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Simultaneous Almost Minimization of Convex Functions and Duals of Results Related to Maximal Monotonicity

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Dedicated to R. T. Rockafellar on his 60th Birthday

1. Introduction

Let E be a real Banach space with dual E^*, and $f : E \to \mathbb{R} \cup \{\infty\}$ be convex. A result *dual* to a given result about f is one which can be obtained by formally interchanging E and E^*, and formally interchanging f and f^*, but leaving statements of the form "$z^* \in \partial f(z)$" unchanged.

The initial motivation for this paper was to establish inequalities dual to some already known about the subdifferential of a proper, convex lower semicontinuous function on E. The technical problem is that if we just apply the known result, we end up with statements about E^{**} and f^{**}, and it is not always clear how to pass back to statements about E and f. Rockafellar tackled this problem in his proof of the maximal monotonicity theorem in [6].

From the point of view of optimization theory, E and E^* are the basic objects to consider, while E^{**} is a much more technical functional-analytic concept. Furthermore, the consideration of weak-* separation in E^{**} requires the development of the theory of locally convex spaces, which imposes yet a further layer of functional analytic complexity. This leads to our second motivation — to provide proofs that stay within the context of E and E^*, and use the smallest number of abstract functional analytic tools.

It seems that the key to such a project is Lemma 3.2, in which we give a (slightly disguised) sufficient condition for proper convex functions $f_1, \ldots, f_m$ to achieve a simultaneous almost minimum close to the intersection of a given family of closed half-spaces. This condition is phrased in terms of the conjugate of $F_m := f_1 + \cdots + f_m$. Our proof of Lemma 3.2 uses the standard tools of optimization theory (as opposed to functional analysis), that is to say the definition of conjugate function, subdifferential, episum, and directional derivative, and Rockafellar's result that the conjugate of a sum is, under suitable circumstances, the episum of the conjugates, (which follows from a separation theorem in $E \times \mathbb{R}$). In

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fact, Lemma 3.2 is much more general than is actually required for our dual results — Corollary 3.4 is quite adequate. Corollary 3.4 seems to be the weakest consequence of Lemma 3.2 which is strong enough to prove Theorem 4.1(b), our first main dual result.

Theorem 4.1(b) is the dual of Theorem 4.1(a), a geometric result which was established in order to give a simple proof of Rockafellar's maximal monotonicity theorem. Corollary 4.3 and Corollary 4.4 give further examples of dual results.

Our second main dual result is contained in Theorem 4.5, which contains two different quantitative generalizations of the maximal monotonicity theorem. The maximal monotonicity theorem tells us that if $h : E \to \mathbb{R} \cup \{\infty\}$ is proper, convex and lower semicontinuous, $q \in E$, $q^* \in E^*$, and $q^* \notin \partial h(q)$, then there exist $z \in E$ and $z^* \in \partial h(z)$ such that $\langle z - q, z^* - q^* \rangle < 0$. Theorem 4.5 tells us that we can find, in different ways, $z \in E$ and $z^* \in \partial h(z)$ such that, among other things, $z - q$ and $z^* - q^*$ are, as nearly as we like, "oppositely aligned". This leads to an open problem, which is stated in Remark 4.6.

In Theorem 4.7, we give another pair of simple dual results, which we have included as a contrast to the situation outlined in Remark 4.8, in which we give a specific result for which we do not know whether the dual result holds. Finally, Remark 4.9 contains an example in which an "obvious" dual result fails.

Complicated though it may be, Lemma 3.2 only tells a partial story in that it gives merely a *sufficient* condition for the functions f_i to have a simultaneous almost minimum. The full story is told by Theorem 5.1, in which we give *necessary and sufficient* conditions for proper convex functions $f_1, \ldots, f_m$ to achieve a simultaneous almost minimum close to the intersection of a given family of closed half-spaces in the vicinity of given convex sets $C_1, \ldots, C_p$. An additional difference between Theorem 5.1 and Lemma 3.2 is that Theorem 5.1 involves the sets $C_1, \ldots, C_p$, which are not present in Lemma 3.2. One of the equivalent conditions is phrased in terms of the directional derivative of the conjugate of $F_m := f_1 + \cdots + f_m$, following an idea already introduced in Corollary 3.4.

2. Preliminaries

Notation 2.1. We shall use the following standard notation. We suppose that E is a real normed space with adjoint E^*. We shall state explicitly when we are assuming that E is complete. If $f : E \to \mathbb{R} \cup \{\infty\}$ is convex, we write

$$\operatorname{dom} f := \{x : x \in E,\ f(x) \in \mathbb{R}\},$$

and we say that f is *proper* if $\operatorname{dom} f \neq \emptyset$. If $f : E \to \mathbb{R} \cup \{\infty\}$ is proper and convex, we define $f^* : E^* \to \mathbb{R} \cup \{\infty\}$ by $f^*(x^*) := \sup_E (x^* - f)$. f^* is the *conjugate* of f. f^* is convex and $w(E^*, E)$-lower semicontinuous. If f is also lower semicontinuous, then f^* is proper. If $x \in E$ we write

$$\begin{aligned}\partial f(x) &:= \{x^* : x^* \in E^*,\ \text{for all } y \in E,\ f(x) + \langle y - x, x^* \rangle \le f(y)\} \\ &= \{x^* : x^* \in E^*,\ f(x) + f^*(x^*) \le \langle x, x^* \rangle\}.\end{aligned}$$

$\partial f(x)$ is the *subdifferential* of f at x. If, further $\eta > 0$, we write

$$\begin{aligned}\partial_\eta f(x) &:= \{x^* : x^* \in E^*,\ \text{for all } y \in E,\ f(x) + \langle y - x, x^* \rangle \le f(y) + \eta\} \\ &= \{x^* : x^* \in E^*,\ f(x) + f^*(x^*) \le \langle x, x^* \rangle + \eta\}.\end{aligned}$$

If $f, g : E \to \mathbb{R} \cup \{\infty\}$, we define $f \underset{e}{+} g$ by $(f \underset{e}{+} g)(x) := \inf_{y \in E} (f(x-y) + g(y))$. $f \underset{e}{+} g$ is the *episum* or *inf-convolution* of f and g. We also write $f \vee g$ for the pointwise maximum of f and g. We shall use the following version of the "sum formula", which follows from Rockafellar's generalization of a finite dimensional result of Fenchel (see [7], Theorem 20, p. 56): if $g_0 : E \to \mathbb{R} \cup \{\infty\}$ is proper and convex, and $g_1, \ldots, g_p : E \to \mathbb{R}$ are convex and continuous then $(g_0 + g_1 + \cdots + g_p)^* = g_0^* \underset{e}{+} g_1^* \underset{e}{+} \cdots \underset{e}{+} g_p^*$. If $x \in \operatorname{dom} f$ and $v \in E$ then we write

$$\mathrm{d}^+ f(x)(v) := \lim_{\theta \to 0+} \frac{f(x + \theta v) - f(x)}{\theta}.$$

$\mathrm{d}^+ f(x)(v)$ is the *directional derivative* of f at x in the direction v. Since the above limit can be replaced by an infimum, it follows that,

$$\text{for all } x \in \operatorname{dom} f \text{ and } v \in E, \quad f(x+v) \geq \mathrm{d}^+ f(x)(v) + f(x). \tag{2.1}$$

To avoid misunderstanding, we state explicitly here that $\mathrm{d}^+ f^*$ is to be interpreted as $\mathrm{d}^+(f^*)$ and not as $(\mathrm{d}^+ f)^*$.

3. The simultaneous almost minimization of convex functions

Lemma 3.1. *Let $m \geq 1$, $f_1, \ldots, f_m : E \to \mathbb{R} \cup \{\infty\}$ be proper and convex and, for all $i = 1, \ldots m$, $x_i^* \in \operatorname{dom} {f_i}^*$. Under these conditions, we shall write $F_m := f_1 + \cdots + f_m$, $X_m^* := x_1^* + \cdots + x_m^*$, and $\tau_m := {f_1}^*(x_1^*) + \cdots + {f_m}^*(x_m^*)$. We note then that*

$${F_m}^*(X_m^*) \leq \tau_m \in \mathbb{R}. \tag{3.1}$$

Let $\eta > 0$ and ${F_m}^*(X_m^*) > \tau_m - \eta$. Then there exists $x \in E$ such that, for all $i = 1, \ldots, m$, $x_i^* \in \partial_\eta f_i(x)$.

Proof. For all $i = 1, \ldots, m$, let $g_i := f_i - x_i^* + {f_i}^*(x_i^*)$. The functions g_i are proper, convex and positive. By hypothesis, $\inf_E (g_1 + \cdots + g_m) < \eta$. Since $0 \leq g_1 \vee \cdots \vee g_m \leq g_1 + \cdots + g_m$ on E, $\inf_E (g_1 \vee \cdots \vee g_m) < \eta$. This completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $m, n \geq 1$. Let $f_1, \ldots, f_m : E \to \mathbb{R} \cup \{\infty\}$ be proper and convex and, for all $i = 1, \ldots, m$, $x_i^* \in \operatorname{dom} {f_i}^*$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $u_1^*, \ldots, u_n^* \in E^*$ and:*

$$\mu_1, \ldots, \mu_n \geq 0 \quad \Longrightarrow \quad {F_m}^*\Big(X_m^* + \sum_{j=1}^n \mu_j u_j^*\Big) \geq \tau_m + \sum_{j=1}^n \mu_j \alpha_j. \tag{3.2}$$

Then

$$\left.\begin{array}{c} \textit{for all } \eta > 0, \textit{ there exists } x \in E \textit{ such that,} \\ \textit{for all } i = 1, \ldots, m,\ x_i^* \in \partial_\eta f_i(x) \textit{ and, for all } j = 1, \ldots, n,\ \langle x, u_j^* \rangle \geq \alpha_j - \eta. \end{array}\right\} \tag{3.3}$$

Proof. Let us suppose that $u_1^*, \ldots, u_n^*$ have been renumbered so that, for all $j = 1, \ldots, p$, $u_j^* \neq 0$ and, for all $j = p+1, \ldots, n$, $u_j^* = 0$. For $j = 1, \ldots, p$, we write $f_{m+j} := (\alpha_j - u_j^*) \vee 0$

and $x_{m+j}^* := 0$. Then, for $j = 1, \ldots, p$, ${f_{m+j}}^*(x_{m+j}^*) = 0$. It follows from the above definitions that, with the notation of Lemma 3.1,

$$X_{m+p}^* = X_m^* \quad \text{and} \quad \tau_{m+p} = \tau_m. \tag{3.4}$$

From (3.2), ${F_m}^*(X_m^*) \geq \tau_m \in \mathbb{R}$, hence F_m is proper and convex. Further, $f_{m+1}, \ldots, f_{m+p}$ are real, convex and continuous. Finally, $F_m + f_{m+1} + \cdots + f_{m+p} = F_{m+p}$. Thus, from the sum formula,

$${F_m}^* \mathbin{\underset{e}{+}} {f_{m+1}}^* \mathbin{\underset{e}{+}} \cdots \mathbin{\underset{e}{+}} {f_{m+p}}^* = {F_{m+p}}^* \tag{3.5}$$

Since $u_{p+1}^* = \cdots = u_n^* = 0$, it follows from (3.2) that:

$$\mu_1, \ldots, \mu_n \geq 0 \quad \Longrightarrow \quad {F_m}^*\Big(X_m^* + \sum_{j=1}^{p} \mu_j u_j^*\Big) \geq \tau_m + \sum_{j=1}^{n} \mu_j \alpha_j. \tag{3.6}$$

Setting $\mu_1 = \cdots = \mu_p = 0$ in (3.6) and using (3.1),

$$\alpha_{p+1}, \ldots, \alpha_n \leq 0, \tag{3.7}$$

and, setting $\mu_{p+1} = \cdots = \mu_n = 0$ in (3.6),

$$\mu_1, \ldots, \mu_p \geq 0 \quad \Longrightarrow \quad {F_m}^*\Big(X_m^* + \sum_{j=1}^{p} \mu_j u_j^*\Big) \geq \tau_m + \sum_{j=1}^{p} \mu_j \alpha_j. \tag{3.8}$$

Now, for $j = 1, \ldots, p$, ${f_{m+j}}^*(w_{m+j}^*) = \infty$ unless there exists $\mu_j \in [0, 1]$ such that $w_{m+j}^* = -\mu_j u_j^*$, in which case ${f_{m+j}}^*(w_{m+j}^*) = -\mu_j \alpha_j$. Thus it follows from (3.8) that:

$$w_m^*, \ldots, w_{m+p}^* \in E^* \text{ and} \sum_{i=m}^{m+p} w_i^* = X_m^* \Longrightarrow \quad {F_m}^*(w_m^*) + \sum_{i=m+1}^{m+p} {f_i}^*(w_i^*) \geq \tau_m,$$

which can be rewritten $({F_m}^* \mathbin{\underset{e}{+}} {f_{m+1}}^* \mathbin{\underset{e}{+}} \cdots \mathbin{\underset{e}{+}} {f_{m+p}}^*)(X_m^*) \geq \tau_m$. From (3.4) and (3.5),

$${F_{m+p}}^*(X_{m+p}^*) \geq \tau_{m+p} > \tau_{m+p} - \eta.$$

From Lemma 3.1 with m replaced by $m + n$,

$$\text{there exists } x \in E \text{ such that, for all } i = 1, \ldots, m+p,\ x_i^* \in \partial_\eta f_i(x).$$

If $j = 1, \ldots, p$ then, since $x_{m+j}^* \in \partial_\eta f_{m+j}(x)$, it follows that $\langle x, u_j^* \rangle \geq \alpha_j - \eta$. If $j = p+1, \ldots, n$ then, from (3.7), $\langle x, u_j^* \rangle = 0 \geq \alpha_j \geq \alpha_j - \eta$. Thus (3.3) is satisfied, which completes the proof of Lemma 3.2. □

Notation 3.3. If C is a nonempty convex subset of E, we define the convex continuous function d_C on E by $d_C(x) := \inf_{y \in C} \|x - y\|$, and the support function σ_C of C on E^* by $\sigma_C(x^*) := \sup_C x^*$. We note that, for all $x^* \in E^*$,

$${d_C}^*(x^*) = \sup_E (x^* - d_C) \geq \sup_C (x^* - d_C) = \sup_C x^* = \sigma_C(x^*) \tag{3.9}$$

Corollary 3.4. *Let $f : E \to \mathbb{R} \cup \{\infty\}$ be proper and convex, $x^* \in \operatorname{dom} f^*$, C be a nonempty convex subset of E, $\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C : E \to \mathbb{R} \cup \{\infty\}$ and $(\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(-x^*) \geq \alpha \in \mathbb{R}$. Then, for all $\eta > 0$,*

$$\text{there exists } x \in E \text{ such that} \quad d_C(x) \leq \eta, \quad x^* \in \partial_\eta f(x) \quad \text{and} \quad -\langle x, x^* \rangle \geq \alpha - \eta.$$

Proof. Let $f_1 := f$, $f_2 := d_C$, $x_1^* := x^*$ and $x_2^* := 0$. Since ${f_2}^*(x_2^*) = 0$,

$$X_2^* = x^* \quad \text{and} \quad f^*(x^*) = \tau_2. \tag{3.10}$$

Since f is proper and convex, and d_C is real, convex and continuous, from the sum formula and (3.9),

$$ {F_2}^* = (f + d_C)^* = f^* \underset{e}{+} {d_C}^* \geq f^* \underset{e}{+} \sigma_C \text{ on } E^*. \tag{3.11}$$

Let $\mu \geq 0$ and $w^* \in E^*$. Then, from (2.1),

$$\begin{aligned} f^*(x^* - \mu x^* - w^*) + \sigma_C(w^*) &\geq f^*(x^*) + \mathrm{d}^+ f^*(x^*)(-\mu x^* - w^*) + \sigma_C(w^*) \\ &\geq f^*(x^*) + (\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(-\mu x^*) \geq f^*(x^*) + \mu\alpha. \end{aligned}$$

(When $\mu = 0$, the last of the inequalities above follows from the fact that $\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C : E \to \mathbb{R} \cup \{\infty\}$ and $\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C$ is positively homogeneous.) Taking the infimum over $w^* \in E^*$,

$$(f^* \underset{e}{+} \sigma_C)(x^* - \mu x^*) \geq f^*(x^*) + \mu\alpha,$$

thus, from (3.10) and (3.11),

$${F_2}^*(X_2^* - \mu x^*) = {F_2}^*(x^* - \mu x^*) \geq f^*(x^*) + \mu\alpha = \tau_2 + \mu\alpha.$$

Since this holds for all $\mu \geq 0$, (3.2) is satisfied with $m := 2$, $n := 1$ and $u_1^* := -x^*$. From Lemma 3.2, (3.3) is satisfied. Since $x_2^* \in \partial_\eta f_2(x) \Longrightarrow d_C(x) \leq \eta$, this completes the proof of Corollary 3.4. □

Remark 3.5. If either $\mathrm{d}^+ f^*(x^*) : E \to \mathbb{R}$ or $\sigma_C : E \to \mathbb{R}$ then it follows from the relation

$$(\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(y^*) + \mathrm{d}^+ f^*(x^*)(-x^* - y^*) \geq (\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(-x^*) \quad (y^* \in E^*)$$

or

$$(\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(y^*) + \sigma_C(-x^* - y^*) \geq (\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(-x^*) \quad (y^* \in E^*)$$

that

$$(\mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C)(-x^*) \geq \alpha \in \mathbb{R} \quad \Longrightarrow \mathrm{d}^+ f^*(x^*) \underset{e}{+} \sigma_C : E \to \mathbb{R} \cup \{\infty\}.$$

In particular, this is true if C is bounded.

4. Dual results related to maximal monotonicity

We suppose throughout this section that E is a Banach space and $g, h : E \to \mathbb{R} \cup \{\infty\}$ are proper, convex and lower semicontinuous. We start this section by giving two known concrete examples of "dual results".

It was proved in [12], Theorem 3.2, p. 1379 that if $x \in \operatorname{dom} g$ and $\varepsilon > 0$ then there exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$\|z - x\| < \varepsilon, \quad \langle z, z^* \rangle \le \langle x, z^* \rangle \text{ and } \quad g(z) \le g(x).$$

It was also proved in [12], Corollary 4.2, p. 1382 that if $x^* \in \operatorname{dom} g^*$ and $\varepsilon > 0$ then there exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$\|z^* - x^*\| < \varepsilon, \quad \langle z, z^* \rangle \le \langle z, x^* \rangle \text{ and } \quad g^*(z^*) \le g^*(x^*).$$

(This latter result generalizes [1], Proposition 4.3, p. 398). This is our first example of what we mean by "dual results".

A second example is provided by the *local maximal monotonicity theorem* (see [10], Main Theorem, p. 466) and the dual *maximal monotone locally theorem* (see [3], Corollary 3.4 and [13], Theorem 3, p. 269):

If U^* is a convex open subset of E^* such that $Gr(\partial g) \cap (E \times U^*) \neq \emptyset$, $q \in E$ and $q^* \in U^* \setminus \partial g(q)$ then there exist $z \in E$ and $z^* \in \partial g(z) \cap U^*$ such that $\langle z - q, z^* - q^* \rangle < 0$.

If U is a convex open subset of E such that $Gr(\partial g) \cap (U \times E^*) \neq \emptyset$, $q \in U$ and $q^* \in E^* \setminus \partial g(q)$ then there exist $z \in U$ and $z^* \in \partial g(z)$ such that $\langle z - q, z^* - q^* \rangle < 0$.

We now come to our first new dual result — Theorem 4.1(b).

Theorem 4.1. *Let E be a Banach space and $g : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous.*

(a) *Let $q \in E$, $\lambda \in \mathbb{R}$ and $\inf_E g < \lambda < g(q)$ $(\le \infty)$. Write*

$$K := \sup\left\{\frac{\lambda - g(y)}{\|q - y\|} : y \in E,\ g(y) < \lambda\right\}.$$

Then $0 < K < \infty$ and, for all $\varepsilon \in (0, 1)$, there exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$z \neq q, \quad \|z^*\| \le (1 + \varepsilon)K \quad \text{and} \quad \langle q - z, z^* \rangle \ge (1 - \varepsilon)K\|q - z\|. \tag{4.1}$$

(b) *Let $q^* \in E^*$, $\lambda \in \mathbb{R}$ and $\inf_{E^*} g^* < \lambda < g^*(q^*)$ $(\le \infty)$. Write*

$$K^* := \sup\left\{\frac{\lambda - g^*(y^*)}{\|q^* - y^*\|} : y^* \in E^*,\ g^*(y^*) < \lambda\right\}.$$

Then $0 < K^* < \infty$ and, for all $\varepsilon \in (0, 1)$, there exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$z^* \neq q^*, \quad \|z\| \le (1 + \varepsilon)K^* \quad \text{and} \quad \langle z, q^* - z^* \rangle \ge (1 - \varepsilon)K^*\|q^* - z^*\|. \tag{4.2}$$

Proof. (a) This was proved in [9], Théorème 2, p. 23.02 with φ replaced by g and x replaced by q, and [11], Main Theorem, p. 328 with φ replaced by g and h replaced by λ.

(b) Let $\beta := \varepsilon/5$. We can follow through the proof of [9], Théorème 2 up to (2.6), or the proof of [11], Main Theorem up to (8), with ε replaced by β and φ replaced by g^*, and derive that there exists y^* (replacing z) $\in \operatorname{dom} g^*$ such that $y^* \neq q^*$ and,

$$\text{for all } v^* \in E^*, \quad \mathrm{d}^+ g^*(y^*)(v^*) + (1+\beta)K^*\|q^* - y^* - v^*\| \geq (1-\beta)K^*\|q^* - y^*\|.$$

We now define $f : E \to \mathbb{R} \cup \{\infty\}$ by $f := g - q^*$ and write $x^* := y^* - q^* \in E^*$ and $\rho := \|x^*\| = \|q^* - y^*\| > 0$. The above inequality then becomes:

$$\text{for all } v^* \in E^*, \quad \mathrm{d}^+ f^*(x^*)(v^*) + (1+\beta)K^*\| - x^* - v^*\| \geq (1-\beta)K^*\rho.$$

from which

$$(\mathrm{d}^+ f^*(x^*) \underset{e}{+} (1+\beta)K^*\|\ \|)(-x^*) \geq (1-\beta)K^*\rho.$$

It now follows from Corollary 3.4 and Remark 3.5, with C the ball in E of radius $(1+\beta)K^*$, $\alpha := (1-\beta)K^*\rho$ and $\eta > 0$ chosen so that $\eta \leq \beta K^*$ and $\eta \leq \beta^2 K^*\rho \leq \beta K^*\rho$, that there exists $x \in E$ such that

$$\|x\| \leq (1+2\beta)K^*, \quad x^* \in \partial_{\beta^2 K^*\rho} f(x) \quad \text{and} \quad -\langle x, x^*\rangle \geq (1-2\beta)K^*\rho. \tag{4.3}$$

From the Brøndsted-Rockafellar Theorem (see [2], p. 608), there exist $z \in E$ and $t^* \in \partial f(z)$ such that

$$\|z - x\| \leq \beta K^* \quad \text{and} \quad \|t^* - x^*\| \leq \beta\rho, \tag{4.4}$$

from which $t^* \neq 0$. Combining this with (4.3) and the definition of ρ,

$$\|z\| \leq (1+3\beta)K^* \leq (1+\varepsilon)K^* \quad \text{and} \quad \|t^*\| \leq (1+\beta)\rho. \tag{4.5}$$

Write $z^* := t^* + q^* \in E^*$. Then

$$z^* \in \partial(f + q^*)(z) = \partial g(z) \quad \text{and} \quad z^* \neq q^*. \tag{4.6}$$

On the other hand, from (4.3), (4.4) and (4.5),

$$\begin{aligned} &\langle z, z^* - q^*\rangle + (1-\varepsilon)K^*\|q^* - z^*\| = \langle z, t^*\rangle + (1-\varepsilon)K^*\|t^*\| \\ &= \langle z, t^* - x^*\rangle + \langle z - x, x^*\rangle + \langle x, x^*\rangle + (1-\varepsilon)K^*\|t^*\| \\ &\leq (1+3\beta)K^*\beta\rho + \beta K^*\rho - (1-2\beta)K^*\rho + (1-5\beta)K^*(1+\beta)\rho = -2\beta^2 K^*\rho \leq 0. \end{aligned}$$

Combining this with (4.5) and (4.6) gives (4.2), which completes the proof of Theorem 4.1(b). □

Remark 4.2. The parts of [9], Théorème 2 and [11], Main Theorem referred to in the proof of Theorem 4.1(b) both use Ekeland's variational principle which, of course, contains the Brøndsted-Rockafellar Theorem referred to later on in the theorem. The first application of Ekeland's variational principle is not, in fact, necessary. We can use instead the fact that a coercive $w(E^*, E)$-lower semicontinuous function on E^* attains a minimum. It is also possible to deduce Theorem 4.1(b) from Theorem 4.1(a) using the sharpening by Gossez (see [5], Theorem 3.1, p. 376) of a result established by Rockafellar in one of his proofs of the maximal monotonicity theorem (see [6], Proposition 1, p. 211). We have used the method presented here (via Corollary 3.4) for the reasons explained in the introduction. In fact, the techniques of this paper can be used to generalize Gossez's result. Details of this will appear elsewhere.

As a consequence of Theorem 4.1, we can deduce the following interpretations of the "slopes" of g and g^*. Corollary 4.3(b) is our second new dual result.

Corollary 4.3. *Let E be a Banach space and $g : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous.*

(a) *Let $q \in E$ and $g(q) > \inf_E g$. Let*

$$M := \sup\left\{\frac{\langle q - z, z^*\rangle}{\|q - z\|} : z \in E,\ g(z) < g(q),\ z^* \in \partial g(z)\right\},$$

$$L := \sup\left\{\frac{g(q) - g(y)}{\|q - y\|} : y \in E,\ g(y) < g(q)\right\},$$

and

$$N := \sup\left\{\frac{\langle q - z, z^*\rangle}{\|q - z\|} : z \in E,\ z \neq q,\ z^* \in \partial g(z)\right\}.$$

Then

$$M = L = N \in (0, \infty].$$

(b) *Let $q^* \in E^*$ and $g^*(q^*) > \inf_{E^*} g^*$. Let*

$$M^* := \sup\left\{\frac{\langle z, q^* - z^*\rangle}{\|q^* - z^*\|} : z \in E, z^* \in \partial g(z),\ g^*(z^*) < g^*(q^*)\right\},$$

$$L^* := \sup\left\{\frac{g^*(q^*) - g^*(y^*)}{\|q^* - y^*\|} : y^* \in E^*,\ g^*(y^*) < g^*(q^*)\right\},$$

and

$$N^* := \sup\left\{\frac{\langle z, q^* - z^*\rangle}{\|q^* - z^*\|} : z \in E, z^* \in \partial g(z),\ z^* \neq q^*\right\}.$$

Then

$$M^* = L^* = N^* \in (0, \infty].$$

Proof. (a) This was proved in [8], Theorem 2.3, p. 132.

(b) We first observe that

$$z^* \in \partial g(z) \quad \Longrightarrow \quad \langle z, q^* - z^*\rangle \le g^*(q^*) - g^*(z^*). \tag{4.7}$$

It follows immediately from this that $M^* \le L^*$.

Next, let $y^* \in E^*$ and $g^*(y^*) < g^*(q^*)$. Let $g^*(y^*) < \lambda < g^*(q^*)$. Let K^* be as in Theorem 4.1(b). Let $\varepsilon \in (0, 1)$. Then, from Theorem 4.1(b),

$$(1 - \varepsilon)\frac{\lambda - g^*(y^*)}{\|q^* - y^*\|} \le (1 - \varepsilon)K^* \le N^*.$$

It follows by letting $\varepsilon \to 0$ and then $\lambda \to g^*(q^*)$ that

$$\frac{g^*(q^*) - g^*(y^*)}{\|q^* - y^*\|} \le N^*.$$

Finally, taking the supremum over y^*, we derive that $L^* \le N^*$.
Since $L^* > 0$, it follows from the above that $N^* > 0$. Now

$$N^* = M^* \vee \sup\left\{\frac{\langle z, q^* - z^*\rangle}{\|q^* - z^*\|} : z \in E, z^* \in \partial g(z),\ z^* \ne q^*,\ g^*(z^*) \ge g^*(q^*)\right\}.$$

If $z \in E$, $z^* \in \partial g(z)$, $z^* \ne q^*$ and $q^*(z^*) \ge g^*(q^*)$ then, from (4.7), $\langle z, q^* - z^*\rangle \le 0$, hence $\langle z, q^* - z^*\rangle / \|q^* - z^*\| \le 0$. Consequently $N^* = M^*$.
This completes the proof of Corollary 4.3(b). □

Corollary 4.4 contains our third new dual result. We have used the symbols "D" and "D^*" to represent the quantities that appear in it because of the similarity of the quotients with those that appear in the definition of Fréchet derivative.

Corollary 4.4. *Let E be a Banach space and $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous. Let $q \in E$, $q^* \in E^*$ and $q^* \notin \partial h(q)$. Then:*

$$D := \inf\left\{\frac{h(y) - h(q) - \langle y - q, q^*\rangle}{\|y - q\|} : y \in \operatorname{dom} h,\ y \ne q\right\} =$$

$$\inf\left\{\frac{\langle z - q, z^* - q^*\rangle}{\|z - q\|} : z \in E,\ z \ne q,\ z^* \in \partial h(z)\right\} \in [-\infty, 0),$$

and

$$D^* := \inf\left\{\frac{h^*(y^*) - h^*(q^*) - \langle q, y^* - q^*\rangle}{\|y^* - q^*\|} : y^* \in \operatorname{dom} h^*,\ y^* \ne q^*\right\} =$$

$$\inf\left\{\frac{\langle z - q, z^* - q^*\rangle}{\|z^* - q^*\|} : z \in E,\ z^* \in \partial h(z),\ z^* \ne q^*\right\} \in [-\infty, 0).$$

Proof. The first assertion was proved (after some minor changes of sign) in [8], Theorem 2.4, p. 133. The second assertion follows from Corollary 4.3(b), using the substitution $g(x) := h(x + q)$. □

As we have already observed in the introduction, our next pair of dual results contain two different quantitative generalizations of the maximal monotonicity theorem. Corollaries 4.3 and 4.4 do not use the inequalities $\|z^*\| \le (1 + \varepsilon)K$ and $\|z\| \le (1 + \varepsilon)K^*$ established in Theorem 4.1. In Theorem 4.5, we use this additional information.

Theorem 4.5. *Let E be a Banach space and $h : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous. Let $q \in E$, $q^* \in E^*$ and $q^* \notin \partial h(q)$, and D and D^* be as in* Corollary 4.4. *Then, for all $n \ge 1$, there exist $z_n \in E$ and $z_n^* \in \partial h(z_n)$ such that*

$$z_n \ne q, \quad \|z_n^* - q^*\| \to -D \text{ in } (0, \infty] \text{ and } \quad \frac{\langle z_n - q, z_n^* - q^*\rangle}{\|z_n - q\|\|z_n^* - q^*\|} \to -1$$

and, for all $n \geq 1$, there also exist $z_n \in E$ and $z_n^* \in \partial h(z_n)$ such that

$$z_n^* \neq q^*, \quad \|z_n - q\| \to -D^* \text{ in } (0,\infty] \text{ and } \frac{\langle z_n - q, z_n^* - q^*\rangle}{\|z_n - q\|\|z_n^* - q^*\|} \to -1.$$

Proof. We write $g := h - q^*$. Then $0 \notin \partial g(q)$, hence $g(q) > \inf_E g$. We define L as in Corollary 4.3. Let $0 < \gamma < 1 < \delta$ and $0 < P < L$. Using the definition of L, we first fix $y \in E$ such that

$$g(y) < g(q) \quad \text{and} \quad \frac{g(q) - g(y)}{\|q - y\|} > P.$$

We next find $\lambda \in \mathbb{R}$ such that

$$\inf_E g < \lambda < g(q) \quad \text{and} \quad \frac{\lambda - g(y)}{\|q - y\|} > P.$$

Finally, we choose $\varepsilon \in (0,1)$ such that

$$\frac{1-\varepsilon}{1+\varepsilon} \geq \gamma, \quad (1-\varepsilon)\frac{\lambda - g(y)}{\|q - y\|} > P \quad \text{and} \quad 1 + \varepsilon \leq \delta. \tag{4.8}$$

Then the hypothesis of Theorem 4.1(a) are satisfied. We now define K and choose $z \in E$ and $z^* \in \partial g(z)$ as in Theorem 4.1(a). Then, from (4.1) and (4.8),

$$\langle q - z, z^*\rangle \geq (1-\varepsilon)K\|q - z\| \geq \gamma(1+\varepsilon)K\|q - z\| \geq \gamma\|q - z\|\|z^*\|. \tag{4.9}$$

From (4.8), (4.9) and the definition of K,

$$P < (1-\varepsilon)\frac{\lambda - g(y)}{\|q - y\|} \leq (1-\varepsilon)K \leq \frac{\langle q - z, z^*\rangle}{\|q - z\|} \leq \|z^*\|. \tag{4.10}$$

Since the supremum defining L has more and larger numbers than the supremum defining K, $K \leq L$. From (4.1) and (4.8) again,

$$\|z^*\| \leq (1+\varepsilon)K \leq (1+\varepsilon)L \leq \delta L. \tag{4.11}$$

Combining (4.9), (4.10), and (4.11), we have:

$$z \neq q, \quad z^* \in \partial g(z), \quad \frac{\langle q - z, z^*\rangle}{\|q - z\|\|z^*\|} \geq \gamma, \quad \text{and} \quad P < \|z^*\| \leq \delta L.$$

Writing $w^* := z^* + q^*$, this translates to:

$$z \neq q, \quad w^* \in \partial h(z), \quad \frac{\langle q - z, w^* - q^*\rangle}{\|q - z\|\|w^* - q^*\|} \geq \gamma, \quad \text{and} \quad P < \|w^* - q^*\| \leq \delta L.$$

Since γ and δ can be chosen arbitrarily close to 1, P can be chosen arbitrarily close to L and, by direct computation, $L = -D$, the first assertion of Theorem 4.5 follows after some sign changes. The second assertion follows in an exactly similar fashion from Theorem 4.1(b), using the substitution $g(x) := h(x+q)$. □

Remark 4.6. Is there a result that unifies the two parts of Theorem 4.5 ? More specifically, if $q \in E$, $q^* \in E^*$ and $q^* \notin \partial h(q)$ write $\mathcal{N}$ for the family of all subsets of $(E \setminus \{q\}) \times (E^* \setminus \{q^*\})$ such that there exists $z_n \in E$ and $z_n^* \in \partial h(z_n)$ such that

$$(z_n, z_n^*) \in \mathcal{N} \quad \text{and} \quad \frac{\langle z_n - q, z_n^* - q^* \rangle}{\|z_n - q\| \|z_n^* - q^*\|} \to -1.$$

Theorem 4.5 then asserts that

$$\delta > 1 \text{ and } 0 < P < -D \quad \Longrightarrow \quad (E \backslash \{q\}) \times \{x^* : x^* \in E^*, P < \|x^* - q^*\| \leq -\delta D\} \in \mathcal{N}$$

and

$$\delta > 1 \text{ and } 0 < P < -D^* \quad \Longrightarrow \quad \{x^* : x^* \in E^*, P < \|x^* - q^*\| \leq -\delta D\} \times (E^* \backslash \{q^*\}) \in \mathcal{N}.$$

Can we find a characterization of $\mathcal{N}$?

Theorem 4.7 contains our last pair of dual results:

Theorem 4.7. *Let E be a Banach space and $g : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous.*

(a) *If $\inf_E g > -\infty$ and $\varepsilon > 0$ then*

$$\textit{there exist } z \in E \textit{ and } z^* \in \partial g(z) \textit{ such that } \|z^*\| \leq \varepsilon.$$

(b) *If $\inf_{E^*} g^* > -\infty$ and $\varepsilon > 0$ then*

$$\textit{there exist } z \in E \textit{ and } z^* \in \partial g(z) \textit{ such that } \|z\| \leq \varepsilon.$$

Proof. (a) is immediate from Ekeland's variational principle. In order to prove (b) we first observe that

$$0 \notin \overline{\operatorname{dom} g} \quad \Longrightarrow \quad \inf_{E^*} g^* = -\infty. \tag{4.12}$$

Indeed, if $0 \notin \overline{\operatorname{dom} g}$ then, from the separation theorem, there exists $x^* \in E^*$ such that $\sup_{\operatorname{dom} g} x^* \leq -1$. Fix $y^* \in \operatorname{dom} g^*$. Then, for all $n \geq 1$,

$$g^*(y^* + nx^*) = \sup_{\operatorname{dom} g} (y^* + nx^* - g) \leq n \sup_{\operatorname{dom} g} x^* + \sup_{\operatorname{dom} g} (y^* - g) \leq -n + g^*(y^*),$$

which establishes (4.12). So if $\inf_{E^*} g^* > -\infty$ then $0 \in \overline{\operatorname{dom} g}$, hence there exists $x \in \operatorname{dom} g$ such that $\|x\| < \varepsilon$. From the Brøndsted-Rockafellar Theorem, there exists $z \in D(\partial g)$ such that $\|z\| < \varepsilon$. This gives the required result. □

In Remark 4.8, we give an example of a specific result for which we do not know if the corresponding dual result holds.

Remark 4.8. Let E be a Banach space and $g : E \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous. Let $N \geq 0$ and $\varepsilon > 0$. It was proved in [12], Theorem 3.1, p. 1379 with $Q := \{0\}$ that if $\inf_E (g + N\| \ \|) > -\infty$ then there exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$\|z^*\| \le N + \varepsilon \quad \text{and} \quad -\langle z, z^*\rangle \ge (N - \varepsilon)\|z\|.$$

The corresponding dual result would be: if $\inf_{E^*} (g^* + N\| \ \|) > -\infty$ then there exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$\|z\| \le N + \varepsilon \quad \text{and} \quad -\langle z, z^*\rangle \ge (N - \varepsilon)\|z^*\|.$$

If $N = 0$, this reduces to the result that we have just proved in Theorem 4.7(b). Suppose, then, that $N > 0$. Scaling N to have the value 1, we arrive at the following question: if $\inf_{E^*} (g^* + \| \ \|) > -\infty$ do there necessarily exist $z \in E$ and $z^* \in \partial g(z)$ such that

$$\|z\| \le 1 + \varepsilon \quad \text{and} \quad -\langle z, z^*\rangle \ge (1 - \varepsilon)\|z^*\|?$$

Remark 4.9. We close this section by exhibiting an example showing that some results do *not* have duals. It follows from Gale's duality theorem, [4], p. 20, or [8], Theorem 2.3, p. 132 that, in Corollary 4.3 (a),

$$L < \infty \quad \Longrightarrow \quad L = \inf\{\|x^*\| : x^* \in \partial g(q)\},$$

and, in fact, the infimum is attained. (This justified the description of the quantity L in [8] as the "least slope" of g at q.) The corresponding dual assertion would be: in Corollary 4.3(b),

$$L^* < \infty \quad \Longrightarrow \quad L^* = \inf\{\|x\| : x \in E,\ \partial g(x) \ni q^*\}.$$

Of course, it would be unreasonable to ask that the infimum be attained. However, even this weaker assertion is *never* true if E is not reflexive. In this case, we use James's theorem to find $t^* \in E^*$ such that t^* does not attain its norm on the unit ball, B, of E, and define $g : E \to \mathbb{R} \cup \{\infty\}$ by $g := \mathrm{I}_B + t^*$, where I_B is the indicator function of B. Then, from the sum formula, $g^*(x^*) = \|x^* - t^*\|$. The conditions of Corollary 4.3(b) are satisfied with $q^* = 0$. Since g^* has Lipschitz constant 1, $L^* \le 1$. On the other hand, g does not attain its minimum on E, so $\{x : x \in E,\ \partial g(x) \ni q^*\} = \emptyset$.

5. A more general existence result

Theorem 5.1. *Let $m, n \ge 1$. Let $f_1, \ldots, f_m : E \to \mathbb{R} \cup \{\infty\}$ be proper and convex, for all $i = 1, \ldots, m$, $x_i^* \in \mathrm{dom}\,(f_i{}^*)$, $\mathrm{d}^+ F_m{}^*(X_m^*) : E \to \mathbb{R} \cup \{\infty\}$ and*

$$F_m{}^*(X_m^*) \ge \tau_m. \tag{5.1}$$

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $u_1^*, \ldots, u_n^* \in E^*$. Let $C_1, \ldots, C_p$ be nonempty convex subsets of E. Then the following four conditions are equivalent:

$$\left.\begin{array}{l} \textit{for all}\ \eta > 0,\ \textit{there exists}\ x \in E\ \textit{such that, for all}\ k = 1, \ldots, p,\ d_{C_k}(x) \le \eta, \\ \textit{for all}\ i = 1, \ldots, m,\ x_i^* \in \partial_\eta f_i(x)\ \textit{and, for all}\ j = 1, \ldots, n, \langle x, u_j^*\rangle \ge \alpha_j - \eta. \end{array}\right\} \tag{5.2}$$

$$\left.\begin{array}{c}\mu_1,\ldots,\mu_n \geq 0 \text{ and } w_1^*,\ldots,w_p^* \in E^* \Longrightarrow \\ {F_m}^*\left(X_m^* + \sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\right) + \sum_{k=1}^p \sigma_C(w_k^*) \geq {F_m}^*(X_m^*) + \sum_{j=1}^n \mu_j \alpha_j.\end{array}\right\} \tag{5.3}$$

$$\left.\begin{array}{c}\mu_1,\ldots,\mu_n \geq 0 \quad \Longrightarrow \\ (\mathrm{d}^+{F_m}^*(X_m^*) \underset{e}{+} \sigma_{C_1} \underset{e}{+} \cdots \underset{e}{+} \sigma_{C_p})\left(\sum_{j=1}^n \mu_j u_j^*\right) \geq \sum_{j=1}^n \mu_j \alpha_j.\end{array}\right\} \tag{5.4}$$

$$\left.\begin{array}{c}\mu_1,\ldots,\mu_n \geq 0 \quad \Longrightarrow \\ ({F_m}^* \underset{e}{+} \sigma_{C_1} \underset{e}{+} \cdots \underset{e}{+} \sigma_{C_p})\left(X_m^* + \sum_{j=1}^n \mu_j u_j^*\right) \geq \tau_m + \sum_{j=1}^n \mu_j \alpha_j.\end{array}\right\} \tag{5.5}$$

Proof. For all $k = 1,\ldots,p$, let $f_{m+k} := d_{C_k}$ and $x_{m+k}^* := 0$. Since ${f_{m+k}}^*(x_{m+k}^*) = 0$,

$$X_m^* = X_{m+p}^* \quad \text{and} \quad \tau_m = \tau_{m+p}. \tag{5.6}$$

It follows from (5.1), (5.6), and (3.1) (twice) that

$$ {F_m}^*(X_m^*) = \tau_m = \tau_{m+p} \geq {F_{m+p}}^*(X_{m+p}^*). \tag{5.7}$$

We first prove that (5.2), (5.3) and (5.4) are equivalent.

((5.2)$\Longrightarrow$(5.3)) Let $\mu_1,\ldots,\mu_n \geq 0$ and $w_1^*,\ldots,w_p^* \in E^*$. Let $\varepsilon > 0$. Choose $\eta > 0$ such that $(m + \sum_{j=1}^n \mu_j + 2\sum_{k=1}^p \|w_k^*\|)\eta < \varepsilon$. Let x be as in (5.2). Since $d_{C_1}(x),\ldots,d_{C_p}(x) \leq \eta$, for all $k = 1,\ldots,p$, we can find $y_k \in C_k$ such that $\|y_k - x\| \leq 2\eta$. Then

$$\begin{aligned}
&{F_m}^*\left(X_m^* + \sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\right) + \sum_{k=1}^p \sigma_{C_k}(w_k^*) \\
&\geq \left\langle x, X_m^* + \sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\right\rangle - F_m(x) + \sum_{k=1}^p \langle y_k, w_k^*\rangle \\
&= \sum_{i=1}^m (\langle x, x_i^*\rangle - f_i(x)) + \sum_{j=1}^n \mu_j \langle x, u_j^*\rangle + \sum_{k=1}^p \langle y_k - x, w_k^*\rangle \\
&\geq \sum_{i=1}^m ({f_i}^*(x_i^*) - \eta) + \sum_{j=1}^n \mu_j(\alpha_j - \eta) - 2\eta \sum_{k=1}^p \|w_k^*\|.
\end{aligned}$$

Thus, from the choice of η,

$$ {F_m}^*\left(X_m^* + \sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\right) + \sum_{k=1}^p \sigma_{C_k}(w_k^*) \geq \tau_m + \sum_{j=1}^n \mu_j \alpha_j - \varepsilon.$$

(5.3) follows by letting $\varepsilon \to 0$ in this and using (5.7). This completes the proof that (5.2)$\Longrightarrow$(5.3).

((5.3)$\Longrightarrow$(5.4)) Let $\mu_1, \ldots, \mu_n \geq 0$ and $w_1^*, \ldots, w_p^* \in E^*$. Let $\lambda > 0$. Replacing μ_j by $\lambda\mu_j$ and w_k^* by λw_k^* in (5.3), it follows that

$$F_m{}^*\Bigg(X_m^* + \lambda\bigg(\sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\bigg)\Bigg) + \lambda\sum_{k=1}^p \sigma_C(w_k^*) \geq F_m{}^*(X_m^*) + \lambda\sum_{j=1}^n \mu_j\alpha_j.$$

Rearranging the terms, dividing by λ and letting $\lambda \to 0$,

$$\mathrm{d}^+F_m{}^*(X_m^*)\bigg(\sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\bigg) + \sum_{k=1}^p \sigma_C(w_k^*) \geq \sum_{j=1}^n \mu_j\alpha_j.$$

(5.4) follows from this by taking the infimum over $w_1^*, \ldots, w_p^* \in E^*$.

((5.4)$\Longrightarrow$(5.2)) From (5.1), F_m is proper and convex. Further, $f_{m+1}, \ldots, f_{m+p}$ are real, convex and continuous. Thus, from the sum formula and (3.9),

$$\left.\begin{aligned} F_{m+p}{}^* =&(F_m + f_{m+1} + \cdots + f_{m+p})^* = F_m{}^* \underset{e}{+} f_{m+1}{}^* \underset{e}{+} \cdots \underset{e}{+} f_{m+p}{}^* \\ =&F_m{}^* \underset{e}{+} d_{C_1}{}^* \underset{e}{+} \cdots \underset{e}{+} d_{C_p}{}^* \geq F_m{}^* \underset{e}{+} \sigma_{C_1} \underset{e}{+} \cdots \underset{e}{+} \sigma_{C_p} \text{ on } E^*. \end{aligned}\right\} \tag{5.8}$$

Let $\mu_1, \ldots, \mu_n \geq 0$ and $w_1^*, \ldots, w_p^* \in E^*$. Then, from (2.1) and (5.4),

$$\begin{aligned} &F_m{}^*\bigg(X_m^* + \sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\bigg) + \sum_{k=1}^p \sigma_C(w_k^*) \\ &\geq F_m{}^*(X_m^*) + \mathrm{d}^+F_m{}^*(X_m^*)\bigg(\sum_{j=1}^n \mu_j u_j^* - \sum_{k=1}^p w_k^*\bigg) + \sum_{k=1}^p \sigma_C(w_k^*) \\ &\geq F_m{}^*(X_m^*) + (\mathrm{d}^+F_m{}^*(X_m^*) \underset{e}{+} \sigma_{C_1} \underset{e}{+} \cdots \underset{e}{+} \sigma_{C_p})\bigg(\sum_{j=1}^n \mu_j u_j^*\bigg) \geq F_m{}^*(X_m^*) + \sum_{j=1}^n \mu_j\alpha_j. \end{aligned}$$

Taking the infimum over $w_1^*, \ldots, w_p^* \in E^*$ and using (5.6) and (5.7),

$$(F_m{}^* \underset{e}{+} \sigma_{C_1} \underset{e}{+} \cdots \underset{e}{+} \sigma_{C_p})\bigg(X_{m+p}^* + \sum_{j=1}^n \mu_j u_j^*\bigg) \geq \tau_{m+p} + \sum_{j=1}^n \mu_j\alpha_j,$$

thus, from (5.8),

$$F_{m+p}{}^*\bigg(X_{m+p}^* + \sum_{j=1}^n \mu_j u_j^*\bigg) \geq \tau_{m+p} + \sum_{j=1}^n \mu_j\alpha_j.$$

Since this holds for all $\mu_1, \ldots, \mu_n \geq 0$, (3.2) is satisfied with m replaced by $m + p$. From Lemma 3.2, (3.3) is satisfied with m replaced by $m + p$. Since $x_{m+k}^* \in \partial_\eta f_{m+k}(x) \Longrightarrow$

$d_{C_k}(x) \leq \eta$, this completes the proof that (5.4)$\Longrightarrow$(5.2), and hence the proof of the equivalence of (5.2), (5.3) and (5.4).

It is immediate from (5.7) that (5.3) $\Longleftrightarrow$ (5.5), which completes the proof of Theorem 5.1. □

Remark 5.2. It follows from Theorem 5.1 ((5.2) $\Longrightarrow$ (5.5)) with $p = 1$ and $C_1 := E$ that if $\mathrm{d}^+ {F_m}^*(X_m^*) : E \to \mathbb{R} \cup \{\infty\}$, ${F_m}^*(X_m^*) \geq \tau_m$, and (3.3) is satisfied then,

$$\mu_1, \ldots, \mu_n \geq 0 \quad \Longrightarrow \quad {F_m}^*\Big(X_m^* + \sum_{j=1}^{n} \mu_j u_j^*\Big) \geq \tau_m + \sum_{j=1}^{n} \mu_j \alpha_j.$$

This provides a partial converse to Lemma 3.2.

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374