Journal of Convex Analysis
Volume 3 (1996), No. 2, 167–194

A Characterization of Sets of Functions and Distributions on $\mathbb{R}^n$ Described by Constraints on the Gradient

Antonio Corbo Esposito
Dipartimento di Ingegneria Industriale, Facoltà di Ingegneria,
Università di Cassino, via Di Biasio 43, 03043 Cassino, Italy.
e-mail: corbo@ing.unicas.it

Riccardo De Arcangelis
Dipartimento di Ingegneria dell'Informazione e Matematica Applicata,
Università degli Studi di Salerno, via Salvador Allende, 84081 Baronissi, Italy.
e-mail: dearcang@matna2.dma.unina.it

Received October 18, 1994
Revised manuscript received March 14, 1996

Let U be a Hausdorff locally convex topological vector subspace of $\mathcal{D}'(\mathbb{R}^n)$ verifying suitable structure conditions. A characterization of the sets $K \subseteq U$ that can be described as $K = \{u \in U : -\langle u, D\varphi\rangle \in C$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi(x)dx = 1\}$ for some closed convex subset C of $\mathbb{R}^n$ is proved. As corollaries characterizations of the sets K that can be described as $K = \{u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) : Du \in C$ for a.e. x in $\mathbb{R}^n\}$ or $K = \{u \in BV_{\mathrm{loc}}(\mathbb{R}^n) : \mathrm{meas}(A)^{-1} \int_A dDu \in C$ for every nonempty bounded open set A of $\mathbb{R}^n\}$ for some closed convex subset C of $\mathbb{R}^n$ are obtained. Similar results for subsets K of $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'$, $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $C^0(\mathbb{R}^n)$ are also proved.

1. Introduction

Some variational problems (cf. for example [3], [8], [9], [12]+[15], [12]+[14], [27]) naturally select some subsets of the Sobolev space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ pointwise constraints on the gradients and of which it would be interesting to look for abstract characterizations.

In a recent paper, cf. [23], some characterizations of the families of subsets of certain function spaces whose elements are subject to pointwise constraints on the gradient have been established.

For example, in the case of $W^{1,p}$ functions, given a family $\{K(\Omega) : \Omega$ bounded open set$\}$ of subsets of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, necessary and sufficient conditions on the family have been given so that

$$K(\Omega) = \{u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n) : Du(x) \in C \text{ for a.e. } x \text{ in } \Omega\}, \quad \text{for every bounded open set } \Omega \text{ with Lipschitz boundary.} \tag{1.1}$$

ISSN 0944-6532 / \$ 2.50

C being a closed convex subset of $\mathbb{R}^n$.

On the other side, fixed an open set Ω in $\mathbb{R}^n$ and given a single subset $K(\Omega)$ of $W^{1,p}_{\rm loc}(\mathbb{R}^n)$, necessary and sufficient conditions in order to characterize $K(\Omega)$ as in (1.1) generally look to be rather elaborated, unless $\Omega = \mathbb{R}^n$.

This is just the case treated in the present paper.

In order to describe the results obtained let us first recall that for every subset E of $\mathbb{R}^n$ the characteristic function χ_E of E is defined by $\chi_E(x) = 1$ if $x \in E$, $\chi_E(x) = 0$ if $x \in \mathbb{R}^n \backslash E$ and that a piecewise affine function u on $\mathbb{R}^n$ is a continuous function that can be expressed as

$$u(x) = \sum_{j=1}^{m} (\langle z_j, x\rangle + s_j)\chi_{P_j}(x) \quad x \in \mathbb{R}^n$$

where $z_1, \dots, z_m \in \mathbb{R}^n$, $s_1, \dots, s_m \in \mathbb{R}$ and $P_1, \dots, P_m$ are pairwise disjoint polyhedra of $\mathbb{R}^n$ with nonempty interiors such that $\bigcup_{j=1}^m P_j = \mathbb{R}^n$.

In the case of $W^{1,p}$ functions with $p \in [1, +\infty[$, we first observe that if C is a closed convex subset of $\mathbb{R}^n$ and

$$K = \{u \in W^{1,p}_{\rm loc}(\mathbb{R}^n) \colon Du(x) \in C \text{ for a.e. } x \text{ in } \mathbb{R}^n\}, \tag{1.2}$$

then K verifies the following conditions

$$\begin{aligned} u \in K, \quad c \in \mathbb{R}, \quad y \in \mathbb{R}^n \Longrightarrow \text{(i)} &\quad u + c \in K, \\ \text{(ii)} &\quad u(\cdot + y) \in K, \\ \text{(iii)} &\quad \frac{1}{t} u(t\cdot) \in K; \end{aligned} \tag{1.3}$$

$$K \quad \text{is convex;} \tag{1.4}$$

$$K \quad \text{is} \quad W^{1,p}_{\rm loc}(\mathbb{R}^n) \quad \text{closed;} \tag{1.5}$$

$$\begin{aligned} &u = \sum_{j=1}^{n} (\langle z_j, \cdot\rangle + s_j)\chi_{P_j} \quad \text{piecewise affine function on } \mathbb{R}^n \\ &\text{with} \quad \langle z_j, \cdot\rangle \in K \quad \text{for every } j \in \{1, \dots, m\} \Rightarrow u \in K. \end{aligned} \tag{1.6}$$

Then we prove that the above conditions are indeed sufficient in order to characterize the subsets of $W^{1,p}_{\rm loc}(\mathbb{R}^n)$ that can be described as in (1.2).

In fact we prove that if K is a subset of $W^{1,p}_{\rm loc}(\mathbb{R}^n)$, with $p \in [1, +\infty[$, verifying (1.3)÷(1.6) and if C is the subset of $\mathbb{R}^n$ defined by

$$C = \{z \in \mathbb{R}^n \colon \langle z, \cdot\rangle \in K\}, \tag{1.7}$$

then C is closed, convex and (1.2) holds.

The above result is deduced as a particular case by a more general characterization result (Theorem 6.3) holding for subsets of a Hausdorff locally convex topological vector subspace of the space $\mathcal{D}'$ of the distributions on $\mathbb{R}^n$ verifying suitable structure conditions.

As corollaries of Theorem 6.3, some characterization results for subsets of $BV_{\mathrm{loc}}(\mathbb{R}^n)$, of $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, of the set of the Radon measures on $\mathbb{R}^n$, of $\mathcal{S}'(\mathbb{R}^n)$ and of $\mathcal{D}'(\mathbb{R}^n)$ are also proved.

For example we prove that if K is a subset of $BV_{\mathrm{loc}}(\mathbb{R}^n)$ verifying (1.3), (1.4), (1.6) and

$$K \text{ is } w^* - BV_{\mathrm{loc}}(\mathbb{R}^n) \text{ sequentially closed,} \tag{1.8}$$

then the set C in (1.7) is closed convex and

$$K = \{u \in BV_{\mathrm{loc}}(\mathbb{R}^n) \colon \frac{1}{\mathrm{meas}(A)} \int_A dDu \in C \text{ for every nonempty bounded open subset } A \text{ of } \mathbb{R}^n\} \tag{1.9}$$

(Theorem 7.4).

Analogously, if K is a subset of $\mathcal{D}'(\mathbb{R}^n)$ verifying (1.3), (1.4), (1.6) and

$$K \text{ is } w^* - \mathcal{D}'(\mathbb{R}^n) \text{ sequentially closed,} \tag{1.10}$$

then the set C in (1.7) is closed convex and

$$K = \{u \in \mathcal{D}'(\mathbb{R}^n) \colon -\langle u, D\varphi\rangle \in C \text{ for every } \varphi \in \mathcal{D}(\mathbb{R}^n) \text{ with } \varphi \geq 0 \text{ a.e., } \int_{\mathbb{R}^n} \varphi = 1\} \tag{1.11}$$

(Theorem 7.1).

In both cases it is observed that, given a closed convex subset C of $\mathbb{R}^n$, the set K given by (1.9) (respectively by (1.11)) verifies conditions (1.3), (1.4), (1.6) and (1.8) (respectively (1.10)).

In conclusion we observe that our results might be applied, for example, to the study of homogenization problems in elastic-plastic torsion theory, we refer to [23] for a discussion of this topic in a context very similar to our one.

2. Notations and preliminary results

For every $p \in [1, +\infty]$ $W^{1,p}_{\mathrm{loc}}$, respectively L^p_{loc}, C^0, denotes the space $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, respectively $L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $C^0(\mathbb{R}^n)$; $\mathcal{D}$ denotes the space $\mathcal{D}(\mathbb{R}^n)$ of the C^∞ functions with compact support in $\mathbb{R}^n$ and $\mathcal{D}'$ the space $\mathcal{D}'(\mathbb{R}^n)$ of the distributions on $\mathbb{R}^n$.

$w^* - W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ (weak$^* - W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$) denotes the projective limit topology on $W^{1,\infty}_{\mathrm{loc}}$ of the spaces $W^{1,\infty}(\Omega)$, Ω bounded open set, endowed with their $w^* - W^{1,\infty}(\Omega)$ topologies with respect to the embedding mappings of $W^{1,\infty}_{\mathrm{loc}}$ in $W^{1,\infty}(\Omega)$.

In a similar manner the topologies $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$, $w-W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ (weak-$W^{1,p}_{\text{loc}}(\mathbb{R}^n)$), $L^p_{\text{loc}}(\mathbb{R}^n)$, $w-L^p_{\text{loc}}(\mathbb{R}^n)$ (weak-$L^p_{\text{loc}}(\mathbb{R}^n)$) and $w^*-L^\infty_{\text{loc}}(\mathbb{R}^n)$ (weak$^*-L^\infty_{\text{loc}}(\mathbb{R}^n)$) are defined.

Moreover $C^0_{\text{loc}}(\mathbb{R}^n)$ denotes the projective limit topology on C^0 of the spaces $C^0(\bar{\Omega})$, Ω bounded open set, endowed with their uniform convergence topologies with respect to the embedding mappings of C^0 in $C^0(\bar{\Omega})$.

As usual $\mathcal{S}$ denotes the set of the rapidly decreasing functions, C^0_0 the set of the continuous functions on $\mathbb{R}^n$ with compact support, C^1_0 the set of the continuously differentiable functions on $\mathbb{R}^n$ with compact support, $\mathcal{S}'$ the set of the tempered distributions and $\mathcal{M}_{\text{loc}}$ the set of the Radon measures on $\mathbb{R}^n$, i.e. the set of the regular measures defined at least for every Borel subset of $\mathbb{R}^n$ and with locally finite variations.

$\mathcal{D}(\mathbb{R}^n)$, respectively $\mathcal{S}(\mathbb{R}^n)$, denotes the usual topology of $\mathcal{D}$, respectively of $\mathcal{S}$, and $w^*-\mathcal{D}'(\mathbb{R}^n)$, respectively $w^*-\mathcal{S}'(\mathbb{R}^n)$, the weak* one of $\mathcal{D}'$, respectively of $\mathcal{S}'$. Moreover $C^0_0(\mathbb{R}^n)$ denotes the usual strict inductive limit topology on C^0_0 that makes it a *LF*-space; we recall that, by using Riesz representation theorem (see e.g. [42] 7.18), the dual space of C^0_0, endowed with the $C^0_0(\mathbb{R}^n)$ topology, is isomorphic to $\mathcal{M}_{\text{loc}}$. The weak* topology of $\mathcal{M}_{\text{loc}}$ is denoted by $w^*-\mathcal{M}_{\text{loc}}(\mathbb{R}^n)$.

Let Ω be an open set, by $BV(\Omega)$ we denote the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are Radon measures with finite total variations on Ω.

$w^*-BV(\Omega)$ denotes the weak$^*-BV(\Omega)$ topology on $BV(\Omega)$ (cf. e.g. [23] for a precise description of this topology).

BV_{loc} denotes the set of functions in L^1_{loc} that are in $BV(A)$ for every bounded open set A and $w^*-BV_{\text{loc}}(\mathbb{R}^n)$ the projective limit topology on BV_{loc} of the spaces $BV(\Omega)$, Ω bounded open set, endowed with their $w^*-BV(\Omega)$ topology with respect to the embedding mappings of BV_{loc} in $BV(\Omega)$.

For a wide exposition about BV functions we refer to [36], here we only recall that BV_{loc}, endowed with its $w^*-BV_{\text{loc}}(\mathbb{R}^n)$ topology, is a sequentially complete Hausdorff locally convex topological vector space.

For every measurable subset E of $\mathbb{R}^n$ $|E|$ denotes the Lebesgue measure of E and, for every z in $\mathbb{R}^n$, u_z the linear function $u_z(x) = \langle z, x\rangle$, $x \in \mathbb{R}^n$.

Furthermore we recall that a subset of $\mathbb{R}^n$ is said to be a polyhedron if it is the intersection of a finite number of half-spaces.

Let (U,τ) be a topological space and let X be a subset of U; $(U,\tau)-\text{cl}(X)$ denotes the closure of X in U, i.e. the set of the points in U that can be approximated by generalized sequences in X, and by $(U,\tau)_{seq}-\text{cl}(X)$ the sequential closure of X in U, i.e. the set of the points in U that can be approximated by sequences in X. Obviously $(U,\tau)_{seq}-\text{cl}(X) \subseteq (U,\tau)-\text{cl}(X)$ for every subset X of U.

Let V be a vector space and let S be a subset of V, $\text{conv}(S)$ denotes the convex hull of S, i.e. the set of the finite convex combinations of points of S.

Given a subset C of $\mathbb{R}^n$, $\Sigma(C)$ denotes the affine hull of C, that is the smallest affine subset of $\mathbb{R}^n$ containing C.

We now recall the notion of integral of a function taking values in a topological vector space (cf. [41]).
Let (U,τ) be a Hausdorff locally convex topological vector space and let $\{p_a\}_{a\in\mathcal{A}}$ be a family of seminorms defining τ.
Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ and let f be a function from E to U.

Definition 2.1. The function f is said to be τ-integrable on E if there exists $u \in U$ such that for every $\eta > 0$ and $a \in \mathcal{A}$ there exists a partition $\Delta_{\eta,a} = \{B_{\eta,a,j}\}_{j=1,\dots,m}$ of E into measurable sets such that

$$\sup\left\{ p_a\left(\sum_{j=1}^{m} f(y_j)|B_{\eta,a,j}| - u\right) : y_j \in B_{\eta,a,j},\ j\in\{1,\dots,m\}\right\} < \eta. \tag{2.1}$$

The vector u is the value of the integral of f on E and is denoted by $\int_E f(y)dy$.

We recall a property about the behaviour of the integral of a vector valued function with respect to duality (cf. [41] Corollary 5.2).

Theorem 2.2. *Let $f\colon E \to U$ be τ-integrable on E and let $L \in U^*$.
Then $\langle L, f(\cdot)\rangle$ is Lebesgue integrable on E and $\int_E \langle L, f(y)\rangle dy = \langle L, \int_E f(y)dy\rangle$.*

By Theorem 2.2 and well known subdifferentiability properties of convex functions (cf. for example [34]), the following Jensen type inequality is soon deduced.

Proposition 2.3. *Let E be a Lebesgue measurable subset of $\mathbb{R}^n$ with nonzero finite measure; let $f\colon E \to U$ be τ-integrable and such that for every open subset A of U $f^{-1}(A)$ is Lebesgue measurable.
Let $\Phi\colon U \to [0,+\infty[$ be convex, then $\Phi\circ f$ is Lebesgue measurable on E and*

$$\Phi\left(\frac{1}{|E|}\int_E f(y)dy\right) \le \frac{1}{|E|}\int_E \Phi(f(y))dy. \tag{2.2}$$

The following result yields an integrability condition (cf. [23] Proposition 2.4).

Proposition 2.4. *Let E be a Lebesgue measurable subset of $\mathbb{R}^n$, let (U,τ) be a sequentially complete Hausdorff locally convex topological vector space and let $f\colon E\to U$ be continuous with compact support.
Then f is τ-integrable on E and $\int_E f(y)dy$ is the limit of the sequences of the Cauchy sums of f, i.e. $\int_E f(y)dy = \lim_h \sum_{j=1}^{m_h} |Q_j^h\cap E| f(y_j^h)$ whenever, for every $h\in\mathbb{N}$, $\{Q_j^h\}_j$ is a partition of $\mathbb{R}^n$ made up by half open cubes with sidelength $\frac{1}{h}$, $\mathrm{spt}(f)\cap Q_j^h \neq \emptyset$ if and only if $j\in\{1,\dots,m_h\}$ and $y_1^h \in Q_1^h\cap E$, ... , $y_{m_h} \in Q_{m_h}^h \cap E$.*

For every distribution u in $\mathcal{D}'$, $y\in\mathbb{R}^n$, $t>0$ the translation of u by the vector y $T[y]u$ and the homothety of u by coefficient t $O_t'u$ are the distributions defined by (cf. [39], Chapter 1, §2.6).

$$\langle T[y]u,\varphi\rangle = \langle u, \varphi(\cdot - y)\rangle \quad \varphi\in\mathcal{D}, \tag{2.3}$$

$$\langle O'_t u, \varphi \rangle = \frac{1}{t^n} \langle u, \varphi \left(\frac{1}{t} \cdot \right) \rangle \quad \varphi \in \mathcal{D}. \tag{2.4}$$

In the following it will be useful to introduce the "rescaled" homothety $O_t u$ of u by coefficient t defined by

$$O_t u = \frac{1}{t} O'_t u. \tag{2.5}$$

Obviously, in the case of real functions on $\mathbb{R}^n$, $T[y]$ and O_t turn out to be the operators defined on the function u by $T[y]u = u(\cdot + y)$, $O_t u = \frac{1}{t} u(t\cdot)$.
For every $r > 0$ B_r denotes the open ball of $\mathbb{R}^n$ centered at the origin and of radius r.
Let α be a symmetric mollifier, i.e. a nonnegative function in $C^\infty(\mathbb{R}^n)$ such that $\mathrm{spt}(\alpha) \subseteq B_1$, $\int_{\mathbb{R}^n} \alpha = 1$ and $\alpha(-x) = \alpha(x)$ for every x in $\mathbb{R}^n$.
For every $\epsilon > 0$ and x in $\mathbb{R}^n$ set $\alpha^{(\epsilon)}(x) = \epsilon^{-n} \alpha(x/\epsilon)$ and define, for every distribution u in $\mathcal{D}'$, the regularization u_ϵ of u by

$$u_\epsilon(x) = (u * \alpha^{(\epsilon)})(x) = \langle u, \alpha^{(\epsilon)}(x - \cdot) \rangle \quad x \in \mathbb{R}^n. \tag{2.6}$$

Obviously if $u \in L^1_{\mathrm{loc}}$, the regularization u_ϵ of u turns out to be given by the classical formula

$$u_\epsilon(x) = \int_{\mathbb{R}^n} \alpha^{(\epsilon)}(x - y) u(y) dy \quad x \in \mathbb{R}^n. \tag{2.7}$$

It is well known, see for example [47] VI.3, that for every $u \in \mathcal{D}'$

$$\begin{gathered} u_\epsilon \in C^\infty(\mathbb{R}^n), \quad D^\theta u_\epsilon = u * (D^\theta \alpha^{(\epsilon)}) = (D^\theta u) * \alpha^{(\epsilon)} \\ \text{for every } \epsilon > 0 \text{ and every multiindex } \theta \in (\mathbb{N} \cup \{0\})^n; \end{gathered} \tag{2.8}$$

$$u_\epsilon \to u \quad \text{in } w^* - \mathcal{D}'(\mathbb{R}^n) \text{ as } \epsilon \to 0^+ \tag{2.9}$$

and

$$\langle u_\epsilon, \varphi \rangle = \langle u, \varphi_c \rangle \quad \text{for every } \varphi \in \mathcal{D},\ \epsilon > 0. \tag{2.10}$$

We also recall that the regularization of a distribution can be described by means of the integral of its translations.

Proposition 2.5. *Let α be the mollifier appearing in* (2.6).
Then for every $u \in \mathcal{D}'$, $\epsilon > 0$ the function $y \in \mathbb{R}^n \mapsto \alpha(y) T[\epsilon y] u \in \mathcal{D}'$ is $w^* - \mathcal{D}'(\mathbb{R}^n)$ integrable on $\mathbb{R}^n$, the integral $\int_{\mathbb{R}^n} \alpha(y) T[\epsilon y] u \, dy$ is indeed a function and

$$\left(\int_{\mathbb{R}^n} \alpha(y) T[\epsilon y] u \, dy \right)(x) = u_\epsilon(x) \quad \text{for a.e. } x \text{ in } \mathbb{R}^n. \tag{2.11}$$

In order to get informations about the left hand side of (2.11) let us consider a Hausdorff locally convex topological vector subspace (U, τ) of $\mathcal{D}'$ such that

$$u \in U,\ y \in \mathbb{R}^n \Rightarrow T[y]u \in U. \tag{2.12}$$

Proposition 2.6. *Let (U, τ) be a sequentially complete Hausdorff locally convex topological vector subspace of $\mathcal{D}'$ verifying* (2.12).
Let α be the mollifier appearing in (2.6), *$u \in U$ and $\epsilon > 0$; assume that* $\mathrm{spt}(\alpha)$ *is convex and that the function $y \in \mathbb{R}^n \mapsto T[y]u \in U$ is τ-continuous on $\mathbb{R}^n$. Then it results*

$$\int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u\, dy \in (U,\tau)_{seq} - \mathrm{cl}(\mathrm{conv}(\{T[y]u, y \in B_\epsilon\})). \tag{2.13}$$

Proof. For every $h \in \mathbb{N}$ let $\{Q_j^h\}_{j\in\mathbb{N}}$ be a sequence of pairwise disjoint half open cubes of $\mathbb{R}^n$ with sidelength $\frac{1}{h}$ and $\bigcup_{j=1}^h Q_j^h = \mathbb{R}^n$ such that for some $m_h \in \mathbb{N}$ $Q_j^h \cap \mathrm{spt}(\alpha) \neq \emptyset$ if and only if $j \in \{1, \ldots, m_h\}$.
Since for every $j \in \{1, \ldots, m_h\}$ the set $Q_j^h \cap \mathrm{spt}(\alpha)$ is connected, let $y_j^h \in Q_j^h \cap \mathrm{spt}(\alpha)$ be such that

$$\int_{Q_j^h \cap \mathrm{spt}(\alpha)} \alpha(y)dy = \alpha(y_j^h)|Q_j^h \cap \mathrm{spt}(\alpha)|, \tag{2.14}$$

then by Proposition 2.4 we get that the integral $\int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u\, dy$ exists and that

$$\sum_{j=1}^{m_h} |Q_j^h \cap \mathrm{spt}(\alpha)|\alpha(y_j^h)T[\epsilon y_j^h]u \to \int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u\, dy. \tag{2.15}$$

By (2.14), once observed that $\sum_{j=1}^{m_h} \int_{Q_j^h \cap \mathrm{spt}(\alpha)} \alpha(y)dy = 1$, we soon deduce that

$$\begin{aligned} &\sum_{j=1}^{m_h} |Q_j^h \cap \mathrm{spt}(\alpha)|\alpha(y_j^h)T[\epsilon y_j^h]u = \\ &= \sum_{j=1}^{m_h} \left(\int_{Q_j^h \cap \mathrm{spt}(\alpha)} \alpha(y)dy \right) \quad T[\epsilon y_j^h]u \in \mathrm{conv}\{T[y]u, y \in B_\epsilon\}, \end{aligned} \tag{2.16}$$

hence by (2.16) and (2.15) condition (2.13) follows. □

In conclusion we recall the following results, see [23] Proposition 4.3 and Proposition 4.4.

Proposition 2.7. *Let C be a convex subset of $\mathbb{R}^n$ and let ψ be a function in $(L^1_{\mathrm{loc}})^n$ such that*

$$\psi(x) \in C \quad \textit{for a.e. } x \textit{ in } \mathbb{R}^n. \tag{2.17}$$

Then for every $\epsilon > 0$ it results

$$\psi_\epsilon(x) \in \bar{C} \quad \textit{for every } x \textit{ in } \mathbb{R}^n. \tag{2.18}$$

Proposition 2.8. *Let G be a convex subset of $\mathbb{R}^m$ such that $\partial G \neq \emptyset$ and $B_r \subseteq G^\circ$ for some $r > 0$.*
Let $t \in]0,1[$, then

$$\operatorname{dist}(tG, \partial G) \geq r(1-t). \tag{2.19}$$

3. Distributions with constraints on the gradient

Let C be a subset of $\mathbb{R}^n$. In this section we study some properties of the sets of the distributions on $\mathbb{R}^n$ verifying a constraint on the gradient determined by C.

Such sets are defined by

$$K_C = \{u \in \mathcal{D}' \colon -\langle u, D\varphi\rangle \in C \text{ for every } \varphi \in \mathcal{D} \text{ with } \varphi \geq 0, \int_{\mathbb{R}^n} \varphi = 1\}. \tag{3.1}$$

In order to describe in some significant cases the sets in (3.1) let us introduce, for every $p \in [1, +\infty]$, the following sets

$$K_C^{1,p} = \{u \in W^{1,p}_{\text{loc}} \colon Du(x) \in C \text{ for a.e. } x \text{ in } \mathbb{R}^n\}, \tag{3.2}$$

$$\begin{aligned} K_C^{BV} =&\{u \in BV_{\text{loc}} \colon \frac{1}{|A|}\int_A dDu \in C \\ &\text{for every nonempty bounded open subset } A \text{ of } \mathbb{R}^n\}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} K_C^0 =&\{u \in C^0 \colon -\int_{\mathbb{R}^n} u\, dD\varphi \in C \\ &\text{for every } \varphi \in BV_{\text{loc}} \text{ with compact support, } \varphi \geq 0 \text{ a.e., } \int_{\mathbb{R}^n} \varphi = 1\} \end{aligned} \tag{3.4}$$

and, being p' the conjugate exponent of p,

$$\begin{aligned} K_C^p =&\{u \in L^p_{\text{loc}} \colon -\int_{\mathbb{R}^n} uD\varphi \in C \\ &\text{for every } \varphi \in W^{1,p'}_{\text{loc}} \text{ with compact support, } \varphi \geq 0 \text{ a.e., } \int_{\mathbb{R}^n} \varphi = 1\}. \end{aligned} \tag{3.5}$$

Moreover we set

$$\begin{aligned} K_C^M =&\{u \in \mathcal{M}_{\text{loc}} \colon -\int_{\mathbb{R}^n} D\varphi du \in C \\ &\text{for every } \varphi \in C_0^1 \text{ with compact support, } \varphi \geq 0, \int_{\mathbb{R}^n} \varphi = 1\} \end{aligned} \tag{3.6}$$

and

$$K_C^S = \{u \in \mathcal{S}' \colon \langle Du, \varphi\rangle \in C \quad \text{for every } \varphi \in \mathcal{S} \text{ with } \varphi \geq 0, \int_{\mathbb{R}^n} \varphi = 1\}. \tag{3.7}$$

We have

Proposition 3.1. *Let C be a closed convex subset of $\mathbb{R}^n$, $p \in [1, +\infty]$ and let K_C and $K_C^{1,p}$ be defined respectively by* (3.1) *and* (3.2).
Then

$$K_C^{1,p} = K_C \cap W^{1,p}_{\text{loc}}. \tag{3.8}$$

Proof. Let u be in $K_C \cap W^{1,p}_{\text{loc}}$, then by the divergence theorem we have

$$\int_{\mathbb{R}^n} \varphi Du \in C \text{ for every } \varphi \in \mathcal{D} \text{ with } \varphi \geq 0, \int_{\mathbb{R}^n} \varphi = 1, \tag{3.9}$$

whence we soon deduce that

$$K_C \cap W^{1,p}_{\text{loc}} \subseteq K_C^{1,p}. \tag{3.10}$$

On the other side, being C closed and convex, there exists a family $\mathcal{F} = \{(a_\sigma, b_\sigma)\}_{\sigma \in S} \subseteq \mathbb{R}^n \times \mathbb{R}$ such that

$$z \in C \Leftrightarrow \langle a_\sigma, z\rangle + b_\sigma \geq 0 \text{ for every } \sigma \in S, \tag{3.11}$$

therefore by (3.11) we get for every u in $K_C^{1,p}$

$$\langle a_\sigma, Du(x)\rangle + b_\sigma \geq 0 \text{ for every } \sigma \in S, \text{ and a.e. } x \in \mathbb{R}^n. \tag{3.12}$$

Let now φ be in $\mathcal{D}$ with $\varphi \geq 0$, $\int_{\mathbb{R}^n} \varphi = 1$, then by multiplying both sides of (3.12) by $\varphi(x)$, integrating over $\mathbb{R}^n$ and the divergence theorem we get

$$-\left\langle a_\sigma, \int_{\mathbb{R}^n} uD\varphi \right\rangle + b_\sigma \geq 0 \text{ for every } \sigma \in S. \tag{3.13}$$

By (3.13) and (3.11) we deduce

$$K_C^{1,p} \subseteq K_C \cap W^{1,p}_{\text{loc}}, \tag{3.14}$$

hence by (3.10) and (3.14) equality (3.8) follows. □

Proposition 3.2. *Let C be a closed and convex subset of $\mathbb{R}^n$ and let K_C and K_C^{BV} be defined respectively by* (3.1) *and* (3.3).
Then

$$K_C^{BV} = K_C \cap BV_{\text{loc}}. \tag{3.15}$$

Proof. Let u be in $K_C \cap BV_{\rm loc}$.
Let A be a nonempty bounded open set and let $\{\varphi_h\}_h \subseteq C_0^\infty(A)$ be such that $0 \le \varphi_h \le \varphi_{h+1}$ and $\varphi_h(x) \to \chi_A(x)$ for every $x \in \mathbb{R}^n$.
Being u in $BV_{\rm loc}$, by the closedness of C we get

$$\frac{1}{|A|}\int_A dDu = \lim_h \frac{1}{|A|}\int_{\mathbb{R}^n} \varphi_h dDu = -\lim_h \int_{\mathbb{R}^n} u dD\left(\frac{1}{\int_A \varphi_h}\varphi_h\right) \in C \tag{3.16}$$

hence by (3.16) we deduce

$$K_C \cap BV_{\rm loc} \subseteq K_C^{BV}. \tag{3.17}$$

In order to prove the reverse inclusion in (3.17) we first observe that if $u \in K_C^{BV}$ then by approximation it follows that

$$\frac{1}{|E|}\int_E dDu \in C \text{ for every bounded Borel set } E \text{ such that } |E| > 0. \tag{3.18}$$

Let $u \in K_C^{BV}$, $\psi \in \mathcal{D}$ be such that $\psi \ge 0$, $\nu \in \mathbb{N}$; let us consider a partition of $\mathbb{R}^n$ made up by half open cubes Q_j^ν, $j \in \mathbb{N}$, of sidelength $\frac{1}{\nu}$ and let $S_\nu = \{j \in \mathbb{N} \colon Q_j^\nu \cap \mathrm{spt}(\psi) \neq \emptyset\}$, then by (3.18) we have that $\frac{1}{|Q_j^\nu|}\int_{Q_j^\nu} dDu \in C$ for every $j \in S_\nu$, hence, being C convex, we have

$$\begin{aligned}\frac{1}{\int_{\mathbb{R}^n}\psi}\int_{\mathbb{R}^n} \sum_{j\in S_\nu}\Big(\frac{1}{|Q_j^\nu|}\int_{Q_j^\nu}\psi\Big)\chi_{Q_j^\nu} dDu = \\ = \sum_{j \in S_\nu} \frac{\int_{Q_j^\nu}\psi}{\int_{\mathbb{R}^n}\psi}\frac{1}{|Q_j^\nu|}\int_{Q_j^\nu} dDu \in C \text{ for every } \nu \in \mathbb{N},\ j \in S_\nu.\end{aligned} \tag{3.19}$$

Let us now observe that the sequence of functions $\sum_{j\in S_\nu}\left(\frac{1}{|Q_j^\nu|}\int_{Q_j^\nu}\psi\right)\chi_{Q_j^\nu}$ converges uniformly on $\mathbb{R}^n$ to ψ as $\nu \to \infty$, hence, being Du a measure with finite total variation on $\mathrm{spt}(\psi)$, by taking the limit as $\nu \to \infty$ in (3.19) we obtain

$$-\langle u, D\varphi\rangle = \int_{\mathbb{R}^n} \varphi dDu \in C \text{ for every } \varphi \in \mathcal{D} \text{ with } \varphi \ge 0, \int_{\mathbb{R}^n}\varphi = 1, \tag{3.20}$$

that is

$$K_C^{BV} \subseteq K_C \cap BV_{\rm loc}. \tag{3.21}$$

By (3.17) and (3.21) equality (3.15) follows. □

Proposition 3.3. *Let C be a closed and convex subset of $\mathbb{R}^n$ and let K_C, K_C^0, K_C^p, $p \in [1, +\infty]$, and K_C^M be defined respectively by* (3.1), (3.4), (3.5) *and* (3.6).
Then

$$K_C^0 = K_C \cap C^0; \tag{3.22}$$

$$K_C^p = K_C \cap L^p_{\text{loc}} \; \textit{for every}\, p \in [1, +\infty]; \tag{3.23}$$

$$K_C^M = K_C \cap \mathcal{M}_{\text{loc}}. \tag{3.24}$$

Proof. Let us prove (3.22), the proof of (3.23) and (3.24) being similar with obvious changes.

It is clear that

$$K_C^0 \subseteq K_C \cap C^0. \tag{3.25}$$

In order to prove the reverse inclusion let u be in $K_C \cap C^0$ and let $\varphi \in BV(\mathbb{R}^n)$ with compact support, $\varphi \geq 0$ a.e. and $\int_{\mathbb{R}^n} \varphi = 1$.
For every $\epsilon > 0$ let φ_ϵ be the regularization of φ given by (2.7), then $\varphi_\epsilon \in \mathcal{D}$, $\varphi_\epsilon \geq 0$ in $\mathbb{R}^n$, $\int_{\mathbb{R}^n} \varphi_\epsilon = 1$ and $\int_{\mathbb{R}^n} \psi D\varphi_\epsilon \to \int_{\mathbb{R}^n} \psi d D\varphi$ for every ψ in C^0.
Being u in $K_C \cap C^0$ we have

$$-\int_{\mathbb{R}^n} u D\varphi_\epsilon \in C \text{ for every } \epsilon > 0, \tag{3.26}$$

hence by the closedness of C, as $\epsilon \to 0^+$, we deduce by (3.26) that

$$K_C \cap C^0 \subseteq K_C^0. \tag{3.27}$$

By (3.25) and (3.27) equality (3.22) follows. □

Proposition 3.4. *Let C be a closed and convex subset of $\mathbb{R}^n$ and let K_C and K_C^S be defined respectively by* (3.1) *and* (3.7)*.*
Then

$$K_C^S = K_C \cap \mathcal{S}'. \tag{3.28}$$

Proof. Let us recall that (see e.g. [47] VI.1)

$$\begin{cases} u \in \mathcal{S}' \Rightarrow Du \in (\mathcal{S}')^n \\ \langle u, D\varphi \rangle = -\langle Du, \varphi \rangle \qquad \text{for every } u \in \mathcal{S}', \varphi \in \mathcal{S}, \end{cases} \tag{3.29}$$

whence we soon have

$$K_C^S \subseteq K_C \cap \mathcal{S}'. \tag{3.30}$$

In order to prove the reverse inclusion in (3.30) let $\{\psi_h\}_h \subseteq \mathcal{D}$ be such that $0 \leq \psi_h \leq 1$, $\psi_h(x) = 1$, if $|x| \leq h$ and $\sup_{x \in \mathbb{R}^n} |D^\theta \psi_h(x)| \leq 1$ for every $h \in \mathbb{N}$ and every multindex $\theta \in \mathbb{N}^n$, then it is easy to see that

$$\varphi\psi_h \to \varphi \quad \text{in} \quad \mathcal{S}(\mathbb{R}^n) \quad \text{for every } \varphi \in \mathcal{S}. \tag{3.31}$$

Let $u \in K_C \cap \mathcal{S}'$, $\varphi \in \mathcal{S}$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi = 1$, then by (3.29) and the closedness of C we deduce that

$$\langle Du, \varphi \rangle = -\langle u, D\varphi \rangle = -\lim_h \left\langle u, D\left(\frac{1}{\int_{\mathbb{R}^n} \psi_h \varphi} \psi_h \varphi\right)\right\rangle \in C \tag{3.32}$$

and

$$K_C \cap \mathcal{S}' \subseteq K_C^S. \tag{3.33}$$

By (3.30) and (3.33) equality (3.28) follows. □

We now go back to the general case.

Proposition 3.5. *Let C be a subset of $\mathbb{R}^n$ and let K_C be given by* (3.1)*. Then*

$$u \in K_C,\ c \in \mathbb{R},\ y \in \mathbb{R}^n,\ t > 0 \Longrightarrow \begin{aligned} &\text{(i) } u + c \in K_C, \\ &\text{(ii) } T[y]u \in K_C, \\ &\text{(iii) } O_t u \in K_C; \end{aligned} \tag{3.34}$$

$$C \textit{ convex} \Rightarrow K_C \textit{ convex}; \tag{3.35}$$

$$C \textit{ closed} \Rightarrow K_C\ w^* - \mathcal{D}'(\mathbb{R}^n) \textit{ closed}; \tag{3.36}$$

$$\begin{aligned} u = \sum_{j=1}^{m} (u_{z_j} + s_j)\chi_{P_j} \textit{ piecewise affine function on } \mathbb{R}^n \\ \textit{such that } u_{z_j} \in K_C \textit{ for every } j \in \{1, \ldots, m\} \Rightarrow u \in K_C. \end{aligned} \tag{3.37}$$

Proof. Conditions (i) and (ii) in (3.34) can be easily obtained.
Let us prove (iii) of (3.34).

Let $u \in K_C$, $t > 0$ and let $\varphi \in \mathcal{D}$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi = 1$.
Let us observe that

$$O_{1/t}(D\varphi) = tD(O_{1/t}\varphi), \tag{3.38}$$

hence by (3.38) we have

$$\begin{aligned} -\langle O_t u, D\varphi \rangle &= -\frac{1}{t^{n+2}} \langle u, O_{1/t}(D\varphi) \rangle = \\ &= -\frac{1}{t^{n+1}} \left\langle u, D\left(O_{1/t}\varphi\right)\right\rangle = \\ &= -\langle u, D\left(\frac{1}{t^{n+1}} O_{1/t}\varphi\right)\rangle. \end{aligned} \tag{3.39}$$

By (3.39), once observed that the function $\frac{1}{t^{n+1}} O_{1/t}\varphi$ is in $\mathcal{D}$ with $\frac{1}{t^{n+1}} O_{1/t}\varphi \geq 0$, $\int_{\mathbb{R}^n} \frac{1}{t^{n+1}} O_{1/t}\varphi = 1$ and once recalled that u is in K_C, we get that

$$-\langle O_t u, D\varphi\rangle \in C \text{ for every } \varphi \in \mathcal{D} \text{ with } \varphi \geq 0 \text{ and } \int_{\mathbb{R}^n} \varphi = 1. \tag{3.40}$$

By (3.40) condition (iii) in (3.34) follows.

Condition (3.35) is trivially verified.

Let us prove (3.36). To this aim let $u \in \mathcal{D}'$ and let $\{u_\lambda\}_{\lambda\in\Lambda}$ be a generalized sequence in $\mathcal{D}'$ that converges to u in $w^* - \mathcal{D}'(\mathbb{R}^n)$.

For every $\varphi \in \mathcal{D}$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi = 1$, by using the closedness of C, it follows that

$$-\langle u, D\varphi\rangle = \lim_\lambda -\langle u_\lambda, D\varphi\rangle \in C \tag{3.41}$$

and (3.36) comes.

In conclusion let us prove (3.37).

Let $u = \sum_{j=1}^m (u_{z_j} + s_j)\chi_{p_j}$ be as in (3.37), then by Proposition 3.1 applied with $p = +\infty$ it follows that

$$z_j \in C \text{ for every } j \in \{1, \dots, m\}, \tag{3.42}$$

therefore by (3.42) we deduce

$$Du(x) \in C \text{ for a.e. } x \text{ in } \mathbb{R}^n. \tag{3.43}$$

By (3.43) and again Proposition 3.1 applied with $p = +\infty$, (3.37) follows. □

4. A representation result

Let C be a closed convex set in $\mathbb{R}^n$ and (U, τ) a topological vector subspace of $\mathcal{D}'$, in the present section we want to prove a representation result for the set $(U,\tau)_{seq} - \mathrm{cl}(K_C^{1,\infty})$, $K_C^{1,\infty}$ being defined by (3.2) with $p = +\infty$.
We assume that

$$\tau \text{ is finer than } w^* - \mathcal{D}'(\mathbb{R}^n) \tag{4.1}$$

and that, being for $u \in \mathcal{D}'$ and $\epsilon > 0$, u_ϵ the regularization of u given by (2.7),

$$\begin{cases} \text{(i) } u_\epsilon \in U \text{ for every } u \in U,\ \epsilon > 0 \text{ small enough,} \\ \text{(ii) } u_\epsilon \to u \text{ in } \tau \text{ as } \epsilon \to 0^+ \text{ for every } u \in U. \end{cases} \tag{4.2}$$

Proposition 4.1. *Let (U, τ) be a topological vector subspace of $\mathcal{D}'$ verifying* (4.1) *and* (4.2).

Let C be a closed convex subset of $\mathbb{R}^n$, $K_C^{1,\infty}$ be defined by (3.2) with $p = +\infty$ and K_C by (3.1).
Then

$$(U,\tau) - \mathrm{cl}(K_C^{1,\infty} \cap U) = (U,\tau)_{seq} - \mathrm{cl}(K_C^{1,\infty} \cap U) = K_C \cap U. \tag{4.3}$$

Proof. Let us preliminarily observe that by Proposition 3.1 with $p = +\infty$ we have

$$K_C^{1,\infty} = K_C \cap W^{1,\infty}_{\mathrm{loc}} \subseteq K_C. \tag{4.4}$$

By Proposition 3.5 the set K_C is $w^* - \mathcal{D}'(\mathbb{R}^n)$ closed, therefore by (4.1) $K_C \cap U$ is τ-closed. From this and (4.4) we infer

$$(U,\tau) - \mathrm{cl}(K_C^{1,\infty} \cap U) \subseteq K_C \cap U. \tag{4.5}$$

In order to prove the reverse inclusion in the sequential case, let u be in $K_C \cap U$ and let, for $\epsilon > 0$, u_ϵ be the regularization of u given by (2.7).

For fixed $\epsilon > 0$ and x in $\mathbb{R}^n$ the function $\alpha^{(\epsilon)}(x - \cdot)$ is in $\mathcal{D}$ with $\alpha^{(\epsilon)}(x - \cdot) \geq 0$, $\int_{\mathbb{R}^n} \alpha^{(\epsilon)}(x-y)dy = 1$ and

$$(D\alpha^{(\epsilon)})(x-y) = -D(\alpha^{(\epsilon)}(x - \cdot))(y) \text{ for every } y \in \mathbb{R}^n, \tag{4.6}$$

the gradient in the left hand side of (4.6) being taken with respect to the set of the variables of $\alpha^{(\epsilon)}$; therefore by (2.8) and (4.6) it results

$$Du_\epsilon(x) = D(u * \alpha^{(\epsilon)})(x) = \langle u, (D\alpha^{(\epsilon)})(x - \cdot)\rangle = -\langle u, D(\alpha^{(\epsilon)}(x - \cdot))\rangle \in C \tag{4.7}$$

that is, by (4.2) (i),

$$u_\epsilon \in K_C^{1,\infty} \cap U \text{ for every } \epsilon > 0 \text{ small enough.} \tag{4.8}$$

As $\epsilon \to 0^+$ by (4.2) we deduce that

$$K_C \cap U \subseteq (U,\tau)_{seq} - \mathrm{cl}(K_C^{1,\infty} \cap U). \tag{4.9}$$

By (4.5) and (4.9) the thesis follows. □

By applying Proposition 4.1 to some particular cases we obtain the following results.

Proposition 4.2. *Let C be a closed convex subset of $\mathbb{R}^n$.*
For every $p \in [1,+\infty]$ let K_C^p be defined by (3.5), K_C^0 by (3.4) and $K_C^{1,\infty}$ by (3.2).
Then

$$(L^p_{\mathrm{loc}}, L^p_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K_C^{1,\infty}) = (L^p_{\mathrm{loc}}, L^p_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K_C^{1,\infty}) = K_C^p \textit{ if } p \in [1,+\infty[; \tag{4.10}$$

$$(L^\infty_{\mathrm{loc}}, w^* - L^\infty_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K_C^{1,\infty}) = (L^\infty_{\mathrm{loc}}, w^* - L^\infty_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K_C^{1,\infty}) = K_C^\infty; \tag{4.11}$$

$$(C^0, C^0_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K^{1,\infty}_C) = (C^0, C^0_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K^{1,\infty}_C) = K^0_C. \tag{4.12}$$

Proof. Let us prove (4.12), the proof of (4.10) and (4.11) being similar with the obvious changes of spaces.
We observe that the space $(C^0, C^0_{\mathrm{loc}}(\mathbb{R}^n))$ satisfies conditions (4.1) and (4.2), hence by Proposition 4.1, once observed that $K^{1,\infty}_C \cap C^0 = K^{1,\infty}_C$, we get that

$$\begin{aligned}&(C^0, C^0_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K^{1,\infty}_C) = (C^0, C^0_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K^{1,\infty}_C) = \\ &= \left\{u \in C^0 \colon - \int_{\mathbb{R}^n} uD\varphi \in C \text{ for every } \varphi \in \mathcal{D} \text{ with } \varphi \geq 0, \int_{\mathbb{R}^n} \varphi = 1\right\}.\end{aligned} \tag{4.13}$$

Therefore the thesis follows by (4.13) and Proposition 3.3, although the first equality in (4.12) can be deduced by standard general topology arguments. □

Proposition 4.3. *Let C be a closed convex subset of $\mathbb{R}^n$.*
For every $p \in [1, +\infty]$ let $K^{1,p}_C$ be defined by (3.2) *and K^{BV}_C by* (3.3).
Then

$$(BV_{\mathrm{loc}}, w^* - BV_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K^{1,\infty}_C) = (BV_{\mathrm{loc}}, w^* - BV_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K^{1,\infty}_C) = K^{BV}_C; \tag{4.14}$$

$$(W^{1,p}_{\mathrm{loc}}, W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K^{1,\infty}_C) = (W^{1,p}_{\mathrm{loc}}, W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K^{1,\infty}_C) = K^{1,p}_C \quad \textit{if} \quad p \in [1, +\infty[; \tag{4.15}$$

$$(W^{1,\infty}_{\mathrm{loc}}, w^* - W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)) - \mathrm{cl}(K^{1,\infty}_C) = (W^{1,\infty}_{\mathrm{loc}}, w^* - W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n))_{seq} - \mathrm{cl}(K^{1,\infty}_C) = K^{1,\infty}_C. \tag{4.16}$$

Proof. Let us prove (4.14).
We observe that $(BV_{\mathrm{loc}}, w^* - BV_{\mathrm{loc}}(\mathbb{R}^n))$ satisfies conditions (4.1) and (4.2) and that $K^{1,\infty}_C \cap BV_{\mathrm{loc}} = K^{1,\infty}_C$, hence (4.14) follows by Proposition 4.1 and Proposition 3.2.
The proof of (4.15) and (4.16) are similar to the one of (4.14) with the obvious changes and by using Proposition 3.1 in place of Proposition 3.2. □

5. The characterization result in a particular case

Let K be a subset of $\mathcal{D}'$, in the present section and in the next one we give necessary and sufficient conditions on K for the existence of a closed convex subset C of $\mathbb{R}^n$ such that $K = K_C$, K_C being defined by (3.1).
Having in mind Proposition 3.5 we assume that K satisfies the following assumptions

$$\begin{aligned} u \in K, c \in \mathbb{R}, y \in \mathbb{R}^n, t > 0 \Longrightarrow &\text{(i) } u + c \in K, \\ &\text{(ii) } T[y]u \in K, \\ &\text{(iii) } O_t u \in K; \end{aligned} \tag{5.1}$$

$$K \text{ convex;} \tag{5.2}$$

$$u = \sum_{j=1}^{m} (u_{z_j} + s_j)\chi_{P_j} \text{ piecewise affine function on } \mathbb{R}^n \text{ such that } u_{z_j} \in K \quad \text{for every } j \in \{1, \dots, m\} \Rightarrow u \in K. \tag{5.3}$$

Let us now observe that, given a closed convex set C in $\mathbb{R}^n$, by Proposition 3.1 with $p = +\infty$ it soon follows that

$$z \in C \Leftrightarrow u_z \in K_C, \tag{5.4}$$

hence it is natural to deduce by the set K the constraint C in the following way

$$C = \{z \in \mathbb{R}^n \colon u_z \in K\}. \tag{5.5}$$

Proposition 5.1. *Let K be a subset of $\mathcal{D}'$ and let C be defined by* (5.5).
Then

a) *if*

$$K \cap W^{1,\infty}_{\text{loc}} \textit{ is } W^{1,\infty}_{\text{loc}}(\mathbb{R}^n) \textit{ closed}, \tag{5.6}$$

C is closed;

b) *if* (5.2) *holds, C is convex.*

Proof. Let $z \in \mathbb{R}^n$ and let $\{z_h\}_h \subseteq C$ with $z_h \to z$, then $u_{z_h} \in K \cap W^{1,\infty}_{\text{loc}}$ for every $h \in \mathbb{N}$ and

$$u_{z_h} \to u_z \text{ in } W^{1,\infty}_{\text{loc}}(\mathbb{R}^n). \tag{5.7}$$

By (5.6) and (5.7) we soon deduce that $u_z \in K$, that is $z \in C$; by virtue of this the closedness of C follows.
Finally the convexity of C trivially follows by (5.2). □

In order to verify that K is indeed a set of distributions defined by the constraint on the gradient described by C in (5.5), we assume that

$$K \cap W^{1,\infty}_{\text{loc}} \text{ is } w^* - W^{1,\infty}_{\text{loc}}(\mathbb{R}^n) \text{ sequentially closed.} \tag{5.8}$$

Proposition 5.2. *Let K be a subset of $\mathcal{D}'$ verifying* (5.1), (5.8)*; let C be defined by* (5.5) *and $K^{1,\infty}_C$ by* (3.2) *with $p = +\infty$.*
Then

$$K \cap W^{1,\infty}_{\text{loc}} \subseteq K^{1,\infty}_C. \tag{5.9}$$

Proof. Let $u \in K \cap W^{1,\infty}_{\text{loc}}$, let us prove that

$$Du(x_0) \in C \text{ for a.e. } x_0 \in \mathbb{R}^n. \tag{5.10}$$

For every $x_0 \in \mathbb{R}^n$, $t > 0$ by (5.1) we have

$$T[-x_0]O_tT[x_0]u - \frac{u(x_0)}{t} = \frac{u(x_0 + t(\cdot - x_0)) - u(x_0)}{t} \in K \cap W^{1,\infty}_{\text{loc}}. \tag{5.11}$$

Being u in $W^{1,\infty}_{\text{loc}}$, by Rademacher theorem, u is differentiable a.e. in $\mathbb{R}^n$, i.e.

$$\begin{aligned} &\frac{u(x_0 + t(\cdot - x_0)) - u(x_0)}{t} \to \langle Du(x_0), \cdot - x_0 \rangle \text{ in } C^0_{\text{loc}}(\mathbb{R}^n) \\ &\text{as } t \to 0^+ \text{ for a.e. } x_0 \in \mathbb{R}^n; \end{aligned} \tag{5.12}$$

moreover, if x_0 is a Lebesgue point for Du, we have

$$D\left(T[-x_0]O_tT[x_0]u - \frac{u(x_0)}{t}\right) \to Du(x_0) \text{ in } w^* - L^\infty_{\text{loc}}(\mathbb{R}^n) \text{ as } t \to 0^+. \tag{5.13}$$

By (5.12) and (5.13) we get

$$\begin{aligned} &T[-x_0]O_tT[x_0]u - \frac{u(x_0)}{t} \to \langle Du(x_0), \cdot - x_0 \rangle \text{ in } w^* - W^{1,\infty}_{\text{loc}}(\mathbb{R}^n) \\ &\text{as } t \to 0^+ \text{ for a.e. } x_0 \in \mathbb{R}^n, \end{aligned} \tag{5.14}$$

hence by (5.14), (5.11) and (5.8) we deduce that

$$\langle Du(x_0), \cdot - x_0 \rangle \in K \cap W^{1,\infty}_{\text{loc}} \text{ for a.e. } x_0 \in \mathbb{R}^n. \tag{5.15}$$

By (5.15) and (i) of (5.1) condition (5.10) soon follows.
Finally by (5.10) we deduce the thesis. □

The remaining part of this section is devoted to the proof of the reverse inclusion in (5.9). A first step in this direction is made by using condition (5.3), in fact if K is a subset of $\mathcal{D}'$ verifying (5.3), C is defined by (5.5) and $K^{1,\infty}_C$ by (3.2) with $p = +\infty$, then

$$u \text{ piecewise affine function on } \mathbb{R}^n \; u \in K^{1,\infty}_C \Rightarrow u \in K \cap W^{1,\infty}_{\text{loc}}. \tag{5.16}$$

In order to extend (5.16) to wider classes of functions let us assume that K verifies (5.1)÷(5.3) and (5.6), then, by virtue of Proposition 5.1, the set C defined in (5.5) is closed and convex.
Let $\Sigma(C)$ be the affine hull of C and let ν ($\leq n$) be its dimension, then C possesses interior points in the topology of $\Sigma(C)$ and, if $C \neq \emptyset$, it is not restrictive to assume that $0 \in C$ (in fact if this is not the case it is sufficient to consider, for $z_0 \in C$, the sets $K - u_{z_0}$ and $C - z_0$). Moreover, by using the same argument as before, it is not restrictive to assume that

$$0 \in C^\circ \quad \text{the interior being taken in the topology of } \Sigma(C). \tag{5.17}$$

Let $R\colon \mathbb{R}^n \to \mathbb{R}^n$ be the identity transformation if $\nu = n$, and, if $\nu < n$, an orthogonal linear transformation such that

$$R(\Sigma(C)) = \mathbb{R}^\nu \times \{0_{n-\nu}\} \tag{5.18}$$

$0_{n-\nu}$ being the origin of $\mathbb{R}^{n-\nu}$.

For every $u \in K_C^{1,\infty}$ we define the functions u' and $\hat{u}$ as

$$u'(y) = u(R^{-1}y) \quad y \in \mathbb{R}^n, \tag{5.19}$$

$$\hat{u}(y_1, \ldots, y_\nu) = \begin{cases} u(y_1, \ldots, y_n) & \text{if } \nu = n \\ u'(y_1, \ldots, y_\nu, 0_{n-\nu}) & \text{if } \nu < n \end{cases} \quad (y_1, \ldots, y_\nu) \in \mathbb{R}^\nu. \tag{5.20}$$

Since $R^{-1} = R^T$ we have that

$$D_y u'(y) = D_x u(R^{-1}y)R^{-1} = D_x u(R^{-1}y)R^T = (RD_x u(R^{-1}y)^T)^T \text{ for a.e. } y \in \mathbb{R}^n, \tag{5.21}$$

that is

$$D_y u'(y) \in \mathbb{R}^\nu \times \{0_{n-\nu}\} \text{ for a.e. } y \in \mathbb{R}^n, \tag{5.22}$$

hence by (5.22) and (5.18) it follows that u' effectively depends only on $(y_1, \ldots y_\nu)$ when $(y_1, \ldots, y_n)$ varies in $\mathbb{R}^n$ and that

$$D\hat{u}(y_1, \ldots, y_\nu) \in \Pr_\nu(RC) \text{ for every } (y_1, \ldots, y_\nu) \in \mathbb{R}^\nu, \tag{5.23}$$

$\Pr_\nu$ being the projection function from $\mathbb{R}^n$ to $\mathbb{R}^\nu$ defined by $\Pr_\nu(y_1, \ldots, y_n) = (y_1, \ldots, y_\nu)$ for every $(y_1, \ldots, y_n) \in \mathbb{R}^n$.

In conclusion by (5.17) and (5.18) we can also assume that 0_ν, the origin of $\mathbb{R}^\nu$, belongs to the interior (in $\mathbb{R}^\nu$) of $\Pr_\nu(RC)$, i.e.

$$0_\nu \in (\Pr_\nu(RC))^\circ. \tag{5.24}$$

We have

Lemma 5.3. *Let K be a subset of $\mathcal{D}'$ verifying* (5.1) (ii), (5.2), (5.3) *and* (5.6)*; let C be defined by* (5.5) *and $K_C^{1,\infty}$ by* (3.2) *with $p = +\infty$.*

Then for every $u \in K_C^{1,\infty}$

$$\hat{u} \in C_0^\infty(\mathbb{R}^\nu) \Rightarrow u \in K. \tag{5.25}$$

Proof. Let u be in $K_C^{1,\infty}$ and t in $]0,1[$.

Let R be the above defined mapping, then by virtue of (5.2) and Proposition 5.1, $\Pr_\nu(RC)$ turns out to be convex; moreover, being $Du(x) \in C$ for a.e. x in $\mathbb{R}^n$, by (5.24) we infer

$$tD\hat{u}(y_1, \ldots, y_\nu) \in (\Pr_\nu(RC))^\circ \text{ for every } (y_1, \ldots, y_\nu) \in \mathbb{R}^\nu. \tag{5.26}$$

By (5.23) and Proposition 2.8 applied with $m = \nu$ and $G = \Pr_\nu(RC)$, using the convention that $\operatorname{dist}(z, \emptyset) = +\infty$ for every $z \in \mathbb{R}^m$, we get that there exists $\delta > 0$ such that

$$\operatorname{dist}(tD\hat{u}(y_1, \ldots, y_\nu), \partial \Pr_\nu(RC)) > \delta \text{ for every } (y_1, \ldots, y_\nu) \in \mathbb{R}^\nu. \tag{5.27}$$

Since $\hat{u} \in C_0^\infty(\mathbb{R}^\nu)$, let $\{\hat{u}_h\}_h$ be a sequence of piecewise affine functions on $\mathbb{R}^\nu$ such that

$$\begin{cases} \hat{u}_h \in L^\infty(\mathbb{R}^\nu) & \text{for every } h \in \mathbb{N}, \\ \hat{u}_h \to t\hat{u} & \text{uniformly on } \mathbb{R}^\nu \\ D\hat{u}_h \to tD\hat{u} & \text{in } (L^\infty(\mathbb{R}^\nu))^\nu \end{cases} \tag{5.28}$$

(see for example Proposition 2.1 at page 309 in [34]), then by (5.26), (5.27) and (5.28) it follows that for h large enough

$$D\hat{u}_h(y_1, \ldots, y_\nu) \in \mathrm{Pr}_\nu(RC) \text{ for a.e. } (y_1, \ldots, y_\nu) \in \mathbb{R}^\nu. \tag{5.29}$$

By (5.28) if we define the functions u_h' and u_h by

$$\begin{cases} u_h'(y_1, \ldots, y_n) = \hat{u}_h(y_1, \ldots, y_\nu) \\ u_h(y) = u_h'(Ry), \quad y \in \mathbb{R}^n \end{cases} \tag{5.30}$$

we deduce that $\{u_h\}_h$ is a sequence of piecewise affine functions on $\mathbb{R}^n$ such that

$$u_h \to tu \text{ in } W^{1,\infty}(\mathbb{R}^n), \tag{5.31}$$

moreover by (5.29) we have that for h large enough

$$Du_h(x) \in C \text{ for a.e. } x \in \mathbb{R}^n. \tag{5.32}$$

By (5.32) and (5.16) we deduce that for h large enough

$$u_h \in K \tag{5.33}$$

hence by (5.33), (5.31) and (5.6) we get

$$tu \in K \text{ for every } t \in]0, 1[. \tag{5.34}$$

As $t \mapsto 1^-$ by (5.34) and (5.6) the thesis follows. □

By Lemma 5.3 we deduce the following result.

Lemma 5.4. *Let K be a subset of $\mathcal{D}'$ verifying* (5.1) (ii), (5.2), (5.3), *and* (5.6)*; let C be defined by* (5.5) *and $K_C^{1,\infty}$ by* (3.2) *with $p = +\infty$.*
Then for every $u \in K_C^{1,\infty}$

$$\hat{u} \in C^\infty(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu) \Rightarrow u \in K. \tag{5.35}$$

Proof. Let u be in $K_C^{1,\infty}$ and t in]0,1[.
As in the proof of the Lemma 5.3 there exists $\delta > 0$ such that (5.27) holds.
For every $h \in \mathbb{N}$ let $\varphi_h \in C_0^\infty(\mathbb{R}^\nu)$ be such that

$$\begin{cases} 0 \leq \varphi_h \leq 1, \varphi_h(x) = 1, \text{ for every } x \text{ such that } |x| \leq h \\ \|D\varphi_h\|_{L^\infty(\mathbb{R}^\nu)^\nu} \leq \dfrac{\delta}{\|\hat{u}\|_{L^\infty(\mathbb{R}^\nu)} + 1} \end{cases} \tag{5.36}$$

and define the functions $\hat{w}_h$ as

$$\hat{w}_h(y_1, \dots, y_\nu) = \varphi_h(y_1, \dots y_\nu)\hat{u}(y_1, \dots, y_\nu) \quad (y_1, \dots, y_\nu) \in \mathbb{R}^\nu. \tag{5.37}$$

By (5.36) we soon have

$$tD\hat{w}_h(y_1, \dots y_\nu) \in \mathrm{Pr}_\nu(RC) \text{ for every } (y_1, \dots, y_\nu) \in \mathbb{R}^\nu, \tag{5.38}$$

hence, if we define the functions w'_h and w_h in the same way of (5.30), by (5.38) and Lemma 5.3 we deduce that

$$tw_h \in K \text{ for every } h \in \mathbb{N}. \tag{5.39}$$

By (5.36) and (5.37) we soon have that

$$w_h \to u \text{ in } W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n), \tag{5.40}$$

hence by (5.40), (5.6) and (5.39) we obtain that

$$tu \in K \text{ for every } t \in]0, 1[. \tag{5.41}$$

As $t \mapsto 1^-$ by (5.41) the thesis follows. □

Lemma 5.5. *Let K be a subset of $\mathcal{D}'$ verifying* (5.1) (ii), (5.2), (5.3) *and* (5.8)*; let C be defined by* (5.5) *and $K^{1,\infty}_C$ by* (3.2) *with $p = +\infty$.*
Then

$$K^{1,\infty}_C \cap L^\infty(\mathbb{R}^n) \subseteq K. \tag{5.42}$$

Proof. Let $u \in K^{1,\infty}_C \cap L^\infty(\mathbb{R}^n)$, $\epsilon > 0$ and let u_ϵ be the regularization of u given by (2.7).
Since $Du(x) \in C$ for a.e. x in $\mathbb{R}^n$, by b) of Proposition 5.1 and Proposition 2.7 it turns out that $Du_\epsilon(x) \in C$ for every x in $\mathbb{R}^n$; therefore we have that

$$u_\epsilon \in K^{1,\infty}_C \cap C^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n). \tag{5.43}$$

By (5.43) it soon follows that $\hat{u}_\epsilon \in C^\infty(\mathbb{R}^\nu) \cap L^\infty(\mathbb{R}^\nu)$, hence by (5.43) and Lemma 5.4 we deduce that

$$u_\epsilon \in K \text{ for every } \epsilon > 0. \tag{5.44}$$

By (5.44) and (5.8) it follows that $u \in K$, that is the thesis. □

We can now prove the characterization result in the case of Lipschitz continuous functions.

Theorem 5.6. *Let K be a subset of $\mathcal{D}'$ verifying* (5.1)÷(5.3), (5.8)*; let C be defined by* (5.5) *and $K^{1,\infty}_C$ by* (3.2) *with $p = +\infty$.*
Then

$$K \cap W^{1,\infty}_{\mathrm{loc}} = K^{1,\infty}_C. \tag{5.45}$$

Proof. By virtue of Proposition 5.2 we only have to prove that

$$K_C^{1,\infty} \subseteq K \cap W_{\mathrm{loc}}^{1,\infty}. \tag{5.46}$$

To this aim take u in $K_C^{1,\infty}$. For every $h \in \mathbb{N}$ we define the functions u_h as

$$u_h(x) = \max\{-h, \min\{u(x), h\}\} \quad x \text{ in } \mathbb{R}^n, \tag{5.47}$$

then we soon have

$$Du_h(x) = Du(x) \text{ if } |u_h(x)| \leq h, \quad Du_h(x) = 0 \text{ if } |u_h(x)| > h. \tag{5.48}$$

By (5.48) and (5.17) we have

$$u_h \in K_C^{1,\infty} \cap L^\infty(\mathbb{R}^n) \text{ for every } h \in \mathbb{N}, \tag{5.49}$$

hence by (5.49) and Lemma 5.5 we deduce

$$u_h \in K \text{ for every } h \in \mathbb{N}. \tag{5.50}$$

By (5.50) and (5.8) inclusion (5.46) and the thesis follows as $h \to +\infty$. □

6. The characterization result in the general case

In the present section we complete the results of the previous one by characterizing the set K when it is contained in a topological vector subspace U of $\mathcal{D}'$.
We assume that (U, τ) is a Hausdorff locally convex topological vector subspace of $\mathcal{D}'$ such that

$$u \in U, c \in \mathbb{R}, y \in \mathbb{R}^n, t > 0 \Rightarrow (i)\ u + c \in U,\ (ii)\ T[y]u \in U,\ (iii)\ O_t u \in U; \tag{6.1}$$

$$\begin{cases} \text{(i)} & \tau \text{ is less fine than } w^* - W_{\mathrm{loc}}^{1,\infty}(\mathbb{R}^n) \text{ on } U \cap W_{\mathrm{loc}}^{1,\infty}, \\ \text{(ii)} & \tau \text{ is finer than } w^* - \mathcal{D}'(\mathbb{R}^n); \end{cases} \tag{6.2}$$

$$\text{for every } u \in U \text{ the function } y \in \mathbb{R}^n \mapsto T[y]u \in U \text{ is continuous.} \tag{6.3}$$

We first need to deduce some properties of the regularizations of the elements of U.

Lemma 6.1. *Let (U, τ) be a sequentially complete Hausdorff locally convex topological vector subspace of $\mathcal{D}'$ verifying* (6.1) (ii), (6.2) (ii) *and* (6.3).
Let α be a mollifier in (2.6), *assume that* $\mathrm{spt}(\alpha)$ *is convex and let, for every $u \in U$, $\epsilon > 0$, u_ϵ be defined by* (2.6).
Then

$$u_\epsilon \in (U, \tau)_{seq} - \mathrm{cl}(\mathrm{conv}(\{T[y]u : y \in B_\epsilon\})) \textit{ for every } u \in U, \epsilon > 0; \tag{6.4}$$

$$u_\epsilon \to u \textit{ in } \tau \textit{ as } \epsilon \to 0^+ \textit{ for every } u \in U. \tag{6.5}$$

Proof. Let us first prove (6.4).
Let $u \in U$ and $\epsilon > 0$, then by (6.3) it follows that the function $y \in \mathbb{R}^n \mapsto \alpha(y)T[\epsilon y]u \in U$ is continuous with compact support hence, by Proposition 2.4, it is τ-integrable on $\mathbb{R}^n$. By (6.2) (ii) it follows that the above function is also $w^* - \mathcal{D}'(\mathbb{R}^n)$ integrable on $\mathbb{R}^n$ hence, by Proposition 2.5, the distributions u_ϵ and $\int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u\, dy$ agree; by virtue of this, (6.1) (ii) and Proposition 2.6 we deduce (6.4).

Let us prove now (6.5).
Let $u \in U$, let $\{p_a\}_{a\in\mathcal{A}}$ be a family of seminorms on U generating the topology τ and let B be a bounded open set containing the support of α.

By using (6.4) for every $a \in \mathcal{A}$ the assumptions of Proposition 2.3 are satisfied with $E = B$, $f = \alpha(\cdot)T[\epsilon\cdot]u$ and $\Phi = p_a$, hence by Proposition 2.3 we deduce

$$\begin{aligned} p_a\left(\int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u \quad dy - u\right) = p_a\left(\int_B \alpha(y)(T[\epsilon y]u - u)dy\right) \leq \\ \leq \int_B \alpha(y) p_a\left(T[\epsilon y]u - u\right) dy \quad \text{for every } a \in \mathcal{A}. \end{aligned} \tag{6.6}$$

By (6.3) we get that

$$\sup_{y\in B} p_a(T[\epsilon y]u - u) \to 0 \text{ as } \epsilon \to 0^+ \text{ for every } a \in \mathcal{A}, \tag{6.7}$$

hence by (6.6) and (6.7) we obtain

$$\int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u\, dy \to u \text{ in } \tau \text{ as } \epsilon \to 0^+. \tag{6.8}$$

At this point we observe that by Proposition 2.5 the distributions u_ϵ and $\int_{\mathbb{R}^n} \alpha(y)T[\epsilon y]u\, dy$ agree, hence (6.5) follows by (6.8) and Proposition 2.5. □

About the set K we assume that K is contained in U and that

$$K \text{ is } \tau\text{-sequentially closed.} \tag{6.9}$$

Lemma 6.2. *Let (U, τ) be a sequentially complete Hausdorff locally convex topological vector subspace of $\mathcal{D}'$ verifying* (6.1) (ii), (6.2) (ii), (6.3) *and let K be a subset of U satisfying* (5.1) (ii), (5.2) *and* (6.9).
Then

$$K = (U, \tau)_{seq} - \mathrm{cl}(K \cap W^{1,\infty}_{\mathrm{loc}}). \tag{6.10}$$

Proof. Since $K \cap W^{1,\infty}_{\mathrm{loc}} \subseteq K$, by (6.9) we soon deduce that

$$(U, \tau)_{seq} - \mathrm{cl}(K \cap W^{1,\infty}_{\mathrm{loc}}) \subseteq K. \tag{6.11}$$

We now prove the reverse inclusion in (6.11).
Let $u \in K$, $\epsilon > 0$ and $y \in B_\epsilon$.

By (5.1) (ii) it follows that

$$T[y]u \in K, \tag{6.12}$$

hence by (6.12) and (5.2) we get

$$\mathrm{conv}(\{T[y]u\colon y \in B_\epsilon\}) \subseteq K, \tag{6.13}$$

therefore by (6.13), (6.1) (ii) and (6.9) we conclude that

$$(U,\tau)_{seq} - \mathrm{cl}(\mathrm{conv}(\{T[y]u\colon y \in B_\epsilon\})) \subseteq K. \tag{6.14}$$

By (6.4) of Lemma 6.1 and (6.14) we obtain, provided that the mollifier in u_ϵ has convex support,

$$u_\epsilon \in K \cap W^{1,\infty}_{\mathrm{loc}}, \tag{6.15}$$

hence by (6.15) and (6.5) of Lemma 6.1 we get as $\epsilon \to 0^+$

$$K \subseteq (U,\tau)_{seq} - \mathrm{cl}(K \cap W^{1,\infty}_{\mathrm{loc}}). \tag{6.16}$$

By (6.16) and (6.11) equality (6.10) follows. □

We now prove the main result of this paper.
Let us recall that given a subset C of $\mathbb{R}^n$, the set K_C is defined by

$$K_C = \left\{ u \in \mathcal{D}'\colon -\langle u, D\varphi\rangle \in C \text{ for every } \varphi \in \mathcal{D} \text{ with } \varphi \geq 0, \int_{\mathbb{R}^n} \varphi = 1 \right\}. \tag{6.17}$$

Theorem 6.3. *Let (U,τ) be a sequentially complete Hausdorff locally convex topological vector subspace of $\mathcal{D}'$ verifying* (6.1)÷(6.3).
Let K be a subset of U satisfying (5.1)÷(5.3) *and* (6.9).
Let C be given by (5.5) *and K_C by* (6.17).
Then C is closed, convex and

$$K = K_C \cap U. \tag{6.18}$$

On the contrary, given a closed convex subset C of $\mathbb{R}^n$ and defined the set K_C by (6.17), *it turns out that conditions* (5.1)÷(5.3) *and* (6.9) *are satisifed by $K = K_C \cap U$.*

Proof. By (6.9) and (6.2) (i) condition (5.6) holds, hence by Proposition 5.1 C turns out to be closed and convex.
Let us now observe that by (6.2) (i) condition (5.8) holds, hence by Theorem 5.6 we deduce that

$$K \cap W^{1,\infty}_{\mathrm{loc}} = K^{1,\infty}_C = K^{1,\infty}_C \cap U, \tag{6.19}$$

$K^{1,\infty}_C$ being given by (3.2) with $p = +\infty$.
At this point, by (6.2) (ii) and Lemma 6.1, the assumptions of Proposition 4.1 are fulfilled, hence by Lemma 6.2, (6.19) and Proposition 4.1 equality (6.18) follows.
Finally the last part of the thesis follows by (6.1), (6.2) (ii) and Proposition 3.5. □

7. Applications to some spaces

In the present section we apply Theorem 6.3 when U agrees with the most common spaces in Analysis.

Theorem 7.1. *Let K be a subset of $\mathcal{D}'$, respectively of $\mathcal{S}'$, verifying (5.1)÷(5.3) and (6.9) with (U,τ) equal to $(\mathcal{D}', w^* - \mathcal{D}'(\mathbb{R}^n))$, respectively to $(\mathcal{S}', w^* - \mathcal{S}'(\mathbb{R}^n))$.*
Let C be defined by (5.5), K_C by (6.17) and K_C^S by (3.7).

Then C is closed, convex and

$$K = K_C, \tag{7.1}$$

respectively

$$K = K_C^S. \tag{7.2}$$

On the contrary, given a closed convex subset C of $\mathbb{R}^n$ and defined the set K_C by (6.17), respectively K_C^S by (3.7), it turns out that conditions (5.1)÷(5.3) and (6.9) with (U,τ) equal to $(\mathcal{D}', w^* - \mathcal{D}'(\mathbb{R}^n))$, respectively to $(\mathcal{S}', w^* - \mathcal{S}'(\mathbb{R}^n))$, are satisfied by $K = K_C$, respectively by $K = K_C^S$.

Proof. The thesis follows by Theorem 6.3 applied with (U,τ) equal to $(\mathcal{D}', w^* - \mathcal{D}'(\mathbb{R}^n))$, respectively to $(\mathcal{S}', w^* - \mathcal{S}'(\mathbb{R}^n))$ and by Proposition 3.4, once observed that the above function spaces verify conditions (6.1)÷(6.3) and are sequentially complete. □

Theorem 7.2. *Let $p \in [1, +\infty]$ and let K be a subset of L^p_{loc} verifying (5.1)÷(5.3) and (6.9) with (U,τ) equal to $(L^p_{\mathrm{loc}}, L^p_{\mathrm{loc}}(\mathbb{R}^n))$ if $p \in [1,+\infty[$ and to $(L^\infty_{\mathrm{loc}}, w^* - L^\infty_{\mathrm{loc}}(\mathbb{R}^n))$ if $p = +\infty$.*
Let C be defined by (5.5), K_C by (6.17) and K_C^p by (3.5).

Then C is closed, convex and

$$K = K_C^p. \tag{7.3}$$

On the contrary, given a closed convex subset C of $\mathbb{R}^n$ and defined the sets K_C^p by (3.5), it turns out that conditions (5.1)÷(5.3) and (6.9) with (U,τ) equal to $(L^p_{\mathrm{loc}}, L^p_{\mathrm{loc}}(\mathbb{R}^n))$ if $p \in [1,+\infty[$ and to $(L^\infty_{\mathrm{loc}}, w^* - L^\infty_{\mathrm{loc}}(\mathbb{R}^n))$ if $p = +\infty$ are satisfied by $K = K_C^p$.

Proof. The thesis follows by Theorem 6.3 applied with (U,τ) equal to $(L^p_{\mathrm{loc}}, L^p_{\mathrm{loc}}(\mathbb{R}^n))$ if $p \in [1,+\infty[$ and to $(L^\infty_{\mathrm{loc}}, w^* - L^\infty_{\mathrm{loc}}(\mathbb{R}^n))$ if $p = +\infty$ and by Proposition 3.3, once observed that the above spaces verify conditions (6.1)÷(6.3) and are sequentially complete. □

Theorem 7.3. *Let K be a subset of C^0, respectively of $\mathcal{M}_{\mathrm{loc}}$, verifying (5.1)÷(5.3) and (6.9) with (U,τ) equal to $(C^0, C^0_{\mathrm{loc}}(\mathbb{R}^n))$, respectively to $(\mathcal{M}_{\mathrm{loc}}, w^* - \mathcal{M}_{\mathrm{loc}}(\mathbb{R}^n))$.*
Let C be defined by (5.5), K_C^0 by (3.4) and K_C^M by (3.6).

Then C is closed, convex and

$$K = K_C^0, \tag{7.4}$$

respectively

$$K = K_C^M. \tag{7.5}$$

On the contrary, given a closed convex subset C *of* $\mathbb{R}^n$ *and defined the set* K_C^0 *by* (3.4) *and the set* K_C^M *by* (3.6), *it turns out that conditions* (5.1)÷(5.3) *and* (6.9) *with* (U,τ) *equal to* $(C^0, C^0_{\text{loc}}(\mathbb{R}^n))$, *respectively to* $(\mathcal{M}_{\text{loc}}, w^* - \mathcal{M}_{\text{loc}}(\mathbb{R}^n))$, *are satisfied by* $K = K_C^0$, *respectively by* $K = K_C^M$.

Proof. The thesis follows by Theorem 6.3 applied with (U,τ) equal to $(C^0, C^0_{\text{loc}}(\mathbb{R}^n))$, respectively to $(\mathcal{M}_{\text{loc}}, w^* - \mathcal{M}_{\text{loc}}(\mathbb{R}^n))$ and by Proposition 3.3, once observed that the above spaces verify conditions (6.1)÷(6.3) and are sequentially complete. □

Theorem 7.4. *Let* K *be a subset of* BV_{loc} *verifying* (5.1)÷(5.3) *and* (6.9) *with* (U,τ) *equal to* $(BV_{\text{loc}}, w^* - BV_{\text{loc}}(\mathbb{R}^n))$.
Let C *be defined by* (5.5) *and* K_C^{BV} *by* (3.3).

Then C *is closed, convex and*

$$K = K_C^{BV}. \tag{7.6}$$

On the contrary, given a closed convex subset C *of* $\mathbb{R}^n$ *and defined the set* K_C^{BV} *by* (3.3), *it turns out that conditions* (5.1)÷(5.3) *and* (6.9) *with* (U,τ) *equal to* $(BV_{\text{loc}}, w^* - BV_{\text{loc}}(\mathbb{R}^n))$ *are satisfied by* $K = K_C^{BV}$.

Proof. The thesis follows by Theorem 6.3 applied with (U,τ) equal to $(BV_{\text{loc}}, w^* - BV_{\text{loc}}(\mathbb{R}^n))$ and by Proposition 3.2, once observed that this space verifies conditions (6.1)÷(6.3) and is sequentially complete. □

Theorem 7.5. *Let* $p \in [1, +\infty]$ *and let* K *be a subset of* $W^{1,p}_{\text{loc}}$ *verifying* (5.1)÷(5.3) *and* (6.9) *with* (U,τ) *equal to* $(W^{1,p}_{\text{loc}}, W^{1,p}_{\text{loc}}(\mathbb{R}^n))$ *if* $p \in [1, +\infty[$ *and to* $(W^{1,\infty}_{\text{loc}}, w^* - W^{1,\infty}_{\text{loc}}(\mathbb{R}^n))$ *if* $p = +\infty$.
Let C *be defined by* (5.5) *and* $K_C^{1,p}$ *by* (3.2).

Then C *is closed, convex and*

$$K = K_C^{1,p}. \tag{7.7}$$

On the contrary, given a closed convex subset C *of* $\mathbb{R}^n$ *and defined the sets* $K_C^{1,p}$ *by* (3.2), *it turns out that conditions* (5.1)÷(5.3) *and* (6.9) *with* (U,τ) *equal to* $(W^{1,p}_{\text{loc}}, W^{1,p}_{\text{loc}}(\mathbb{R}^n))$ *if* $p \in [1, +\infty[$ *and to* $(W^{1,\infty}_{\text{loc}}, w^* - W^{1,\infty}_{\text{loc}}(\mathbb{R}^n))$ *if* $p = +\infty$ *are satisfied by* $K = K_C^p$.

Proof. Let us observe that if $p \in [1, +\infty[$, by (5.2), the set K is $W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ closed if and only if it is $w - W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ closed, hence the thesis follows by Theorem 6.3 applied with

(U, τ) equal to $(W^{1,p}_{\text{loc}}, w - W^{1,p}_{\text{loc}}(\mathbb{R}^n)$ if $p \in [1, +\infty[$ and to $(W^{1,\infty}_{\text{loc}}, w^* - W^{1,\infty}_{\text{loc}}(\mathbb{R}^n))$ if $p = +\infty$ and by Proposition 3.1, once observed that the above spaces verify conditions (6.1)÷(6.3) and are sequentially complete. □

References

[1] H. Attouch: Variational Convergence for Functions and Operators, Pitman, London, 1984.

[2] C. Baiocchi, A. Capelo: Disequazioni variazionali e quasivariazionali. Applicazioni a problemi di frontiera libera. Volume 1: Problemi variazionali, Quaderni Unione Matematica Italiana, Pitagora Editrice, Bologna, 1978.

[3] A. Bensoussan, J. L. Lions, G. Papanicolaou: Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.

[4] G. Birkhoff: Integration of Functions with Values in a Banach Space, Trans. Amer. Mat. Soc. 38 (1935) 357–378.

[5] G. Bouchitté: Convergence et relaxation de fonctionnelles du calcul des variations à croissance linéaire. Applications à l'homogénéisation en plasticité, Ann. Fac. Sci. Univ. Toulose Math. 8 (1986/87) 7–36.

[6] G. Bouchitté, G. Dal Maso: Integral Representation and Relaxation of Convex Local Functionals on $BV(\Omega)$, Ann. Sc. Norm. Sup. Pisa 20 (1993) 483–533.

[7] G. Bouchitté, M. Valadier: Integral Representation of Convex Functionals on a Space of Measures, J. Funct. Anal. 80 (1988) 398–420.

[8] H. Brezis: Multiplicateur de Lagrange en Torsion Elasto-Plastique, Arch. Rational Mech. Anal. 49 (1973) 32–40.

[9] H. Brezis, G. Stampacchia: Sur la régularité de la solution d'inéquations elliptiques, Bull. Soc. Math. France 96 (1968) 153–180.

[10] G. Buttazzo: Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations, Longman, Scientific & Technical, 1989.

[11] G. Buttazzo, G. Dal Maso: Integral Representation and Relaxation of Local Functionals, Nonlinear Anal. 9 (1985) 515–532.

[12] L. Carbone: Sur la convergence des intégrales du type de l'énergie sur des fonctions a gradient borné, J. Math. Pures Appl. 56 (1977) 79–84.

[13] L. Carbone: Γ-convergence d'intégrales sur des fonctions avec des contraintes sur le gradient, Comm. Part. Diff. Eq. 2 (1977) 627–651.

[14] L. Carbone: Sull'omogeneizzazione di un problema variazionale con vincoli sur gradiente, Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Natur. 63 (1977) 10–14.

[15] L. Carbone: Sur un problème d'homogénéisation avec des contraintes sur le gradient, J. Math. Pures Appl. 58 (1979) 275–297.

[16] L. Carbone, S. Salerno: On a Problem of Homogenization with Quickly Oscillating Constraints on the Gradient, J. Math. Anal. Appl. 90 (1982) 219–250.

[17] L. Carbone, S. Salerno: Further Results on a Problem of Homogenization with Constraints on the Gradient, J. Analyse Math. 44 (1984/85) 1–20.

[18] L. Carbone, S. Salerno: Homogenization with Unbounded Constraints on the Gradient, Nonlinear Anal. 9 (1985) 431–444.

[19] L. Carbone, C. Sbordone: Some Properties of Γ-limits of Integral Functionals, Ann. Mat. Pura Appl. 122 (1979) 1–60.

[20] D. Ciorandescu, J. Saint Jean Paulin: Homogenization in Open Sets with Holes, J. Math. Anal. Appl. 71 (1979) 590–607.

[21] A. Corbo Esposito, R. De Arcangelis: Comparison Results for Some Types of Relaxation of Variational Integral Functions, Ann. Mat. Pura Appl. 164 (1993) 155–193.

[22] A. Corbo Esposito, R. De Arcangelis: The Lavrentieff Phenomenon and Different Processes of Homogenization, Comm. Part. Diff. Eq. 17 (1992) 1503–1538.

[23] A. Corbo Esposito, R. De Arcangelis: A Characterization of Families of Function Sets Described by Constraints on the Gradient, Ann. Inst. Henri Poincaré – Analyse non linéaire 11 (1994) 553–609.

[24] G. Dal Maso: An Introduction to Γ-convergence, Birkhäuser-Verlag, Boston, Basel, Berlin, 1993.

[25] G. Dal Maso, L. Modica: A General Theory of Variational Functions, in Topics in Functional Analysis 1980/81, Quaderni della Scuola Normale Superiore, Pisa (1982) 149–221.

[26] R. De Arcangelis: A General Homogenization Result for Almost Periodic Functions, J. Math. Anal. Appl. 156 (1991) 358–380.

[27] R. De Arcangelis, A. Vitolo: Some Cases of Homogenization with Unbounded Oscillating Constraints on the Gradient, Asymptotic Anal. 5 (1992) 397–428.

[28] E. De Giorgi: Sulla convergenza di alcune successioni di integrali del tipo dell'area, Rend. Mat., Roma 8 (1975) 277–294.

[29] E. De Giorgi: Convergence Problems for Functionals and Operators, Proceed. Int. Meeting on "Recent Methods in Nonlinear Analysis", Rome 8–12 May 1975, Pitagora ed., Bologna (1979) 131–188.

[30] E. De Giorgi, F. Colombini, L. C. Piccinini: Frontiere orientate di misura minima e questioni collegate, Quaderni della Scuola Normale Superiore, Pisa, 1972.

[31] E. De Giorgi, G. Letta: Une notione generale de convergence faible pour des fonctions croissantes d'ensemble, Ann. Sc. Norm. Sup. Pisa 4 (1977) 61–99.

[32] F. Demengel, R. Temam: Convex Functions of a Measure and Applications, Indiana Univ. Math. J. 33 (1984) 673–709.

[33] N. Dunford, J. T. Schwartz: Linear Operators, Wiley-Interscience Publications, 1957.

[34] I. Ekeland, R. Temam: Convex Analysis and Variational Problems, North-Holland American Elsevier, 1976.

[35] N. Fusco: Γ-convergenza unidimensionale, Boll. Un. Mat. Ital. B, 16-B (1979) 74–86.

[36] E. Giusti: Minimal Surfaces and Functions of Bounded Variations, Birkhäuser-Verlag, Basel, 1984.

[37] R. Glowinski, H. Lanchon: Torsion élastoplastique d'une barre cylindrique de section multiconnexe, J. Mécanique 12 (1973) 151–171.

[38] C. Goffman, J. Serrin: Sublinear Functions of Measures and Variational Integrals, Duke Math. J. 31 (1964) 159–178.

[39] I. M. Guelfand, G. E. Chilov: Les Distributions, vol. 1, Dunod, Paris, 1972.

[40] P. Marcellini, C. Sbordone: Homogenization of Non Uniformly Elliptic Operators, Applicable Analysis 8 (1978) 101–113.

[41] R. S. Phillips: Integration in a Convex Linear Topological Space, Trans. Amer. Mat. Soc. 47 (1940) 114–145.

[42] W. Rudin: Analisi Reale e Complessa, Boringhieri, Torino, 1980.

[43] L. Schwartz: Theorie des distributions, Hermann, Paris, 1973.

[44] J. Serrin: On the Definition and Properties of Certain Variational Integrals, Trans. Amer. Math. Soc. 101 (1961) 139–167.

[45] T. W. Ting: Elastic-Plastic Torsion, Arch. Rational Mech. Anal. 24 (1969) 228–244.

[46] F. Treves: Topological Vector Spaces, Distributions and Kernels, Academic Press, New York, 1967.

[47] K. Yoshida: Functional Analysis, Springer-Verlag, Berlin, 1978.