcal{L}(\Omega)\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^n)$-measurable, namely for every open set $U \subset \mathbb{R}^n$, $a^{-1}(U) := \{(x,s,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^n\colon U \cap a(x,s,\xi) \neq \emptyset\} \in \mathcal{L}(\Omega)\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(\mathbb{R}^n)$.
For a single or multivalued map F we also denote by $\Gamma(F)$ its graph.
We assume $p \in (1,n)$ and $\alpha, \beta \in \mathbb{R}_+$ such that

$$(1.1.1)\quad \beta < p-1, \quad (1.1.2)\quad \frac{\alpha}{p-1-\beta} < \frac{p^*}{p}, \quad (1.1.3)\quad \frac{\alpha}{p-1-\beta} < p' \qquad (1.1)$$

with $p^* = np/(n-p)$, $p' = p/(p-1)$. Besides we suppose a to be closed and convex valued and satisfying the following conditions:

i) for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $a(x,s,\cdot)\colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is maximal monotone;

ii) there exist $\mu \in L^{p'}(\Omega)$, $\nu \in L^{n/(n-1)}(\Omega)$, $c, c_1, c_2 \in \mathbb{R}_+$ such that

$$|\eta| \leq \mu(x) + c_1|\xi|^{p-1} + c_2|s|^\alpha|\xi|^\beta \qquad \text{(ii1)}$$

$$\langle \eta, \xi \rangle \geq \nu(x) + c|\xi|^p \qquad \text{(ii2)}$$

for a.e. $x \in \Omega$, for every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\eta \in a(x,s,\xi)$.

iii) if $u, v, (u_h)_{h\in N}$, η are given, being $u \in H_0^{1,p}(\Omega)$, $v \in H^{1,p}(\Omega)$, $(u_h)_{h\in N}$ in $H_0^{1,p}(\Omega)$ such that $u_h \to u$ a.e. in Ω and η an $\mathcal{L}(\Omega)$-measurable selection of the map

$$x \in \Omega \mapsto a(x, u(x), Dv(x)) \subset \mathbb{R}^n,$$

then there exists $(\eta_h)_{h\in N}$ converging a.e. to η in Ω, such that, for every $h \in N$, η_h is an $\mathcal{L}(\Omega)$-measurable selection of the map $x \in \Omega \mapsto a(x, u_h(x), Dv(x)) \subset \mathbb{R}^n$.

From now on "measurability" will have the meaning adopted above for the multivalued function a.

Remark 1.1. The assumption iii) is for instance satisfied by all multivalued functions of the form: $a(x,s,\xi) = a_0(x,\xi) + a_1(x,s,\xi)$ with $a_0\colon \Omega\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$, $a_1\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ measurable and $a_1(x,\cdot,\xi)$ continuous for a.e. $x\in\Omega$ and every $\xi\in\mathbb{R}^n$. Indeed if η is an $\mathcal{L}(\Omega)$-measurable selection of the map $x\in\Omega\mapsto a(x,u(x),Dv(x))\in\mathbb{R}^n$, the sequence $(\eta_h)_{h\in N}$ defined by

$$\eta_h(x) = \eta(x) - a_1(x,u(x),Dv(x)) + a_1(x,u_h(x),Dv(x))$$

satisfies iii).
Another case in which iii) is true is given by $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to 2^{\mathbb{R}^n}$ defined by $a(x,s,\xi) = \{\lambda a_0(x,s,\xi) + (1-\lambda)a_1(x,s,\xi),\ \lambda\in[0,1]\}$, where $a_i\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$, $i = 1,2$, are measurable functions, continuous in the second variable for a.e. $x\in\Omega$ and every $\xi\in\mathbb{R}^n$.

Remark 1.2. If $u\colon\Omega\to\mathbb{R}$, $w\colon\Omega\to\mathbb{R}^n$ are $\mathcal{L}(\Omega)$-measurable,

$$x\in\Omega\mapsto a(\cdot,u,w)(x) = a(x,u(x),w(x))\in 2^{\mathbb{R}^n}$$

turns out to be measurable as well.
Indeed, if $U\subset\mathbb{R}^n$ is an open set,

$$\begin{aligned} & a^{-1}(U) \\ = & \{x\in\Omega\colon \exists\eta\in U\cap a(x,u(x),w(x))\} \\ = & \{x\in\Omega\colon \exists\eta\in U, s\in\mathbb{R}, \xi\in\mathbb{R}^n \mid (x,s,\xi,\eta)\in\Gamma(a)\cap(\Gamma((u,w))\times\mathbb{R}^n)\} \\ = & \operatorname{pr}_\Omega(\Gamma(a)\cap(\Gamma((u,w))\times\mathbb{R}^n)\cap(\Omega\times\mathbb{R}\times\mathbb{R}^n\times U)), \end{aligned}$$

so that the assertion follows by Theorem III.23 in [4] (here pr_Ω denotes the projection from $\Omega\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$ into Ω).

Remark 1.3. If $g\colon\Omega\to 2^{\mathbb{R}^n}$ is a measurable selection of $a(\cdot,u,Dv)$, $u,v\in H_0^{1,p}(\Omega)$, then $g\in L^{p'}(\Omega)^n$. Indeed, by hypothesis (ii1) and Young's inequality we have:

$$|g(x)| \le \mu(x) + c_1|Dv(x)|^{p-1} + c_2\frac{p-1-\beta}{p-1}|u(x)|^{\alpha(p-1)/(p-1-\beta)} + c_2\frac{\beta}{p-1}|Dv(x)|^{p-1} \quad (1.2)$$

where by (1.1.2) we have $\alpha p'(p-1)/(p-1-\beta) < p^*$, so that our assertion follows by Sobolev's inequality.

Problem. Let $A\colon H_0^{1,p}(\Omega)\to 2^{L^{p'}(\Omega)^n}$ be defined by

$$A(u) = \{g\in(L^{p'}(\Omega))^n\colon g(x)\in a(x,u(x),Du(x)) \quad \text{for a.e.} \quad x\in\Omega\},$$

and $\mathcal{A}\colon H_0^{1,p}(\Omega)\to 2^{H^{-1,p'}(\Omega)}$ by $\mathcal{A}(u) = \{-\operatorname{div} g\colon g\in A(u)\}$.
If a non-empty closed convex set $K\subset H_0^{1,p}(\Omega)$ and $f\in H^{-1,p'}(\Omega)$ are given, our first problem is stated as follows:

$$\begin{cases} u\in K \\ \exists\zeta\in\mathcal{A}(u) \quad \text{such that} \quad \langle\zeta, v-u\rangle \ge \langle f, v-u\rangle \quad \text{for every} \quad v\in K. \end{cases} \tag{I}$$

Remark 1.4. It is easy to show that problem (I) is equivalent to the relation

$$f \in (\mathcal{A} + \partial I_K)u,$$

where ∂I_K is the subdifferential of the indicator function I_K of K.

2. Existence of solutions for the problem (I)

Lemma 2.1. *If $((u_h, f_h))_{h\in N}$ is a sequence in $H_0^{1,p}(\Omega)\times H^{-1,p'}(\Omega)$ such that $f_h \in \mathcal{A}(u_h)$ for $h \in N$, $u_h \to u \in H_0^{1,p}(\Omega)$ strongly and $f_h \rightharpoonup f \in H^{-1,p'}(\Omega)$ weakly, then $f \in \mathcal{A}(u)$.*

Proof. By the definition of $\mathcal{A}$, for every $h \in N$ there exists $g_h \in L^{p'}(\Omega)^n$ such that $g_h(x) \in a(x, u_h(x), Du_h(x))$ for a.e. $x \in \Omega$ and $f_h = -\operatorname{div} g_h$. Taking (1.2) and the boundedness of $(u_h)_{h\in N}$ in $H_0^{1,p}(\Omega)$ into account, we obtain that the sequence $(g_h)_{h\in N}$ is bounded in $L^{p'}(\Omega)^n$.
Therefore there exists $g \in L^{p'}(\Omega)^n$ such that, by passing to a subsequence if necessary, $g_h \rightharpoonup g$ in $L^{p'}(\Omega)^n$. Now we prove that $f = -\operatorname{div} g$ and $g(x) \in a(x, u(x), Du(x))$ for a.e. $x \in \Omega$. Since $\langle f, v\rangle = \lim_{h\to\infty}\langle -\operatorname{div} g_h, v\rangle = \lim_{h\to\infty}\int_\Omega \langle g_h, Dv\rangle dx = \int_\Omega \langle g, Dv\rangle dx = \langle -\operatorname{div} g, v\rangle$ for every $v \in H_0^{1,p}(\Omega)$, we get $f = -\operatorname{div} g$.
To conclude we show that for every $\xi \in \mathbb{R}^n$ and $\zeta \in a(x, u(x), \xi)$ is $\langle g(x)-\zeta, Du(x)-\xi\rangle \geq 0$ for a.e. $x \in \Omega$. Then from maximal monotonicity of the map $\xi \in \mathbb{R}^n \mapsto a(x, u(x), \xi) \in 2^{\mathbb{R}^n}$ for $x \in \Omega\backslash\Omega_0$ where Ω_0 has Lebesgue measure zero, it follows that $g(x) \in a(x, u(x), Du(x))$. Let η be a measurable selection of $a(\cdot, u, \xi)$. Since, by passing to a subsequence if necessary, $(u_h)_{h\in N}$ converges a.e. to u, by hypothesis iii), there exists a sequence $(\eta_h)_{h\in N}$ converging a.e. to η on Ω, such that for every $h \in N$ η_h is an $\mathcal{L}(\Omega)$-measurable selection of the map $x \in \Omega \mapsto a(x, u_h(x), \xi) \in 2^{\mathbb{R}^n}$. We note first that from (ii1) it follows that $\eta_h \to \eta$ in $L^{p'}(\Omega)^n$, indeed for every $h \in N$ and a.e. $x \in \Omega$ we have:

$$|\eta_h(x)| \leq \mu(x) + c_1|\xi|^{p-1} + c_2|u_h(x)|^\alpha|\xi|^\beta$$

where the right hand side converges in $L^{p'}(\Omega)^n$ as $\alpha p' < p^*$. If $\varphi \in C(\Omega)$, $\varphi \geq 0$, since the map $\xi \in \mathbb{R}^n \mapsto a(x, u_h(x), \xi) \in 2^{\mathbb{R}^n}$ is monotone for a.e. $x \in \Omega$ and for every $h \in N$, we have $0 \leq \int_\Omega \langle g_h - \eta_h, Du_h - \xi\rangle \varphi dx$. Taking the convergence of $(g_h)_{h\in N}$, $(\eta_h)_{h\in N}$, $(Du_h)_{h\in N}$ into account, we have $\lim_{h\to\infty}\int_\Omega \langle g_h - \eta_h, Du_h - \xi\rangle\varphi dx = \int_\Omega \langle g - \eta, Du - \xi\rangle \varphi dx$ and since φ is arbitrary, we obtain $\langle g(x) - \eta(x), Du(x) - \xi\rangle \geq 0$ for a.e. $x \in \Omega$. But the selection η is also arbitrary, so that from Theorem III.9 in [4], which ensures the existence of a sequence $(\sigma_h)_{h\in N}$ of measurable selections of $a(\cdot, u, \xi)$ such that $\{\sigma_h(x) : h \in N\}$ is dense in $a(x, u(x), \xi)$, it follows that $\langle g(x) - \zeta, Du(x) - \xi\rangle \geq 0$ for a.e. $x \in \Omega$ and every $\zeta \in a(x, u(x), \xi)$. □

Lemma 2.2. *If $u \in H_0^{1,p}(\Omega)$ and $\mathcal{A}$ is the operator defined in 1.1, then $\mathcal{A}(u)$ is closed, convex, nonempty and bounded. Moreover $\mathcal{A}: H_0^{1,p}(\Omega) \to 2^{H^{-1,p'}(\Omega)}$ is a bounded operator.*

Proof. From hypothesis (ii1) about a we immediately get boundedness of $\mathcal{A}(u)$: indeed if $-\operatorname{div} g \in \mathcal{A}(u)$, $g \in A(u)$, taking (1.2) and $\alpha p/(p-1-\beta) < p^*$ into account, from Sobolev's inequality it follows that for a suitable constant K

$$\|g\|_{L^{p'}} \leq K(\|\mu\|_{L^{p'}} + (\|Du\|_{L^p})^{p/p'} + (\|Du\|_{L^p})^{\alpha(p-1)/(p-1-\beta)}). \tag{2.1}$$

Since $\|-\operatorname{div} g\|_{H^{-1,p'}} \leq \|g\|_{L^{p'}}$, by (2.1) we obtain also boundedness of $\mathcal{A}$. Moreover a is convex valued, so that the same is true for $\mathcal{A}$.
Now let us prove that $\mathcal{A}(u)$ is nonempty. $a(x,u(x),\cdot)$, and consequently $a(x,u(x),\cdot)^{-1}$, is maximal monotone for a.e. $x \in \Omega$. Moreover, by (ii1), $(a(x,u(x),\cdot)^{-1})^{-1}$ is locally bounded according to definition in [8] 2.2 ch.III, hence the theorem on page 147 in [8] ensures that $a(x,u(x),\cdot)^{-1}$ is surjective. Then for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^n$ is $a(x,u(x),\xi) \neq \emptyset$. Hence for a.e. $x \in \Omega$ is $a(x,u(x),Du(x)) \neq \emptyset$ and by Theorem 9 in [4] ch.III there exists a measurable selection $g: \Omega \to \mathbb{R}^n$ of $a(\cdot,u,Du)$. By Remark 1.2 $-\operatorname{div} g \in \mathcal{A}(u)$.
Lastly, $\mathcal{A}(u)$ is closed in $H^{-1,p'}(\Omega)$: indeed if $(f_h)_{h\in N}$ is a sequence in $\mathcal{A}(u)$ such that $f_h \rightharpoonup \gamma \in H^{-1,p'}(\Omega)$ applying Lemma 2.1 we get $\gamma \in \mathcal{A}(u)$. □

Remark 2.3. Given $u \in H_0^{1,p}(\Omega)$, as in Remark 1.3 we can see that if $g: \Omega \to 2^{\mathbb{R}^n}$ is a measurable selection of $a(\cdot,u,Dv)$, $v \in H_0^{1,p}(\Omega)$, we have $g \in L^{p'}(\Omega)^n$.

Therefore we may define the operator $\mathcal{B}_u: H_0^{1,p}(\Omega) \to 2^{H^{-1,p'}(\Omega)}$, by

$$H_0^{1,p}(\Omega) \ni v \mapsto \mathcal{B}_u(v) := \{-\operatorname{div}\eta : \eta(x) \in a(x,u(x),Dv(x)) \quad \text{for a.e.} \quad x \in \Omega\}.$$

Lemma 2.1 and 2.2 hold, with analogous and more simple proof (hypothesis iii) doesn't need in this case), if we replace $\mathcal{A}$ by $\mathcal{B}_u$.

Definition 2.4. ([8]) If X is a Banach space and X^* its dual, then $T: X \to 2^{X^*}$ is *coercive* if for every selection $\tau: X \to X^*$, $\tau(x) \in T(x)$ for every $x \in X$, we have

$$\lim_{\|x\|\to\infty} \frac{\langle \tau(x), x\rangle}{\|x\|} = \infty.$$

If T is single valued, "coercivity" is given by the same condition with $\tau(x) = T(x)$.

Definition 2.5. ([8]) Let X be a reflexive Banach space.
$T: X \to 2^{X^*}$ is said to have the *generalized pseudo monotone property* if for any sequence $((x_h, f_h))_{h\in N}$ in $\Gamma(T)$ such that $x_h \rightharpoonup x$ in X, $f_h \rightharpoonup f$ in X^* and for which $\limsup\langle f_h, x_h - x\rangle \leq 0$, we have $(x,f) \in \Gamma(T)$ and $\langle f_h, x_h\rangle \to \langle f, x\rangle$.

Definition 2.6. ([8]) Let X be a strictly convex reflexive Banach space and X^* its dual.

a) $P: X \to X^*$ is said to be *smooth* if it is bounded, coercive, maximal monotone.
b) $T: X \to 2^{X^*}$ with the generalized pseudo monotone property is called *regular* if $T + P$ is surjective for any smooth operator $P: X \to X^*$.

Definition 2.7. ([8]) Let X be a reflexive Banach space and $T: X \to 2^{X^*}$.

a) The map T is *upper semicontinuous* in $x \in D(T) := \{x \in X : Tx \neq \emptyset\}$, if for any sequence $(x_h)_{h\in N}$ in $D(T)$, with $x_h \to x$ in X and for any neighbourhood V of $T(x)$, there exists $h_0 \in N$ such that $T(x_h) \subset V$ for $h > h_0$.
b) The map T is said to be of *type* (M) provided that:

M$_1$) $T(x)$ is bounded, convex, closed and nonempty for each $x \in X$;
M$_2$) for each sequence $((x_h, f_h))_{h\in N}$ in $\Gamma(T)$ such that $x_h \rightharpoonup x$ in X, $f_h \rightharpoonup f$ in X^* and for which $\limsup\langle f_h, x_h - x\rangle \leq 0$, we have $(x, f) \in \Gamma(T)$;
M$_3$) T is upper semicontinuous from the finite-dimensional subspaces of X to X^* endowed with the weak topology.

Remark 2.8. If X is a reflexive Banach space, $T: X \to 2^{X^*}$ is bounded and satisfies condition M$_2$), then it satisfies condition M$_3$).
Indeed let x_h, $x \in Y$, $h \in N$, Y be a finite dimensional subspace of X, $x_h \to x$, V be a weakly open neighbourhood of $T(x)$ in X^*. If for infinitely many values of $h \in N$ we have $T(x_h)\backslash V \neq \emptyset$, then there exists an increasing sequence $(h_k)_{k\in N}$ in N such that for every $k \in N$ there exists $\lambda_k \in T(x_{h_k})\backslash V$. By boundedness of T, by passing to a further subsequence if necessary, the sequence $(\lambda_k)_{k\in N}$ converges weakly in X^*, so that $\lim_{k\to\infty}\langle\lambda_k, x_{h_k} - x\rangle = 0$ and by M$_2$) $\lim_{k\to\infty}\lambda_k \in T(x) \subset V$. This contradicts the fact that, as $\lambda_k \notin V$ for every $k \in N$, $\lim_{k\to\infty}\lambda_k \notin V$.

Lemma 2.9. *If $u_0 \in H_0^{1,p}(\Omega)$, define $\mathcal{A}_0: H_0^{1,p}(\Omega) \to 2^{H^{-1,p'}(\Omega)}$ by*

$$\mathcal{A}_0(v) = \mathcal{A}(v + u_0).$$

$\mathcal{A}$ being the operator in (I). *If $P: H_0^{1,p}(\Omega) \to H^{-1,p'}(\Omega)$ is any smooth operator, then $P + \mathcal{A}_0$ is of type* (M).

Proof. From Lemma 2.2 it follows that M$_1$) is satisfied by $P + \mathcal{A}_0$.
In order to obtain M$_2$) we first establish the following propositions:

I) If $u_0 \in H_0^{1,p}(\Omega)$ is given, the operator $\mathcal{B}_{u_0}$ defined in 2.3 is maximal monotone.
II) $\mathcal{A}_0$ satisfies M$_2$).

Proof of I). On account of 2.3, if $v \in H_0^{1,p}(\Omega)$ then $\mathcal{B}_{u_0}(v)$ is closed, convex, non empty and bounded. Moreover it is upper semicontinuous from the line segments in $H_0^{1,p}(\Omega)$ to $H^{-1,p'}(\Omega)$ endowed with the weak topology: if V is a weakly open neighborhood of $\mathcal{B}_{u_0}(v)$ and $(v_h)_{h\in N}$ strongly converges to v in $H_0^{1,p}(\Omega)$, we can find $k \in N$ such that $\mathcal{B}_{u_0}(v_h) \subset V$ when $h > k$. On the contrary let us suppose that for an increasing sequence $(h_k)_{k\in N}$ in N, be $f_k \in \mathcal{B}_{u_0}(v_{h_k})\backslash V$. Being $\mathcal{B}_{u_0}$, as remarked in 2.3, a bounded operator, we can extract a subsequence from $(f_k)_{k\in N}$, whose limit, Lemma 2.1 being true with $\mathcal{B}_{u_0}$ in place of $\mathcal{A}$, belongs to $\mathcal{B}_{u_0}(v)$. Then it belongs to V, contrary to the fact that $f_k \notin V$ for each $k \in N$. Monotonicity of $\mathcal{B}_{u_0}$ follows from the fact that $a(x, u_0(x), \cdot)$ is monotone for a.e. $x \in \Omega$. Then $\mathcal{B}_{u_0}$ verifies all the hypothesis of [8] theorem in 2.3 ch.III, so that it is maximal monotone.

Proof of II). Let $((u_h, f_h))_{h\in N}$ be a given sequence in $H_0^{1,p}(\Omega) \times H^{-1,p'}(\Omega)$ such that $f_h \in \mathcal{A}(u_h)$ for $h \in N$, $u_h \rightharpoonup u$ in $H_0^{1,p}(\Omega)$, $f_h \rightharpoonup f$ in $H^{-1,p'}(\Omega)$ and $\limsup\langle f_h, u_h - u\rangle \leq 0$.

If $g_h \in L^{p'}(\Omega)^n$, $g_h(x) \in a(x, u_h(x), Du_h(x))$ for a.e. $x \in \Omega$ and $f_h = -\operatorname{div} g_h$, inequality (2.1) and boundedness of $(u_h)_{h\in N}$ in $H_0^{1,p}(\Omega)$ guarantee that $(g_h)_{h\in N}$ is bounded in $L^{p'}(\Omega)^n$. Therefore there exists $g \in L^{p'}(\Omega)^n$ such that, by extracting a subsequence if necessary, $g_h \rightharpoonup g$ in $L^{p'}(\Omega)^n$. In analogy to the proof in 2.1, we obtain $-\operatorname{div} g = f$.

Now we conclude by proving that if $v \in H_0^{1,p}(\Omega)$ and $-\operatorname{div}\eta \in \mathcal{B}_u(v)$, then $0 \le \langle -\operatorname{div} g - (-\operatorname{div}\eta), u - v\rangle$; this fact implies $-\operatorname{div} g \in \mathcal{B}_u(u) = \mathcal{A}(u)$ by maximal monotonicity of $\mathcal{B}_u$. Let $\eta \in L^{p'}(\Omega)$ be such that $\eta(x) \in a(x, u(x), Dv(x))$ for a.e. $x \in \Omega$. By Rellich's theorem there exists a subsequence of $(u_h)_{h\in N}$ converging to u strongly in $L^p(\Omega)$. Hence, by passing to a further subsequence if necessary, we can suppose $(u_h)_{h\in N}$ converging a.e. on Ω to u. Let $(\eta_h)_{h\in N}$ be given by hypothesis iii) in connection with $u, v, \eta, (u_h)_{h\in N}$, such that $\eta_h \to \eta$ a.e. on Ω and η_h is, for each $h \in N$, an $\mathcal{L}(\Omega)$-measurable selection of $x \in \Omega \mapsto a(x, u_h(x), Dv(x)) \in \mathbb{R}^n$. By (1.2) we have for $h \in N$ and a.e. $x \in \Omega$:

$$|\eta_h(x)| \le \mu(x) + c_1|Dv(x)|^{p-1} + c_2\frac{p-1-\beta}{p-1}|u_h(x)|^{\alpha(p-1)/(p-1-\beta)} + c_2\frac{\beta}{p-1}|Dv(x)|^{p-1}.$$

The sequence on the right hand side converges in $L^{p'}(\Omega)$ strongly as (1.1.1) and (1.1.2) give $\alpha p'(p-1)/(p-1-\beta) < p^*$. So $\eta_h \to \eta$ in $L^{p'}(\Omega)^n$ strongly.

Having $\eta_h(x) \in a(x, u_h(x), Dv(x))$, from monotonicity we get:

$$\begin{aligned} 0 &\le \limsup \int_\Omega \langle g_h(x) - \eta_h(x), D(u_h - v)(x)\rangle dx \\ &= \limsup \left(\int_\Omega \langle g_h, D(u_h - u)\rangle dx + \int_\Omega \langle g_h, D(u - v)\rangle dx - \int_\Omega \langle \eta_h, D(u_h - v)\rangle dx \right) \\ &= \limsup \langle f_h, u_h - u\rangle + \int_\Omega (\langle g, D(u - v)\rangle - \langle \eta, D(u - v)\rangle) dx \\ &\le \langle -\operatorname{div} g - (-\operatorname{div}\eta), u - v\rangle. \end{aligned}$$

This concludes that $\mathcal{A}$, and consequently $\mathcal{A}_0$, satisfies condition M_2).

We return to the proof of the lemma.

Now we prove property M_2) for $P + \mathcal{A}_0$, where P is assumed to be smooth.

Let $((v_h, f_h))_{h\in N}$ in $H_0^{1,p}(\Omega) \times H^{-1,p'}(\Omega)$ be such that $f_h \in (P + \mathcal{A}_0)(v_h)$ for each $h \in N$, $v_h \rightharpoonup v$ in $H_0^{1,p}(\Omega)$, $f_h \rightharpoonup f$ in $H^{-1,p'}(\Omega)$ and $\limsup\langle f_h, v_h - v\rangle \le 0$. We will show that $f \in (P + \mathcal{A}_0)(v)$. Taking $b_h \in \mathcal{A}_0(v_h)$ so that $f_h = P(v_h) + b_h$, from boundedness of P and $\mathcal{A}_0$, due to Lemma 2.2, we get $b, d \in H^{-1,p'}(\Omega)$ such that, by extracting a subsequence if necessary, $b_h \rightharpoonup b$ and $P(v_h) \rightharpoonup d$. Hence from monotonicity of P we get:

$$\begin{aligned} &\limsup\langle b_h, v_h - v\rangle = \\ &= \limsup(\langle P(v_h) + b_h, v_h - v\rangle - \langle P(v_h) - P(v), v_h - v\rangle - \langle P(v), v_h - v\rangle) \\ &\le \limsup(\langle P(v_h) + b_h, v_h - v\rangle - \langle P(v), v_h - v\rangle \\ &= \limsup\langle f_h, v_h - v\rangle \\ &\le 0. \end{aligned}$$

Proposition II) gives $b \in \mathcal{A}_0(v)$, hence we may take $\gamma\colon \Omega \to \mathbb{R}^n$ measurable such that $\gamma(x) \in a(x, (v+u_0)(x), D(v+u_0)(x))$ for a.e. $x \in \Omega$ and $b = -\operatorname{div}\gamma$. Extracting a subsequence if necessary, we can suppose $(v_h + u_0)_{h\in N}$ a.e. convergent to $v + u_0$. Then, like in the similar case shown in the proof of II) above, by iii) there exists, in connection with $v+u_0$, $v+u_0$, γ, $(v_h+u_0)_{h\in N}$, a sequence $(\gamma_h)_{h\in N}$ converging strongly to γ in $L^{p'}(\Omega)^n$ and such that γ_h is, for each $h \in N$, a measurable selection of the map

$$x \in \Omega \mapsto a(x, (v_h+u_0)(x), D(v+u_0)(x)) \in \mathbb{R}^n.$$

Thus, if $g_h \in L^{p'}(\Omega)^n$, $g_h(x) \in a(x, (v_h+u_0)(x), D(v_h+u_0)(x))$ for a.e. $x \in \Omega$ and $b_h = -\operatorname{div} g_h$ for $h \in N$, from monotonicity of $a(x, (v_h+u_0)(x), \cdot)$, we get:

$$\begin{aligned}\langle P(v_h), v_h - v\rangle &= \langle P(v_h) + b_h, v_h - v\rangle - \langle b_h - (-\operatorname{div}\gamma_h), v_h - v\rangle + \langle \operatorname{div}\gamma_h, v_h - v\rangle \\ &= \langle P(v_h) + b_h, v_h - v\rangle - \int_\Omega \langle g_h - \gamma_h, D(v_h + u_0 - (v+u_0))\rangle dx \\ &\quad - \int_\Omega \langle \gamma_h, D(v_h - v)\rangle dx \\ &\le \langle f_h, v_h - v\rangle - \int_\Omega \langle \gamma_h, D(v_h - v)\rangle dx\end{aligned}$$

and therefore $\limsup\langle P(v_h), v_h - v\rangle \le 0$. On the other hand by Proposition in 5.2 ch.III of [8], the operator P, being maximal monotone, satisfies M_2), consequently the previous inequality ensures that $P(v) = d$. Thus we conclude that $f = d + b \in (P + \mathcal{A}_0)(v)$ and $P + \mathcal{A}_0$ satisfies M_2).
Property M_3) follows from boundedness of $P + \mathcal{A}_0$ and Remark 2.8. □

Lemma 2.10. *If $u_0 \in H_0^{1,p}(\Omega)$, the operator $\mathcal{A}_0$, defined in 2.9, is coercive.*

Proof. Let $G\colon H_0^{1,p}(\Omega) \to H^{-1,p'}(\Omega)$ be such that $G(v) \in \mathcal{A}_0(v)$ for each $v \in H_0^{1,p}(\Omega)$, and $g_v \in L^{p'}(\Omega)^n$, $g_v(x) \in a(x, (v+u_0)(x), D(v+u_0)(x))$ for a.e. $x \in \Omega$, be such that $G(v) = -\operatorname{div} g_v$. By (2.1) we have

$$\|g_v\|_{L^{p'}} \le K(\|\mu\|_{L^{p'}} + (\|v+u_0\|_{H_0^{1,p}})^{p/p'} + (\|v+u_0\|_{H_0^{1,p}})^{\alpha(p-1)/(p-1-\beta)}).$$

Then by using coercivity ii2), Sobolev's and Hölder's inequalities:

$$\begin{aligned}\langle G(v), v\rangle =&\ \int_\Omega \langle g_v, D(v+u_0)\rangle dx - \int_\Omega \langle g_v, Du_0\rangle dx \\ \ge&\ \int_\Omega (\nu + c|D(v+u_0)|^p)dx - \|g_v\|_{L^{p'}}\|Du_0\|_{L^p} \\ \ge&\ -\|\nu\|_{L^1} + \overline{c}(\|v+u_0\|_{H_0^{1,p}})^p - K\|Du_0\|_{L^p}(\|\mu\|_{L^{p'}} + (\|v+u_0\|_{H_0^{1,p}})^{p/p'} + \\ &+ (\|v+u_0\|_{H_0^{1,p}})^{\alpha(p-1)/(p-1-\beta)}),\end{aligned}$$

where $\overline{c} > 0$ is a suitable constant.

Since from condition (1.1.3) it follows that $\alpha(p-1)/(p-1-\beta) < p$, the above inequality gives the desired result. □

Definition 2.11. ([8]) Let X be a normed space and X^* its dual. A map $T\colon X \to 2^{X^*}$ is *quasi bounded* if to each $M > 0$ there corresponds a $C > 0$ such that for each $x \in X$, $\|x\| \leq M$, if $f \in Tx$ satisfies $\langle f, x\rangle \leq M\|x\|$ then $\|f\| \leq C$.

Theorem 2.12. ([8] theorem 3.5 ch.III) *Let X be a reflexive strictly convex Banach space and X^* its dual. If $T\colon X \to 2^{X^*}$ is maximal monotone, $0 \in D(T)$ and $H\colon X \to 2^{X^*}$ is quasi bounded, regular and coercive, then $T + H$ is surjective.*

Theorem 2.13. *Assuming the hypotheses described in section* 1*, then there exists a solution of the problem* (I)*.*

Proof. By Remark 1.4 it suffices to prove that if $f \in H^{-1,p'}(\Omega)$ is given, there exists $u \in H_0^{1,p}(\Omega)$ such that $f \in (\mathcal{A} + \partial I_K)u$. For this purpose we show that if $u_0 \in K$ is given, then Theorem 2.12 can be applied to the case $X = H_0^{1,p}(\Omega)$, $T(v) = \partial I_K(v + u_0)$, $H(v) = \mathcal{A}(v + u_0)$ for every $v \in H_0^{1,p}(\Omega)$.
First we show that $\mathcal{A}$ has the generalized pseudo-monotone property (Def. 2.5). Let $((v_h, f_h))_{h\in N}$ be a sequence in $H_0^{1,p}(\Omega) \times H^{-1,p'}(\Omega)$ such that $f_h \in \mathcal{A}(v_h)$ for $h \in N$, $v_h \rightharpoonup v$ in $H_0^{1,p}(\Omega)$, $f_h \rightharpoonup f$ in $H^{-1,p'}(\Omega)$ and $\limsup\langle f_h, v_h - v\rangle \leq 0$. We proved in Lemma 2.9 that $\mathcal{A}$ satisfies M_2), hence $f \in \mathcal{A}(v)$. For every $h \in N$ we may write $f_h = -\operatorname{div} g_h$, $f = -\operatorname{div} g$, $g_h(x) \in a(x, v_h(x), Dv_h(x))$, $g(x) \in a(x, v(x), Dv(x))$ for a.e. $x \in \Omega$. Like in the similar case shown in the proof of II) in Lemma 2.9, by hypothesis iii) there exists a sequence $(\gamma_h)_{h\in N}$ in connection with v, v, g, $(v_h)_{h\in N}$, such that $\gamma_h \to g$ in $L^{p'}(\Omega)^n$ and γ_h is a measureable selection of the map $x \in \Omega \mapsto a(x, v_h(x), Dv(x)) \in \mathbb{R}^n$; then, by using monotonicity of $a(x, v_h(x), \cdot)$ we get:

$$
\begin{aligned}
\langle -\operatorname{div} g_h, v_h\rangle &= \int_\Omega \langle g_h, Dv_h\rangle dx = \\
&= \int_\Omega \langle g_h - \gamma_h, Dv_h - Dv\rangle dx + \int_\Omega \langle \gamma_h, Dv_h - Dv\rangle dx + \int_\Omega \langle g_h, Dv\rangle dx \\
&\geq \int_\Omega \langle \gamma_h, Dv_h - Dv\rangle dx + \langle f_h, v\rangle.
\end{aligned}
$$

It follows that $\liminf\langle f_h, v_h\rangle \geq \langle f, v\rangle$ and thus $\lim_{h\to\infty}\langle f_h, v_h\rangle = \langle f, v\rangle$.

It can be easily seen that the operator H, defined by $H(v) = \mathcal{A}(v+u_0)$, has the generalized pseudo-monotone property, too.

We finally see that H is regular. If $P\colon H_0^{1,p}(\Omega) \to H^{-1,p'}(\Omega)$ is smooth, by Lemma 2.9, $P + \mathcal{A}_0$ is of type (M); moreover it is coercive and bounded because P and $\mathcal{A}_0$ are.
Theorem in [8] 5.4, ch.III and subsequent remarks ([8] page 156) applied to $P+\mathcal{A}_0$ ensure its surjectivity, so that H is regular. On the other hand H is bounded and coercive by Lemmas 2.2 and 2.10 so that it satisfies the hypotheses in 2.12. The domain of ∂I_K is K, then by applying Proposition 2.13 of [8] ch.III, we obtain that ∂I_K is maximal monotone.

Thus the same is true for T, defined by $T(u) = \partial I_K(u + u_0)$, and Theorem 2.12 can be applied to conclude the proof. □

3. Existence theorems for problems with measure or L^1 data

Notation 3.1. If any $\psi \in L^\infty(\Omega)$ is given, let:

$$K(\psi) = \{v \in H_0^{1,1}(\Omega): v \geq \psi \quad \text{a.e. on} \quad \Omega\}$$

and $V_0^\infty(\Omega, \psi) = \{\varphi \in \mathcal{D}(\Omega): \forall (v_h)_{h\in N}$ in $H_0^{1,p}(\Omega) \cap K(\psi)$, $\exists (\varphi_h)_{h\in N}$ in $\mathcal{D}(\Omega)$ such that

$$\varphi_h \to \varphi \quad \text{in} \quad \mathcal{D}(\Omega) \quad \text{and} \quad v_h + \varphi_h \in K(\psi) \forall h \in N\}.$$

For $k > 0$ denote $\tau_k: \mathbb{R} \to \mathbb{R}: \tau_k(s) = (s \wedge k) \vee (-k)$ and $v^k := \tau_k \circ v$ for any $v \in H^{1,1}_{\text{loc}}(\Omega)$.

Remark 3.2. Let $g: \Omega \to \mathbb{R}^n$ be a measurable selection of $a(\cdot, u, Du)$. If, together with the assumptions in section 1 on the multivalued map a, we suppose $p \in \left(2 - \frac{1}{n}, n\right)$ and $u \in H_0^{1,r}(\Omega)$ for every $r \in \left[1, \frac{n(p-1)}{n-1}\right)$, then by (1.2) and (1.1.2), it turns out that $g \in L^1(\Omega)$.

Weakly formulated problems

Let $\psi \in L^\infty(\Omega)$ and $K(\psi)$ like in Notation 3.1; we assume $W_0^{1,\infty}(\Omega) \cap K(\psi)$ to be non empty. We denote $p_0 = n(p-1)/(n-1)$ and assume $p \in \left(2 - \frac{1}{n}, n\right)$. We suppose moreover that for $a: \Omega \times \mathbb{R} \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ all conditions stated in section 1 are true but replacing (1.1.2) by:

$$\frac{\alpha}{p - 1 - \beta} < \frac{n-1}{n-p}. \tag{3.1}$$

Problem with measure data. Let $f: \mathcal{B}(\Omega) \to \mathbb{R}$ be a bounded Radon measure and $V_0^\infty(\Omega, \psi)$ like in Notation 3.1. We consider the following problem:

$$\begin{cases} \text{find } u \in K(\psi), u \in H_0^{1,r}(\Omega) \text{ for each } r \in [1, p_0) \text{ and} \\ g \text{ selection of } a(\cdot, u, Du), g \in L^{r/(p-1)}(\Omega)^n \text{ for each } r \in [(p-1) \vee 1, p_0) \\ \text{such that } \displaystyle\int_\Omega \langle g, D\varphi \rangle \geq \int_\Omega \varphi df \text{ for each } \varphi \in V_0^\infty(\Omega, \psi). \end{cases} \tag{II}$$

Problem with L^1 data. Let $f \in L^1(\Omega)$. We consider the following problem:

$$\begin{cases} \text{find } u \in K(\psi), u \in H_0^{1,r}(\Omega) \text{ for each } r \in [1, p_0) \text{ and } g \text{ selection of } a(\cdot, u, Du) \\ g \in L^{r/(p-1)}(\Omega)^n \text{ for each } r \in [(p-1) \vee 1, p_0) \text{ such that when } k \geq \|\psi\|_{L^\infty}: \\ u^k \in H_0^{1,p}(\Omega), \langle g, Du^k \rangle \in L^1(\Omega), \displaystyle\int_\Omega \langle g, D(u^k - v) \rangle dx \leq \int_\Omega f(u^k - v) dx \\ \text{for } v \in W_0^{1,\infty}(\Omega) \cap K(\psi). \end{cases} \tag{III}$$

Theorem 3.3. *With the assumptions in subsection* 3.1*, let* $(f_h)_{h\in N}$ *be a sequence in* $H^{-1,p'}(\Omega)\cap L^1(\Omega)$ *such that* $\sup\{\|f_h\|_{L^1}: h\in N\}<\infty$*. For each* $h\in N$ *let* $u_h\in H_0^{1,p}(\Omega)\cap K(\psi)$ *be a solution of the problem (I) relative to* $K=H_0^{1,p}(\Omega)\cap K(\psi)$ *and* $f=f_h$*. Then* $(u_h)_{h\in N}$ *is bounded in* $H_0^{1,r}(\Omega)$ *for each* $r\in[1,p_0)$*; besides if* $g_h\in A(u_h)$*, (A as defined in our first problem* (I)*), satisfies* $\langle -\operatorname{div} g_h, v-u_h\rangle \geq \langle f_h, v-u_h\rangle$ *for every* $v\in H_0^{1,p}(\Omega)\cap K(\psi)$*, then* $(|g_h|^{r/(p-1)})_{h\in N}$ *is bounded in* $L^1(\Omega)$*. If moreover* $k\geq\|\psi\|_{L^\infty}$ *then* $(u_h^k)_{h\in N}$ *is bounded in* $H_0^{1,p}(\Omega)$*.*

Proof. Let $\Omega(h,k)=\{x\in\Omega: k\leq|u_h(x)|<k+1\}$, we prove that if $k\geq\|\psi\|_{L^\infty}$ it follows that

$$\int_{\Omega(h,k)}|Du_h|^p dx\leq F^p \quad\text{where}\quad F^p=\frac{1}{c}\left(\int_\Omega|\nu|dx+\sup\left\{\int_\Omega|f_h|dx: h\in N\right\}\right); \tag{3.2}$$

and:

$$\int_{\Omega_{h,k}}|Du_h|^r dx\leq H(r,k)\left(\int_\Omega|u_h|^{r^*}dx\right)^{(p-r)/p}, \quad r\in[1,p_0) \tag{3.3}$$

where $\Omega_{h,k}=\{x\in\Omega: |u_h(x)|\geq k\}$ and $H(r,k)=F^r\left(\sum_{j=k}^\infty\left(\frac{1}{j^{r^*}}\right)^{(p-r)/r}\right)^{r/p}$ ($H(r,k)$ is a positive real number because $r^*(p-r)/r>1$ if $r\in[1,p_0)$).

For $k\geq 0$ let $\varphi_k:\mathbb{R}\to\mathbb{R}$ be the odd function defined by

$$\varphi_k(t)=\begin{cases}0 & \text{if } t\in[0,k]\\ t-k & \text{if } t\in(k,k+1]\\ 1 & \text{if } t\in(k+1,\infty)\end{cases}.$$

Observing that $u_h-\varphi_k(u_h)\in H_0^{1,p}(\Omega)\cap K(\psi)$ if $k\geq\|\psi\|_{L^\infty}$, from the inequality $\int_\Omega\langle g_h, D(u_h-v)\rangle dx\leq\int_\Omega f_h(u_h-v)dx$ which holds for $v\in H_0^{1,p}(\Omega)\cap K(\psi)$, $h\in N$, we get $\int_\Omega\langle g_h, D\varphi_k(u_h)\rangle dx\leq\int_\Omega f_h\varphi_k(u_h)dx\leq\int_\Omega|f_h|dx$. Moreover by (ii2) we have

$$\begin{aligned}\int_\Omega\langle g_h, D\varphi_k(u_h)\rangle dx &= \int_\Omega\langle g_h, \varphi_k'(u_h)Du_h\rangle dx\\ &= \int_{\Omega(h,k)}\langle g_h, Du_h\rangle dx\\ &\geq \int_{\Omega(h,k)}(\nu+c|Du_h|^p)dx,\end{aligned}$$

hence $\int_{\Omega(h,k)}|Du_h|^p dx\leq\frac{1}{c}\left(\int_\Omega|\nu|dx+\sup\left\{\int_\Omega|f_h|dx: h\in N\right\}\right)$, namely (3.2).

Now from (3.2):

$$\int_{\Omega(h,k)} |Du_h|^r dx \leq \left(\int_{\Omega(h,k)} |Du_h|^p dx\right)^{r/p} |\Omega(h,k)|^{(p-r)/p}$$
$$\leq F^r \left(\int_{\Omega(h,k)} |u_h|^{r^*} dx\right)^{(p-r)/p} \left(\frac{1}{k^{r^*}}\right)^{(p-r)/p}.$$

It follows that:

$$\int_{\Omega_{h,k}} |Du_h|^r dx = \sum_{j=k}^{\infty} \int_{\Omega(h,j)} |Du_h|^r dx$$
$$\leq F^r \sum_{j=k}^{\infty} \left(\int_{\Omega(h,j)} |u_h|^{r^*} dx\right)^{(p-r)/p} \left(\frac{1}{j^{r^*}}\right)^{(p-r)/p}$$
$$\leq F^r \left(\sum_{j=k}^{\infty} \int_{\Omega(h,j)} |u_h|^{r^*} dx\right)^{(p-r)/p} \left(\sum_{j=k}^{\infty} \left(\frac{1}{j^{r^*}}\right)^{(p-r)/r}\right)^{r/p}$$
$$\leq H(r,k) \left(\int_{\Omega} |u_h|^{r^*} dx\right)^{(p-r)/p},$$

and (3.3) is proved.

Now observing that $u_h - u_h^k + w \in H_0^{1,p}(\Omega) \cap K(\psi)$ when $k \geq \|\psi\|_{L^\infty}$, we have $\int_\Omega \langle g_h, D(u_h^k - w)\rangle dx \leq \int_\Omega f_h(u_h^k - w)dx$, thus $\int_\Omega \langle g_h, Du_h^k\rangle dx \leq \int_\Omega \langle g_h, Dw\rangle dx + (k + \|w\|_{L^\infty}) \sup\{\|f_h\|_{L^1}: h \in N\}$. As $p-1$ is less than p_0, to prove our theorem we may suppose $r > p-1$, so that by Holder's inequality:

$$\int_\Omega \langle g_h, Du_h^k\rangle dx \leq \left(\int_\Omega |g_h|^{\frac{r}{p-1}} dx\right)^{\frac{p-1}{r}} \left(\int_\Omega |Dw|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}} + \tag{3.4}$$
$$+ (k + \|w\|_{L^\infty}) \sup\{\|f_h\|_{L^1}: h \in N\} \qquad \text{for any} \quad k \geq \|\psi\|_{L^\infty}.$$

By (1.2) there exists $K_1, K_2 \in \mathbb{R}_+$ and $m \in L^1(\Omega)$ such that

$$|g_h(x)|^{p'} \leq m(x) + K_1|Du_h(x)|^p + K_2|u_h(x)|^{\alpha p/(p-1-\beta)} \quad \text{for a.e.} \quad x \in \Omega$$

which by means of (ii2), letting $\omega = m - \nu\frac{1}{c}K_1$ and $K = \frac{1}{c}K_1$, gives

$$|g_h(x)|^{p'} \leq \omega(x) + K\langle g_h(x), Du_h(x)\rangle + K_2|u_h(x)|^{\alpha p/(p-1-\beta)} \quad \text{for a.e.} \quad x \in \Omega.$$

Let $k \geq \|\psi\|_{L^\infty}$ and $r \in ((p-1) \vee 1, p_0)$ be fixed and $\Omega^{h,k} = \{x \in \Omega: |u_h(x)| \leq k\}$, hence:

$$\int_{\Omega^{h,k}} |g_h|^{p'} dx \leq \int_{\Omega^{h,k}} (\omega + K\langle g_h, Du_h^k\rangle + K_2|u_h|^{\alpha p/(p-1-\beta)})dx$$

and by (3.4):

$$\begin{aligned}&\int_{\Omega^{h,k}} |g_h|^{p'} dx \leq \\ &\leq \int_\Omega (|\omega| + K_2 k^{\alpha p/(p-1-\beta)}) dx + K(k + \|w\|_{L^\infty}) \sup\{\|f_h\|_{L^1} : h \in N\} + \\ &\quad + K \left(\int_\Omega |g_h|^{\frac{r}{p-1}} dx\right)^{\frac{p-1}{r}} \left(\int_\Omega |Dw|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{r}}.\end{aligned} \tag{3.5}$$

On the other hand by $r < p_0$ and $p < n$ we have $r/(p-1) < p'$, thus letting

$$c(k,r) = |\Omega|^{(p-r)/p} (K(k + \|w\|_{L^\infty}) \sup\{\|f_h\|_{L^1} : h \in N\} + \int_\Omega (|\omega| + K_2 k^{\alpha p/(p-1-\beta)}) dx)^{r/p},$$

$c'(r) = |\Omega|^{(p-r)/p} K^{r/p} \left(\int_\Omega |Dw|^{\frac{r}{r-p+1}} dx\right)^{\frac{r-p+1}{p}}$, it turns out that

$$\int_{\Omega^{h,k}} |g_h|^{\frac{r}{p-1}} dx \leq \left(\int_{\Omega^{h,k}} |g_h|^{p'} dx\right)^{r/p} |\Omega|^{(p-r)/p} \leq c(k,r) + c'(r) \left(\int_\Omega |g_h|^{\frac{r}{p-1}} dx\right)^{\frac{1}{p'}}.$$

Again by (1.2) there exist $M_1, M_2 \in \mathbb{R}_+$ such that

$$|g_h(x)|^{\frac{r}{p-1}} \leq |\mu(x)|^{\frac{r}{p-1}} + M_1 |Du_h(x)|^r + M_2 |u_h(x)|^{\alpha r/(p-1-\beta)} \quad \text{for a.e.} \quad x \in \Omega.$$

Since (3.1) gives $n\left(1 - \frac{p-1-\beta}{\alpha}\right) < p_0$, by choosing $r > n\left(1 - \frac{p-1-\beta}{\alpha}\right)$, we have $\frac{\alpha r}{p-1-\beta} < r^*$. Hence by Holder's and Sobolev's inequalities:

$$\begin{aligned}&\left(\int_\Omega |g_h|^{\frac{r}{p-1}} dx\right)^{\frac{1}{p'}} \leq \\ &\leq \left(\int_\Omega (|\mu|^{\frac{r}{p-1}} + M_1 |Du_h|^r) dx + M_3 \left(\int_\Omega |u_h|^{r^*} dx\right)^{\frac{\alpha r}{(p-1-\beta) r^*}}\right)^{\frac{1}{p'}} \\ &\leq M_0 + M \left(\left(\int_\Omega |Du_h|^r dx\right)^{\frac{1}{p'}} + \left(\int_\Omega |Du_h|^r dx\right)^{\frac{\alpha}{(p-1-\beta)p'}}\right),\end{aligned} \tag{3.6}$$

where $M_3, M, M_0 \in \mathbb{R}_+$ are suitable constants.
In the coercivity condition (ii2) we may suppose $\frac{|\nu|}{c} \geq 1$, so that

$|Du_h(x)|^r \leq \left(\frac{2}{c}\right)^{r/(p-1)} (|g_h(x)|^{r/(p-1)} + |\nu(x)|^{r/(p-1)})$ for a.e. $x \in \Omega$. Then by (3.6)

$$\int_{\Omega^{h,k}} |Du_h|^r dx \ \leq$$

$$\leq \left(\frac{2}{c}\right)^{r/(p-1)} [c(k,r) + c'(r) \left(\int_\Omega |g_h|^{\frac{r}{p-1}} dx\right)^{\frac{1}{p'}} + \int_\Omega |\nu|^{\frac{r}{p-1}} dx]$$

$$\leq \left(\frac{2}{c}\right)^{\frac{r}{p-1}} [c(k,r) + c'(r) \left(M_0 + M \left(\left(\int_\Omega |Du_h|^r dx\right)^{\frac{1}{p'}} + \left(\int_\Omega |Du_h|^r dx\right)^{\frac{\alpha}{(p-1-\beta)p'}}\right)\right)$$

$$+ \int_\Omega |\nu|^{\frac{r}{p-1}} dx].$$

By (3.3) and Sobolev's inequality: $\int_{\Omega_{h,k}} |Du_h|^r dx \leq H(r,k) S \left(\int_\Omega |Du_h|^r dx\right)^{r^*(p-r)/rp}$, so that by adding the above inequality:

$$\int_\Omega |Du_h|^r dx \ \leq$$

$$\leq \left(\frac{2}{c}\right)^{\frac{r}{p-1}} [c(k,r) + c'(r) \left(M_0 + M \left(\left(\int_\Omega |Du_h|^r dx\right)^{\frac{1}{p'}} + \left(\int_\Omega |Du_h|^r dx\right)^{\frac{\alpha}{(p-1-\beta)p'}}\right)\right) +$$

$$+ \int_\Omega |\nu|^{\frac{r}{p-1}} dx] + H(r,k) S \left(\int_\Omega |Du_h|^r dx\right)^{r^*(p-r)/rp}.$$

Since $\frac{\alpha}{(p-1-\beta)p'} < 1$ by (1.1.3) and $r^*(p-r)/rp < 1$ as $p < n$, the last inequality gives boundedness of $\left(\int_\Omega |Du_h|^r dx\right)_{h \in N}$. Hence by Sobolev's inequality, $(u_h)_{h\in N}$ is bounded in $H_0^{1,r}(\Omega)$ if $r \in [1, p_0)$.
By using (1.2) again, if $M_4 > 0$ is a suitable constant:

$$\int_\Omega |g_h|^{r/(p-1)} dx \leq M_4 \int_\Omega (|\mu|^{r/(p-1)} + |Du_h|^r + |u_h|^{\alpha r/(p-1-\beta)}) dx.$$

Now (3.1) involves that $\alpha r/(p-1-\beta) < p_0^*$, so that $(|g_h|^{r/(p-1)})_{h \in N}$ is bounded in $L^1(\Omega)$ if $r \in [1, p_0)$ as $(u_h)_{h\in N}$ is bounded in $L^q(\Omega)$ for every $q \in [1, p_0^*)$.
Finally from (3.5), being $k \geq \|\psi\|_{L^\infty}$, we obtain that $(g_h 1_{\Omega^{h,k}})_{h\in N}$ is bounded in $(L^{p'}(\Omega))^n$. By coercivity condition ii2), the same is true for $(u_h^k)_{h \in N}$ in $H_0^{1,p}(\Omega)$. □

Definition 3.4. ([1]) Let Ω be a bounded open set in $\mathbb{R}^n$, the *norm-capacity* of $H_0^{1,p}(\Omega)$ is the map $c_p\colon 2^\Omega \to [0, \infty]$ defined as follows:

$$c_p(K) = \inf\{\|u\|_{H_0^{1,p}} \colon u \in C_0^1(\Omega), u \geq 0 \text{ on } \Omega, u \geq 1 \text{ on } K\} \quad \text{if} \quad K \subset \Omega \quad \text{is compact,}$$

$$c_p(U) = \sup\{c_p(K) \colon K \text{ compact } \subset U\} \quad \text{if} \quad U \subset \Omega \quad \text{is open,}$$

$$c_p(E) = \inf\{c_p(U) : U \text{ open } \supset E\} \quad \text{for arbitrary} \quad E \subset \Omega.$$

Definition 3.5. ([1]) We say that $u: \Omega \to \mathbb{R}$ is c_p-*quasicontinuous*, with c_p defined in 3.4, if for every $\epsilon > 0$ there exists an open set $U_\epsilon \subset \Omega$, with $c_p(U_\epsilon) < \epsilon$, such that $u_{|\Omega \setminus U_\epsilon}$ is continuous.

Proposition 3.6. *Let Ω be a bounded open set in $\mathbb{R}^n$ and c_p the norm-capacity as in* definition 3.4.

i) *If $E \in \mathcal{L}(\Omega)$ and $c_p(E) = 0$, then $|E| = 0$.*

ii) *If $u \in H_0^{1,p}(\Omega)$ there exists $\tilde{u}: \Omega \to \mathbb{R}$ c_p-quasi continuous, such that $u = \tilde{u}$ a.e. on Ω.*

Proof. i) If $E \in \mathcal{L}(\Omega)$ and $|E| > 0$, there exists a compact $K \subset E$ with $|K| > 0$, hence by definition of c_p and its monotonicity $c_p(E) \geq c_p(K) \geq |K|^{1/p}$.
ii) For this proof we refer to [1], Proposition 2.8, or, for a more immediate statement, to [6] Proposition 7.7. Indeed it is possible to verify that, in the case of $H_0^{1,p}(\Omega)$, the two definitions of capacity given in [1] and in [6] coincide with c_p introduced above and moreover the two quasi-continuous representatives coincide except on a set of zero capacity at most. See also [11] and [7] for a general overview on the notion of capacity and relative properties. □

Notation 3.7. If $u \in H_0^{1,p}(\Omega)$ we denote henceforth by $\tilde{u}$ a c_p-quasi continuous representative of u.

Definition 3.8. ([1]) A Radon measure $\mu: \mathcal{L}(\Omega) \to \mathbb{R}$ is said to be of *finite energy* relative to $H_0^{1,p}(\Omega)$ if it is continuous on $(C_0^1(\Omega), \|\|_{H^{1,p}})$.

Proposition 3.9.

i) *If $\mu: \mathcal{L}(\Omega) \to \mathbb{R}$ is a positive and finite energy Radon measure relative to $H_0^{1,p}(\Omega)$ then $\mu(E) = 0$ if $E \in \mathcal{L}(\Omega)$ and $c_p(E) = 0$. Moreover $\tilde{u} \in L^1(\Omega, \mu)$ for any $u \in H_0^{1,p}(\Omega)$.*

ii) *Let $\varphi \in H^{-1,p'}(\Omega)$ be a positive functional. Then there exists a positive finite energy Radon measure μ_φ such that $\langle \varphi, u \rangle = \int_\Omega \tilde{u} d\mu_\varphi$ for every $u \in H_0^{1,p}(\Omega)$.*

Proof. It follows from [1], Propositions 2.20, 2.21, 2.22. □

Definition 3.10. ([1]) A convex set $K \subset H_0^{1,p}(\Omega)$ is *unilateral* if it is closed, nonempty and:

- $u \wedge v \in K$ for every $u, v \in K$;
- $u + \tilde{v} \in K$ for every $u \in K$, $v \in H_0^{1,p}(\Omega)$, with $\tilde{v} \geq 0$ $c_p -$ a.e.

Theorem 3.11. *Let $K \subset H_0^{1,p}(\Omega)$ be an unilateral convex set.*

a) *There exists $\chi: \Omega \to [-\infty, \infty]$ c_p-quasi upper semicontinuous (i.e. for every $\epsilon > 0$ there is an open set $A \subset \Omega$ with $c_p(A) < \epsilon$ such that $\chi|_{\Omega \setminus A}$ is upper semicontinuous) such that*

$$K = \{u \in H_0^{1,p}(\Omega) : \tilde{u} \geq \chi \quad c_p - \text{ a.e. on } \Omega\}.$$

b) *Let $f \in H^{-1,p'}(\Omega)$ be given and $u \in H_0^{1,p}(\Omega)$ be a solution of (I) where K is supposed to be unilateral. Then if $g \in A(u)$ is such that $\langle -\operatorname{div} g, v-u\rangle \geq \langle f, v-u\rangle$ for every $v \in K$, and χ is related to K as in* a), *there exists a positive and finite energy Radon measure μ such that:*

$$\int_\Omega \tilde{v} d\mu = \langle -\operatorname{div} g - f, v\rangle \quad \text{for every} \quad v \in H_0^{1,p}(\Omega) \quad \text{and} \quad \int_\Omega (\tilde{u}-\chi)d\mu = 0.$$

Proof. See [1] Théorème 3.2 and Généralization. □

Lemma 3.12. *Let $f \in H^{-1,p'}(\Omega)$ be given and $u \in H_0^{1,p}(\Omega)$ be a solution of the problem* (I) *where $K \subset H_0^{1,p}(\Omega)$ is assumed to be an unilateral convex set. If $g \in A(u)$ satisfies $\langle -\operatorname{div} g, v-u\rangle \geq \langle f, v-u\rangle$ for every $v \in K$, then:*

i) $\int_\Omega \langle g, D(\varphi(u-v))\rangle dx \leq \langle f, \varphi(u-v)\rangle$ *for every $v \in K$ and $\varphi \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, $\varphi \geq 0$ on Ω.*

ii) *If $v \in K$, $u - v \in L^\infty(\Omega)$, $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$ a.e. on Ω, then $\varphi(u-v) \in H_0^{1,p}(\Omega)$ and*

$$\int_\Omega \langle g, D(\varphi(u-v))\rangle dx \leq \langle f, \varphi(u-v)\rangle.$$

Proof. i) Let χ and μ be as in theorem 3.11, $\varphi \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, $\varphi \geq 0$ and $v \in K$. It can be easily seen that $\varphi(u-v) \in H_0^{1,p}(\Omega)$, thus if χ and μ are given by Theorem 3.11:

$$\begin{aligned}\int_\Omega \langle g, D(\varphi(u-v))\rangle dx &= \langle -\operatorname{div} g, \varphi(u-v)\rangle \\ = \langle -\operatorname{div} g - f, \varphi(u-v)\rangle + \langle f, \varphi(u-v)\rangle &= \int_\Omega \varphi(\tilde{u}-\tilde{v})d\mu + \langle f, \varphi(u-v)\rangle \\ = \int_\Omega \varphi(\tilde{u}-\chi)d\mu + \int_\Omega \varphi(\chi - \tilde{v})d\mu + \langle f, \varphi(u-v)\rangle &\leq \langle f, \varphi(u-v)\rangle.\end{aligned}$$

The last inequality depends also on Proposition 3.9i) which, by $\tilde{v} \geq \chi$ c_p-a.e. on Ω, gives $\tilde{v} \geq \chi$ μ-a.e. on Ω.

ii) If $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$ a.e. on Ω, there exists a sequence $(\varphi_i)_{i\in N}$ in $C_0^1(\mathbb{R}^n)$, $\varphi_i \to \varphi$ in $H^{1,p}(\Omega)$, with $\|\varphi_i\|_{L^\infty} \leq \|\varphi\|_{L^\infty}$, $\varphi_i \geq 0$ for every $i \in N$. To see this we may take $\varphi_i = J_{\epsilon_i} * \varphi$ where J_{ϵ_i} are the usual mollifiers and $\epsilon_i \searrow 0$. Therefore, by passing to a subsequence if necessary, $\varphi_i(u-v) \to \varphi(u-v)$ in $H_0^{1,p}(\Omega)$. Hence $\langle f, \varphi_i(u-v)\rangle \to \langle f, \varphi(u-v)\rangle$, $\int_\Omega \langle g, D(\varphi_i(u-v))\rangle dx \to \int_\Omega \langle g, D(\varphi(u-v))\rangle dx$ and from i) follows ii). □

Remark 3.13. With the assumptions in subsection 3.1, for a given $(f_h)_{h\in N}$ such that $\sup\{\|f_h\|_{L^1}: h \in N\} < \infty$ let us take $(u_h)_{h\in N}$, $(g_h)_{h\in N}$ as in Theorem 3.3 and fix $r \in ((p-1)\vee 1, p_0)$ and $k \geq \|\psi\|_{L^\infty}$. Then let $u \in H_0^{1,r}(\Omega)$ be the weak limit of a subsequence of $(u_h)_{h\in N}$, $\Omega^k = \{x \in \Omega: |u(x)| < k\}$, $\Omega^{h,k} = \{x \in \Omega: |u_h(x)| < k\}$, $g_{h,k} = g_h 1_{\Omega^{h,k}}$.

From boundedness of $(u_n^k)_{h\in N}$ in $H_0^{1,p}(\Omega)$, due to Theorem 3.3, by using (1.2) it follows that $(g_{h,k})_{h\in N}$ is bounded in $L^{p'}(\Omega)^n$. Moreover if $(h_j)_{j\in N}$ is an increasing sequence in N such that $g_{h_j} \rightharpoonup g$ in $L^{r/(p-1)}(\Omega)^n$, $u_{h_j} \rightharpoonup u$ in $H_0^{1,r}(\Omega)$ and $g_{h_j,k} \rightharpoonup \tilde{g}_k$ in $L^{p'}(\Omega)^n$, then

$$\tilde{g}_k(x) = g(x) \quad \text{for a.e.} \quad x \in \Omega^k.$$

By Rellich's theorem we may assume also $u_{h_j} \to u$ a.e. in Ω. Thus, if $\epsilon, \delta \in \mathbb{R}_+$ there exists $\Omega(\epsilon) \subset \Omega^{k-\delta}$ such that $u_{h_j} \to u$ uniformly on $\Omega(\epsilon)$ and $|\Omega^{k-\delta}\setminus\Omega(\epsilon)| < \epsilon$. Let $j_{\epsilon,\delta} \in N$ be such that $|u_{h_j}(x)| < k$ when $j > j_{\epsilon,\delta}$ and $x \in \Omega(\epsilon)$, hence $\Omega(\epsilon) \subset \Omega^{h_j,k}$ if $j > j_{\epsilon,\delta}$. Since if $x \in \Omega(\epsilon)$ is $g_{h_j,k}(x) = g_{h_j}(x)$, it follows that $\int_E g dx = \lim_{j\to\infty} \int_E g_{h_j,k} dx = \int_E \tilde{g}_k dx$ for any $E \subset \Omega(\epsilon)$ measurable. Therefore $g = \tilde{g}_k$ a.e. on $\Omega(\epsilon)$ and from the arbitrariness of ϵ, δ it easily follows that $\tilde{g}_k = g$ a.e. on Ω^k.

Theorem 3.14. *With the assumptions in subsection* 3.1*, let* $(f_h)_{h\in N}$*,* $(u_h)_{h\in N}$*,* $(g_h)_{h\in N}$ *as in Theorem 3.3. If* $u \in H_0^{1,r}(\Omega)$*,* $g \in L^{r/(p-1)}(\Omega)^n$ *for some* $r \in ((p-1)\vee 1, p_0)$*, and* $u_h \rightharpoonup u$ *in* $H_0^{1,r}(\Omega)$*,* $g_h \rightharpoonup g$ *in* $L^{r/(p-1)}(\Omega)^n$*, then*

$$u \in K(\psi) \quad \textit{and} \quad g(x) \in a(x, u(x), Du(x)) \quad \textit{for a.e.} \quad x \in \Omega.$$

Proof. By passing to a subsequence if necessary, by Rellich's theorem we may suppose that $u_h \to u$ a.e. on Ω, thus, being $(u_h)_{h\in N}$ in $K(\psi)$, we get $u \in K(\psi)$.

Let $w \in W_0^{1,\infty}(\Omega) \cap K(\psi)$, $\vartheta \in C_0^1(\Omega)$, $\vartheta \geq 0$, $k \geq \|\psi\|_{L^\infty}$ and Ω^k, $\Omega^{h,k}$, $g_{h,k}$ be like in 3.13.

Moreover let us consider, for $\epsilon \in \mathbb{R}_+$, $\tau_\epsilon: \mathbb{R} \to \mathbb{R}$ defined as in Notation 3.1 and, for $k \in (1,\infty)$, the even function $\sigma_k: \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma_k(t) = \begin{cases} 1 & \text{if} \quad 0 \leq t < k-1 \\ 0 & \text{if} \quad t \geq k \\ -t+k & \text{if} \quad k-1 \leq t < k \end{cases}$$

We observe that $u_h^k = \tau_k \circ u_h \rightharpoonup u^k = \tau_k \circ u$ in $H_0^{1,p}(\Omega)$. Indeed as $(u_h^k)_{h\in N}$ is bounded in $H_0^{1,p}(\Omega)$, due to Theorem 3.3, every subsequence has a subsequence converging in $H_0^{1,p}(\Omega)$, whose limit is u^k because $u_h \to u$ a.e. on Ω.

Now let $\xi \in \mathbb{R}^n$, η be a measurable selection of $a(\cdot, u^k, \xi)$ and $(\eta_h)_{h\in N}$ be given by hypothesis iii) in connection with u^k, $(u_h^k)_{h\in N}$, $v(x) = \langle \xi, x\rangle$ and η, such that $\eta_h \to \eta$ a.e. in Ω and η_h is a measurable selection of $a(\cdot, u_h^k, \xi)$ for every $h \in N$.

Letting $\varphi_{h,k} = (\sigma_k \circ u_h)(\sigma_k \circ u)\vartheta$, from monotonicity of $a(x, u_h(x), \cdot)$ for a.e. $x \in \Omega$ and from definition of $\varphi_{h,k}$ it follows that:

$$\begin{aligned} 0 &\leq \int_\Omega \langle g_h - \eta_h, Du_h - \xi\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx = \\ &= \int_\Omega \langle g_h, D(u_h - u)\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx + \int_\Omega \langle g_h, Du\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx + \\ &\quad - \int_\Omega \langle g_h, \xi\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx - \int_\Omega \langle \eta_h, Du_h - \xi\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx. \end{aligned} \tag{3.7}$$

Now, by Proposition 3.6i), $K(\psi) \cap H^{1,p}_0(\Omega)$ is an unilateral convex set.
Since $u_h - \tau_\epsilon \circ (u_h - u) \in K(\psi)$, Lemma 3.12ii) may be applied with $\varphi_{h,k}$, u_h, $u_h - \tau_\epsilon \circ (u_h - u)$ instead of φ, u and v respectively, getting:

$$\begin{aligned} &\int_\Omega \langle g_h, D(u_h - u)\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx \;=\; \int_\Omega \langle g_h, D\tau_\epsilon \circ (u_h - u)\rangle \varphi_{h,k} dx \\ &= \int_\Omega \langle g_h, D(\varphi_{h,k}\tau_\epsilon \circ (u_h - u))\rangle dx - \int_\Omega \langle g_{h,k}, D\varphi_{h,k}\rangle \tau_\epsilon \circ (u_h - u) dx \\ &\leq \int_\Omega \langle f_h, \varphi_{h,k}\tau_\epsilon \circ (u_h - u)\rangle dx - \int_\Omega \langle g_{h,k}, (\sigma'_k \circ u_h) Du_h\rangle (\sigma_k \circ u)\vartheta \tau_\epsilon \circ (u_h - u) dx \\ &\quad - \int_\Omega \langle g_{h,k}, (\sigma'_k \circ u) Du\rangle (\sigma_k \circ u_h)\vartheta \tau_\epsilon \circ (u_h - u) dx + \\ &\quad - \int_\Omega \langle g_{h,k}, D\vartheta\rangle (\sigma_k \circ u_h)(\sigma_k \circ u)\tau_\epsilon \circ (u_h - u) dx \\ &\leq \epsilon \sup_{h\in N} \|f_h\|_{L^1}\|\vartheta\|_{L^\infty} + \epsilon \sup_{h\in N} \|g_{h,k}\|_{L^{p'}} \sup_{h\in N} \|Du^k_h\|_{L^p}\|\vartheta\|_{L^\infty} + \\ &\quad + \epsilon \sup_{h\in N} \|g_{h,k}\|_{L^{p'}} \|Du^k\|_{L^p}\|\vartheta\|_{L^\infty} + \|\tau_\epsilon \circ (u_h - u) D\vartheta\|_{L^p} \sup_{h\in N} \|g_{h,k}\|_{L^{p'}}, \end{aligned}$$

where, by Remark 3.13, $\sup_{h\in N} \|g_{h,k}\|_{L^{p'}} < \infty$.
We observe that $Du^k \tau'_\epsilon \circ (u_h - u)(\sigma_k \circ u_h)(\sigma_k \circ u)\vartheta \to Du^k(\sigma_k \circ u)^2\vartheta$ a.e. on Ω and hence strongly in $L^p(\Omega)^n$. Thus if $\tilde{g}_k$ is the weak limit in $L^{p'}(\Omega)^n$ of a subsequence of $(g_{h,k})_{h\in N}$ which we shall denote by $(g_{h,k})_{h\in N}$ as well, by Remark 3.13, we have:

$$\begin{aligned} &\lim_{h\to\infty} \int_\Omega \langle g_h, Du\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx \\ &= \lim_{h\to\infty} \int_\Omega \langle g_{h,k}, Du^k\rangle \tau'_\epsilon \circ (u_h - u)(\sigma_k \circ u_h)(\sigma_k \circ u)\vartheta dx \\ &= \int_\Omega \langle \tilde{g}_k, Du^k\rangle (\sigma_k \circ u)^2\vartheta dx = \int_\Omega \langle g, Du^k\rangle (\sigma_k \circ u)^2 \vartheta dx. \end{aligned}$$

Now since $g_h \rightharpoonup g$ in $L^{r/(p-1)}(\Omega)^n$, $\xi\tau'_\epsilon \circ (u_h - u)\varphi_{h,k} \to \xi(\sigma_k \circ u)^2\vartheta$ a.e. on Ω and consequently in the norm of $L^q(\Omega)^n$ for every $q \in (1, \infty)$, it follows that:

$$\lim_{h\to\infty} \int_\Omega \langle g_h, -\xi\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx = \int_\Omega \langle g, -\xi\rangle (\sigma_k \circ u)^2 \vartheta dx.$$

The growth condition ii1) involves that

$$|\eta_h \tau'_\epsilon \circ (u_{h_j} - u)\varphi_{h,k}| \le (\mu + c_1|\xi|^{p-1} + c_2|u_h^k|^\alpha |\xi|^\beta)\vartheta 1_{\Omega^{h,k}} \quad \text{a.e. on} \quad \Omega.$$

Thus since $\eta_h \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} \to \eta(\sigma_k \circ u)^2 \vartheta$ a.e. on Ω such a convergence is also strong in $L^{p'}(\Omega)^n$. On the other hand $(Du_h^k - \xi) \rightharpoonup (Du^k - \xi)$ in $L^p(\Omega)^n$, so that:

$$\begin{aligned}
&\lim_{h\to\infty} \int_\Omega \langle -\eta_h, Du_h - \xi\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx \\
&= \lim_{h\to\infty} \int_\Omega \langle -\eta_h, Du_h^k - \xi\rangle \tau'_\epsilon \circ (u_h - u)\varphi_{h,k} dx \\
&= \int_\Omega \langle -\eta, Du^k - \xi\rangle (\sigma_k \circ u)^2 \vartheta dx = \int_\Omega \langle -\eta, Du - \xi\rangle (\sigma_k \circ u)^2 \vartheta dx.
\end{aligned}$$

From (3.7) and all above inequalities, being $\lim_{h\to\infty} \|\tau_\epsilon \circ (u_h - u) D\vartheta\|_{L^p} = 0$, we get:

$$\begin{aligned}
0 \le &\int_\Omega \langle g - \eta, Du - \xi\rangle (\sigma_k \circ u)^2 \vartheta dx + \epsilon(\sup_{h\in N} \|f_h\|_{L^1} + \sup_{h\in N} \|g_{h,k}\|_{L^{p'}} \sup_{h\in N} \|Du_h^k\|_{L^p} + \\
&+ \sup_{h\in N} \|g_{h,k}\|_{L^{p'}} \|Du^k\|_{L^p}) \|\vartheta\|_{L^\infty}.
\end{aligned}$$

Since ϵ and ϑ are arbitrary it follows that: $0 \le \langle g - \eta, Du^k - \xi\rangle (\sigma_k \circ u)^2$ a.e. on Ω, hence $0 \le \langle g - \eta, Du^k - \xi\rangle$ a.e. on Ω^{k-1}. Also $\xi \in \mathbb{R}^n$ and the measurable selection η of $a(\cdot, u^k, \xi)$ are arbitrarily choosen, so that, like in the last part of the proof of Lemma 2.1, we get $\langle g(x) - \zeta, Du(x) - \xi\rangle \ge 0$ for a.e. $x \in \Omega^{k-1}$, every $\xi \in \mathbb{R}^n$ and every $\zeta \in a(x, u^k(x), \xi)$. Finally the maximal monotonicity of $a(x, u(x), \cdot)$ for a.e. $x \in \Omega$, ensures that $g(x) \in a(x, u(x), Du(x))$ for a.e. $x \in \Omega^{k-1}$, which concludes the proof as k is also arbitrary. □

Theorem 3.15. *With the assumptions in subsection* 3.1*, there exists a solution of the problem* (II) *where* $f: \mathcal{B}(\Omega) \to \mathbb{R}$ *is a bounded Radon measure.*

Proof. Let us define $f_h: \Omega \to \mathbb{R}$ by $f_h(x) = \int_\Omega J_{\epsilon_h}(x - y) df(y)$, where J_{ϵ_h} are the usual mollifiers and $\epsilon_h \searrow 0$. Then $(f_h)_{h\in N}$ is a sequence of $H^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that $\sup\{\|f_h\|_{L^1}: h \in N\} < \infty$ and $f_h \to f$ in the distributional sense. Corresponding to each f_h, by Theorem 2.13 there exist $u_h \in H_0^{1,p}(\Omega) \cap K(\psi)$ and $g_h \in A(u_h)$, solving the problem (I), i.e. $\langle -\operatorname{div} g_h, v - u_h\rangle \ge \langle f_h, v - u_h\rangle$ for every $v \in H_0^{1,p}(\Omega) \cap K(\psi)$. By Theorem 3.3, fixing $r \in ((p-1) \vee 1, p_0)$, there exist $u \in H_0^{1,r}(\Omega) \cap K(\psi)$, $g \in L^{r/(p-1)}(\Omega)^n$ and an increasing sequence $(h_j)_{j\in N}$ in N such that $u_{h_j} \rightharpoonup u$ in $H_0^{1,r}(\Omega)$ and $g_{h_j} \rightharpoonup g$ in $L^{r/(p-1)}(\Omega)^n$. Then for any $s \in ((p-1) \vee 1, p_0)$ we have $u_{h_j} \rightharpoonup u$ in $H_0^{1,s}(\Omega)$ and $g_{h_j} \rightharpoonup g$ in $L^{s/(p-1)}(\Omega)^n$. Indeed from every subsequence of $(u_{h_j})_{j\in N}$ we can extract a further subsequence which weakly converges in $H_0^{1,s}(\Omega)$, whose limit still is u, as $H^{-1,r'}(\Omega) \subset H^{-1,s'}(\Omega)$. We can apply the same argument to $(g_{h_j})_{j\in N}$.

Moreover by Theorem 3.14, $g(x) \in a(x, u(x), Du(x))$ for a.e. $x \in \Omega$. Now if some $\varphi \in V_0^\infty(\Omega, \psi)$ is given, by definition of $V_0^\infty(\Omega, \psi)$ there exists $(\varphi_h)_{h\in N}$ in $\mathcal{D}(\Omega)$, $\varphi_h \to \varphi$ in the topology of $\mathcal{D}(\Omega)$, such that for every $h \in N$ $u_h + \varphi_h \in K(\psi)$. Thus from $\int_\Omega \langle g_h, D\varphi_h\rangle dx \geq \int_\Omega f_h \varphi_h dx$ letting $h \to \infty$, we get $\int_\Omega \langle g, D\varphi\rangle dx \geq \int_\Omega \varphi df$. □

Theorem 3.16. *With the assumptions made in subsection* 3.1*, there exists a solution of problem* (III) *where f is supposed to be an element of $L^1(\Omega)$ and $\psi \in L^\infty(\Omega)$.*

Proof. Let $(f_h)_{h\in N}$ be a sequence in $H^{-1,p'}(\Omega) \cap L^1(\Omega)$ such that $f_h \to f$ in $L^1(\Omega)$. For every $h \in N$ we consider a solution $u_h \in H_0^{1,p}(\Omega) \cap K(\psi)$ of problem (I) corresponding to f_h, which exists by Theorem 2.13. Then, for every $h \in N$, let $g_h \in A(u_h)$ be such that $\langle -\operatorname{div} g_h, v - u_h\rangle \geq \langle f_h, v - u_h\rangle$ for any $v \in H_0^{1,p}(\Omega) \cap K(\psi)$. For a fixed $k \geq \|\psi\|_{L^\infty}$, like in the proof of the previous theorem, in virtue of Theorems 3.3 and 3.14 and of Remark 3.13, there exist $u \in H_0^{1,r}(\Omega)$ and a selection $g \in L^{r/(p-1)}(\Omega)^n$ of $a(\cdot, u, Du)$, such that, by passing to a subsequence if necessary, $u_h \rightharpoonup u$ in $H_0^{1,r}(\Omega)$ and a.e. in Ω, $g_h \rightharpoonup g$ in $L^{r/(p-1)}(\Omega)^n$ for every $r \in ((p-1) \vee 1, p_0)$. Moreover, if $u_h^k = \tau_k \circ u_h$ and $u^h = \tau_k \circ u$, with τ_k defined as in 3.1, then $u_h^k \rightharpoonup u^k$ in $H_0^{1,p}(\Omega)$ and a.e. on Ω. When $k \geq \|\psi\|_{L^\infty}$ then $u_h - u_h^k + v \in H_0^{1,p}(\Omega) \cap K(\psi)$, so that for any $v \in W_0^{1,\infty}(\Omega) \cap K(\psi)$, $h \in N$:

$$\int_\Omega \langle g_h, D(u_h^k - v)\rangle dx \leq \int_\Omega f_h(u_h^k - v)dx. \tag{3.8}$$

Like in Remark 3.13 let $\Omega^{h,k} = \{x \in \Omega : |u_h(x)| < k\}$, $g_{h,k} = g_h 1_{\Omega^{h,k}}$, so that

$$\int_\Omega \langle g_h, Du_h^k\rangle dx = \int_\Omega \langle g_h - g_{h,k}, Du_h^k\rangle dx + \int_\Omega \langle g_{h,k}, Du_h^k\rangle dx = \int_\Omega \langle g_{h,k}, Du_h^k\rangle dx.$$

Let now γ be a measurable selection of $a(\cdot, u^k, Du^k)$ such that $\gamma|_{\Omega^k} = g|_{\Omega^k}$, where $\Omega^k = \{x \in \Omega : |u(x)| < k\}$. By hypothesis (iii) for every $h \in N$ we may take a measurable selection γ_h of $a(\cdot, u_h^k, Du^k)$ such that $\gamma_h \to \gamma$ a.e. in Ω.

Moreover taking monotonicity of $a(x, u_h(x), \cdot)$ into account:

$$\begin{aligned}
&\int_\Omega \langle g_{h,k} Du_h^k\rangle dx \\
&= \int_{\Omega^{h,k}} \langle g_{h,k} - \gamma_h, Du_h^k - Du^k\rangle dx + \int_\Omega \langle g_{h,k}, Du^k\rangle dx + \int_{\Omega^{h,k}} \langle \gamma_h, Du_h^k - Du^k\rangle dx \\
&\geq \int_{\Omega^{h,k}} \langle \gamma_h, Du_h^k - Du^k\rangle dx + \int_\Omega \langle g_{h,k}, Du^k\rangle dx.
\end{aligned}$$

By Remark 3.13 there exists $\tilde{g}_k \in L^{p'}(\Omega)^n$ such that, by passing to a further subsequence if necessary, $g_{h,k} \rightharpoonup \tilde{g}_k$ in $L^{p'}(\Omega)^n$ and $\tilde{g}_k(x) = g(x)$ for a.e. $x \in \Omega^k$.
Hence $\langle g, Du^k\rangle \in L^1(\Omega)$ and $\int_\Omega \langle g_{h,k}, Du^k\rangle dx \to \int_\Omega \langle g, Du^k\rangle dx$.

Now we see that $(\gamma_h)_{h\in N}$ is strongly convergent to γ in $L^{p'}(\Omega)^n$: indeed from growth condition (ii1) it follows that $|\gamma_h| \le \mu + c_1|Du^k|^{p-1} + c_2|u_h^k|^\alpha|Du^k|^\beta$ a.e. on Ω and $|Du^k|^\beta \in L^{p'}(\Omega)$ as, by (1.1.3), $\beta p' < p$.
Moreover $(Du_h^k - Du^k)1_{\Omega^{h,k}} \rightharpoonup 0$ in $L^p(\Omega)^n$ as $Du^k(1_{\Omega^{h,k}} - 1_{\Omega^k}) \to 0$ a.e. on Ω. Therefore

$$\lim_{h\to\infty}\int_{\Omega^{h,k}}\langle\gamma_h, Du_h^k - Du^k\rangle dx = 0.$$

Then from (3.8) it follows that for a suitable increasing sequence $(h_j)_{j\in N}$ in N we get:

$$\begin{aligned}
&\int_\Omega \langle g, D(u^k - v)\rangle dx \\
&= \lim_{h\to\infty}\int_{\Omega^{h,k}}\langle\gamma_h, Du_h^k - Du^k\rangle dx + \int_\Omega\langle g_{h,k}, Du^k\rangle dx - \int_\Omega\langle g_h, Dv\rangle dx \le \\
&\le \lim_{j\to\infty}\int_\Omega\langle g_{h_j}, D(u_{h_j}^k - v)\rangle dx \le \lim_{j\to\infty}\int_\Omega f_{h_j}(u_{h_j}^k - v)dx = \int_\Omega f(u^k - v)dx.
\end{aligned}$$

□

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