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Existence of Solutions for Degenerate Sweeping Processes

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In this paper, the existence of solutions to a class of non-standard sweeping processes of the form $-u'(t) \in \partial\delta_{C(t)}(Au(t))$ is established. In contrast to the "classical" sweeping process with $A = \mathrm{id}$, these problems may be degenerated, since there might be no solutions at all.

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1. Introduction

As a motivation of our work, we consider the evolution differential inclusion

$$-q'(t) \in \partial\delta_\Gamma(g(t, q(t))) \quad \text{a.e. in} \quad [0, T]\,, \quad q(0) = q_0\,, \tag{1}$$

where Γ is a closed and convex subset of the real Hilbert space H and $\partial\delta_\Gamma$ denotes the cone of normals to Γ. Moreover, g is assumed to be a continuous function with some additional properties. Inclusions of the type (1) arise in plastic flow problems, i.e. in quasistatic elastoplasticity, where one often encounters problems of the form

$$-\frac{\partial q}{\partial t}(t, x) \in \partial\delta_{E(x)}\Big(\frac{\partial W}{\partial q}(t, x, q(t, x))\Big)\,, \tag{2}$$

cf. e.g. [4], and also [11]. In this formulation, every $E(x) \subset \mathbb{R}^n$ is a closed convex set (sometimes called the "rigidity set"), $t \geq 0$ denotes the process time, $x \in \Omega \subset \mathbb{R}^n$ is the material point, $q \in \mathbb{R}^n$ an internal variable (like plastic strain or hardening), and W denotes a (modified) stored energy function. For our purpose of motivation, the main thing to note is that for every fixed x, the inclusion (2) is of type (1).

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In fact (1) may be reduced to an autonomous problem, as is shown by the following lemma (the proof of which is given in Section 2 below).

Lemma 1.1. *If $q : [0,T] \to H$ is a solution of* (1), *then $u : [0,T] \to \mathbb{R} \times H =: \mathcal{H}$, defined by $u(t) = (t, q(t))$, is a solution of*

$$-u'(t) \in \partial\delta_{C(t)}(Au(t)) \quad a.e.\ in \quad [0,T]\,, \quad u(0) = (0, q_0) \tag{3}$$

in the Hilbert space $\mathcal{H}$, where $C(t) := [t, \infty[\times \Gamma$ for $t \in [0,T]$ and $A(t,x) := (t, g(t,x))$ for $(t,x) \in \mathcal{H}$. Conversely, if $u = (\theta, q) : [0,T] \to \mathcal{H}$ solves (3), *then $q : [0,T] \to H$ solves* (1).

Observe that $C(\cdot)$ is Lipschitz with constant $L = 1$ w.r.t. d_H (the Hausdorff-distance) in the previous lemma. Hence the differential inclusion (3) is a generalization of the well-known sweeping process

$$-u'(t) \in \partial\delta_{C(t)}(u(t))\,, \tag{4}$$

cf. e.g. [12, 10], which corresponds to $A = \mathrm{id}$. However, contrary to this "classical" sweeping process, the problems (1) or (3) may be degenerated, i.e. there are simple examples showing that it is possible that there are no solutions at all (cf. Example 4.1 in Section 4 below).

To obtain positive results about the existence of solutions, we will consider, more generally than (3), non-standard evolution equations of the form

$$-u'(t) \in \partial\delta_{C(t)}(v(t))\,, \quad v(t) \in Au(t) \quad \text{a.e. in} \quad [0,T]\,, \quad u(0) = u_0 \in D(A)\,, \tag{5}$$

with some (possibly multivalued) operator $A : D(A) \to 2^H \setminus \{\emptyset\}$ and a Lipschitz-continuous moving set $C(\cdot)$ with nonempty closed convex values $C(t)$ for $t \in [0,T]$. Note that (5) also requires $v(t) \in C(t)$ a.e. in $[0,T]$, and we will additionally suppose that $v_0 := A^0 u_0 \in C(0)$. Here $A^0 x \in Ax$ is defined through $|A^0 x| = \min\{|y| : y \in Ax\}$ for $x \in D(A)$.

It is interesting to note that problems falling under a similar general scheme as (5) were arrived at in [8] or [9] (cf. also the references therein for additional information), but in those papers the authors quite differently had in mind elliptic-parabolic partial differential equations as a motivation for the study of

$$-u'(t) \in \partial\varphi^t(v(t))\,, \quad v(t) \in Au(t) \quad \text{a.e. in} \quad [0,T]\,, \quad u(0) = u_0 \in D(A)\,, \tag{6}$$

with a maximal monotone and strongly monotone $A = \partial\psi$. Moreover, φ^t had to depend on t in a somehow regular way, and also some compactness and coercivity assumptions like $(a1)$ and $(a2)$ of [9, p. 1183] had to be satisfied. Both these conditions stated that each of the sublevel sets $\{z \in H : \varphi^t(z) \le r\}$ should be compact and that $\varphi^t(z) \ge c\,|z|^2$. (In fact, one should also have $v_0 = A^0 u_0 \in D(\varphi^0)$ in (6), cf. (17) below.) Since these conditions do not hold for (5), which corresponds to the case $\varphi^t = \delta_{C(t)}$, they had to be replaced by suitable substitutes in case of the sweeping processes (5).

After giving some preliminaries in Section 2, we will prove the corresponding theorem on the existence of solutions to (5) in Section 3. Afterwards this result will be applied in Section 4 to obtain solutions also for (1).

Let us finally remark that there are other non-standard variants of the classical sweeping process. In [5] the author considered problems of the type $-A_t u'(t) \in \partial f(t, u(t))$, $u(t) \in \Gamma(t)$, in $C(X)$ instead of the Hilbert space H, whereas in [4] the problem $\frac{d}{dt}[A(t, u(t))] + f(t) \ni -\partial\varphi(u(t))$ will be investigated under conditions similar to the ones of Theorem 4.2 below. Furthermore, in [14] and [2] the modification was neither in the underlying space nor in the left-hand side of (4), but in the fact that $C(t)$ was not necessarily assumed to be convex.

2. Preliminaries

Let H be a Hilbert space with norm $|x|$ and inner product $x \cdot y$ or $\langle x, y\rangle$. For sets $C_1, C_2 \subset H$

$$d_H(C_1, C_2) = \max\Big\{ \sup_{x \in C_2} \operatorname{dist}(x, C_1),\ \sup_{x \in C_1} \operatorname{dist}(x, C_2)\Big\}$$
$$\text{with} \quad \operatorname{dist}(x, C_1) = \inf\{|x - y| : y \in C_1\}$$

is the Hausdorff-distance between C_1 and C_2. For $\emptyset \neq C \subset H$ closed and convex, $\operatorname{proj}(x, C)$ denotes the projection of $x \in H$ onto C, i.e. $y = \operatorname{proj}(x, C)$ iff $y \in C$ and $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$. Moreover, $N_C(x) = \partial\delta_C(x) = \{y \in H : 0 \geq \langle y, z - x\rangle \text{ for all } z \in C\}$ is the normal cone at $x \in C$, and $\delta^*(x, C) = \sup\{x \cdot y : y \in C\}$ stands for the support function of C.

If $A : D(A) \to 2^H \setminus \{\emptyset\}$ is a maximal monotone operator (mmop), then $J_\lambda = (I + \lambda A)^{-1}$ resp. $A_\lambda = \lambda^{-1}(I - J_\lambda)$ will denote the resolvent resp. the Yosida-approximation of A for $\lambda > 0$. For properties and other terminology related to mmops (and also to convex functionals) we refer to standard references like [1], [3] or [6]. In particular, it will be repeatedly used that $A_\lambda x \in AJ_\lambda x$ for $x \in H$.

We start with the definition of solutions.

Definition 2.1. (a) Let $q_0 \in H$ be given such that $g(0, q_0) \in \Gamma$. A function $q \in W^{1,2}([0, T]; H)$ is called a solution of (1) if $g(t, q(t)) \in \Gamma$ for a.e. $t \in [0, T]$ and if (1) is satisfied.

(b) Let $u_0 \in D(A)$ be given such that $v_0 = A^0 u_0 \in C(0)$. A pair (u, v) is called a solution of (5) if $u \in W^{1,2}([0, T]; H)$, $v \in L^\infty([0, T]; H)$, $v(t) \in Au(t) \cap C(t)$ for a.e. $t \in [0, T]$ and if (5) is satisfied.

We remark that one is not forced to define solutions with the regularity as we have above (an alternative is e.g. to require only $u \in W^{1,1}([0, T]; H)$ and v measurable), but since the solutions we obtain in Theorem 3.1 below have the stated regularity, we prefer this definition.

Next we carry out the

Proof of Lemma 1.1. By (1) for a.e. $t \in [0, T]$ and all $(s, y) \in [t, \infty[\times \Gamma$

$$\begin{aligned} 0 &\leq \langle q'(t),\, y - g(t, q(t))\rangle \\ &\leq (s - t) + \langle q'(t),\, y - g(t, q(t))\rangle = \langle (1, q'(t)),\, (s, y) - (t, g(t, q(t)))\rangle_{\mathcal{H}}\,, \end{aligned}$$

and that means (3). On the other hand, let $u = (\theta, q)$ be a solution of (3). Then for a.e. $t \in [0, T]$ we have $Au(t) \in C(t)$, i.e. $\theta(t) \geq t$ and $g(\theta(t), q(t)) \in \Gamma$, and for $s \geq t$ and

$y \in \Gamma$

$$0 \le \theta'(t)(s - \theta(t)) + \langle q'(t),\, y - g(\theta(t), q(t))\rangle\,. \tag{7}$$

In particular $0 \le \theta'(t)(t - \theta(t))$ for a.e. $t \in [0,T]$. Hence $\varphi(t) = \theta(t) - t \ge 0$ is continuous and has $\varphi(0) = 0$ as well as $\varphi'(t)\varphi(t) \le -\varphi(t) \le 0$ a.e., and therefore $\theta(t) = t$ in $[0,T]$. Consequently, (7) with $s = t$ shows that q is a solution of (1). □

Lemma 2.2. *Let A be a mmop in H such that $\langle Ax - Ay, x - y\rangle \ge c\,|x - y|^2$ in $D(A) \times D(A)$ for some $c > 0$. Then*

(a) $\lambda x + A^{-1}x = A_\lambda^{-1}x$ *for $x \in H$ and $\lambda > 0$, and*

(b) *for $x, y \in H$ and $\lambda > 0$*

$$\langle A_\lambda x - A_\lambda y, x - y\rangle \ge \frac{c}{1 + \lambda\, c}\,|x - y|^2\,.$$

Proof. Ad (a): Since A^{-1} is locally bounded, A is onto by [3, Thm. 2.3]. Thus $A^{-1} : H \to D(A)$ is single-valued and monotone. Moreover, because $A_\lambda : H \to H$ is a single-valued mmop, the cited result and part (b) of this lemma imply that A_λ is bijective. As a consequence of $A_\lambda y \in AJ_\lambda y$ for $y \in H$ we have $A^{-1}(A_\lambda y) = J_\lambda y$, and therefore, with $y = A_\lambda^{-1}x$ for $x \in H$, we obtain $A^{-1}x = J_\lambda(A_\lambda^{-1}x) = A_\lambda^{-1}x - \lambda x$, the latter following from $J_\lambda + \lambda A_\lambda = I$.
Ad (b): Cf. [9, Lemma 2.1(iii)]. The proof did not rely on A being a subdifferential. □

The following result from [3, Ex. 2.8.2] will be useful later.

Lemma 2.3. *For $t \in [0,T]$ let $\varphi^t = \delta_{C(t)}$. Then $\partial\varphi^t = N_{C(t)}$ and for $\lambda > 0$ and $x \in H$*

$$\varphi^t_\lambda(x) = \frac{1}{2\lambda}|x - \mathrm{proj}(x, C(t))|^2 \quad \textit{and} \quad (\partial\varphi^t)_\lambda(x) = \partial\varphi^t_\lambda(x) = \frac{1}{\lambda}\,[x - \mathrm{proj}(x, C(t))]\,.$$

From this we obtain

Lemma 2.4. *Let* (H2) *below be satisfied, i.e. $C(\cdot)$ is Lipschitz-continuous with constant L w.r.t. d_H and has nonempty closed convex values. Then for $\lambda > 0$ and $x \in H$ the function $t \mapsto \varphi^t_\lambda(x)$, with φ^t as in* Lemma 2.3, *is differentiable a.e. on $[0,T]$ and*

$$\frac{d}{dt}\,\varphi^t_\lambda(x) \le L\,|\partial\varphi^t_\lambda(x)| \quad \textit{a.e. on} \quad [0,T]\,.$$

Proof. The proof is an appropriate modification of [9, Lemma 2.3]. For $s, t \in [0,T]$ with $s \le t$ and $y \in C(t)$ we obtain from Lemma 2.3 and from the properties of a projection

$$\begin{aligned}
\varphi^t_\lambda(x) - \varphi^s_\lambda(x) &= \frac{1}{2\lambda}|x - \mathrm{proj}(x, C(t))|^2 - \frac{1}{2\lambda}|x - \mathrm{proj}(x, C(s))|^2 \\
&\le \frac{1}{2\lambda}|x - y|^2 - \frac{1}{2\lambda}|x - \mathrm{proj}(x, C(s))|^2 \\
&= \frac{1}{\lambda}\,\langle\,\mathrm{proj}(x, C(s)) - y,\, x - \mathrm{proj}(x, C(s))\rangle + \frac{1}{2\lambda}|y - \mathrm{proj}(x, C(s))|^2 \\
&\le |y - \mathrm{proj}(x, C(s))|\,|\partial\varphi^t_\lambda(x)| + \frac{1}{2\lambda}|y - \mathrm{proj}(x, C(s))|^2\,.
\end{aligned} \tag{8}$$

Since $\mathrm{proj}(x, C(s)) \in C(s)$ we find $y \in C(t)$ such that $|y - \mathrm{proj}(x, C(s))| \leq L|t-s|$. Inserting this into (8) we arrive at

$$\varphi_\lambda^t(x) - \varphi_\lambda^s(x) \leq L|t-s|\,|\partial\varphi_\lambda^t(x)| + \frac{L^2}{2\lambda}|t-s|^2\,. \tag{9}$$

This implies that the function $t \mapsto \varphi_\lambda^t(x)$ may be written as a sum of a nonincreasing function and an a.c. function, and therefore is differentiable a.e. Dividing (9) by $(t-s)$ and taking the limit $s \to t^-$ at a.e. fixed t thus yields the claim. $\square$

Lemma 2.5. *Let $\psi : H \to \mathbb{R} \cup \{\infty\}$ be lsc, convex and proper.*

(a) *If $u : [0,T] \to H$ is differentiable at $t \in\,]0,T[$ and $u(t) \in D(\partial\psi)$, then*

$$\frac{d}{dt}[\psi \circ u](t) = (u'(t), z) \quad \textit{for all} \quad z \in \partial\psi(u(t)).$$

(b) *If $\lambda_n \to 0^+$, $x_n \to x \in H$ and $\{\partial\psi_{\lambda_n}(x_n) : n \in \mathbb{N}\} \subset H$ is bounded, then $\psi_{\lambda_n}(x_n) \to \psi(x)$.*

Proof. Ad (a): Cf. [3, Lemma 3.3] and formula (ψ8) of [9, p. 1187]. Ad (b): This is (ψ6) in [9, p. 1187]. $\square$

3. Existence of solutions of (5)

In this section we generally will assume that the following hypotheses are satisfied:

(H1) $A : D(A) \to 2^H \setminus \{\emptyset\}$ is a mmop such that for some $c > 0$

$$\langle Ax - Ay, x - y\rangle \geq c\,|x-y|^2 \quad \text{for all} \quad x, y \in D(A). \tag{10}$$

Moreover, for some function $M : [0,\infty[\to [0,\infty[$ which maps bounded sets into bounded sets,

$$\|Ax\| := \sup\{|y| : y \in Ax\} \leq M(|x|) \quad \text{for} \quad x \in D(A). \tag{11}$$

(H2) $C(\cdot)$ is Lipschitz-continuous with constant $L > 0$ w.r.t. d_H and $C(t)$ is nonempty, closed and convex for every $t \in [0,T]$,

and we intend to prove

Theorem 3.1. *Let* (H1) *and* (H2) *be satisfied, and fix $T > 0$. Suppose further that in addition*

(H3a) *$A = \partial\psi$ for some lsc, convex and proper $\psi : H \to \mathbb{R} \cup \{\infty\}$, or*

(H3b) *If $\mu_n \to 0^+$, $u_n \to u$ in $C([0,T];H)$ and $v_n = A_{\mu_n}u_n \rightharpoonup v$ in $L^2([0,T];H)$, then even $v_n \to v$ in $L^2([0,T];H)$,*

and

(H4) *$C(t) \cap \overline{B}_r(0)$ is compact for all $t \in [0,T]$ and $r > 0$.*

Then (5) *has a solution on $[0,T]$ in the sense of* Definition 2.1 *for every $u_0 \in D(A)$ with $v_0 = A^0u_0 \in C(0)$.*

We remark that some of the assumptions might be relaxed by using a complete discretization instead of Yosida-Moreau regularization as below. But since we are mainly interested in an application of Theorem 3.1 to the finite-dimensional problem (1), but not to PDEs, the above assumptions are easily seen to be satisfied in this case, cf. Theorem 4.2 below. The proof of the theorem will be carried out along the lines of [8] or [9], with appropriately modified arguments, since (as was already explained above) we neither can assume the uniform coercivity of $\varphi^t = \delta_{C(t)}$ nor the compactness of the sublevel sets in the present case of the sweeping process.

Firstly, we will derive uniform bounds for a sequence $(u_{\lambda,\mu})$ of approximate solutions, obtained by Yosida-approximation of A through A_μ and by approximation of $\partial\varphi^t = \partial\delta_{C(t)} = N_{C(t)}$ through $\partial\varphi^t_\lambda$ (cf. Lemma 2.3). To get these uniform bounds, only (H1) and (H2) will be needed. Then a solution u of (5) will be obtained by letting jointly $\lambda \to 0^+$ and $\mu \to 0^+$. Conditions (H3a) or (H3b) and (H4) will be made use of to prove this convergence.

So, we assume that (H1) and (H2) hold and choose for every fixed $\lambda, \mu > 0$ a solution $u_{\lambda,\mu} \in C^1([0,T]; H)$ of

$$-u'_{\lambda,\mu}(t) = \frac{1}{\lambda}\left[A_\mu u_{\lambda,\mu}(t) - \mathrm{proj}(A_\mu u_{\lambda,\mu}(t), C(t))\right] \quad \text{for all } t \in [0,T],\ u_{\lambda,\mu}(0) = u^0_\mu, \tag{12}$$

where $A_\mu u^0_\mu = v_0$. Note that u^0_μ exists since $A_\mu : H \to H$ is single-valued and onto, cf. Lemma 2.2 and [6, Thm. 11.6]. Thus by Lemma 2.2 (b)

$$\begin{aligned} |u^0_\mu| &\le |u_0| + |u^0_\mu - u_0| \le |u_0| + \left(\frac{1+c\mu}{c}\right)|A_\mu u^0_\mu - A_\mu u_0| \\ &\le |u_0| + \left(\frac{1+c\mu}{c}\right)(|v_0| + |A^0 u_0|) =: c_1 \end{aligned} \tag{13}$$

for every $\mu > 0$, in particular for $\mu \in\,]0, 3/2c]$ (this condition will be needed below), because $u_0 \in D(A)$. Moreover, since the right-hand side of the differential equation (12) is continuous, (12) in fact has a C^1-solution on $[0,T]$.

Lemma 3.2. *There exists a constant $c_2 > 0$ (which depends only on T, L, and c) such that for all $\mu \in\,]0, 3/2c]$ and all $\lambda > 0$*

$$|u_{\lambda,\mu}|_\infty + |u'_{\lambda,\mu}|_{L^1([0,T];H)} + |u'_{\lambda,\mu}|_{L^2([0,T];H)} \le c_2$$

and

$$|\, v_{\lambda,\mu}(t) - \mathrm{proj}(v_{\lambda,\mu}(t), C(t))|^2 \le c_2\, \lambda$$

for $t \in [0,T]$, where $v_{\lambda,\mu} = A_\mu u_{\lambda,\mu}$.

Proof. To differentiate $\psi(t) = \frac{1}{2}\mathrm{dist}^2(A_\mu u_{\lambda,\mu}(t), C(t)) = \lambda\, \varphi^t_\lambda(A_\mu u_{\lambda,\mu}(t))$ a.e. on $[0,T]$ note first that $A_\mu u_{\lambda,\mu}$ is differentiable a.e., since A_μ is Lipschitz continuous. Moreover, for a fixed closed and convex $C \subset H$ we have $\frac{1}{2}(\rho^2)'(x) = x - \mathrm{proj}(x, C)$, where $\rho(x) = \mathrm{dist}(x, C)$, cf. e.g. [7, Prop. 7.1], which does not depend on $\dim X < \infty$. Next, as a consequence of Lemma 2.4, we obtain that for all $x \in H$ and a.e. $t \in [0,T]$

$$\frac{d}{dt}\,\varphi^t_\lambda(x) \le L\,|\partial\varphi^t_\lambda(x)|\,. \tag{14}$$

Because $v_{\lambda,\mu} = A_\mu u_{\lambda,\mu}$ is differentiable a.e., it follows from Lemma 2.2 (b) (take $x = u_{\lambda,\mu}(t+h)$ and $y = u_{\lambda,\mu}(t)$ there, and let $h \to 0^+$) that a.e.

$$\Big(\frac{c}{1+c\mu}\Big)\, |u'_{\lambda,\mu}(t)|^2 \le \langle v'_{\lambda,\mu}(t),\, u'_{\lambda,\mu}(t)\rangle\,.$$

Thus we obtain a.e. on $[0,T]$ by means of (12) and (14)

$$\begin{aligned} \psi'(t) &= \lambda\Big(\frac{d}{dt}\,\varphi^t_\lambda\Big)(v_{\lambda,\mu}(t)) + \Big\langle v'_{\lambda,\mu}(t),\, v_{\lambda,\mu}(t) - \mathrm{proj}(v_{\lambda,\mu}(t), C(t))\Big\rangle \\ &= \lambda\Big(\frac{d}{dt}\,\varphi^t_\lambda\Big)(v_{\lambda,\mu}(t)) - \lambda\,\langle v'_{\lambda,\mu}(t),\, u'_{\lambda,\mu}(t)\rangle \\ &\le \lambda\, L\,\Big|\partial\varphi^t_\lambda(v_{\lambda,\mu}(t))\Big| - \lambda\Big(\frac{c}{1+c\mu}\Big)\,|u'_{\lambda,\mu}(t)|^2 \\ &= \lambda\, L\,|u'_{\lambda,\mu}(t)| - \lambda\Big(\frac{c}{1+c\mu}\Big)\,|u'_{\lambda,\mu}(t)|^2\,, \end{aligned} \tag{15}$$

so that $\psi'(t) \le \lambda L^2(1+c\mu)/4c$, since $\alpha w^2 + \beta \le \gamma|w|$ for $\alpha,\gamma > 0$ and $\beta, w \in \mathbb{R}$ implies $\beta \le \gamma^2/4\alpha$. Hence $\psi(0) = 0$, integration, and $\varphi^t_\lambda \le \varphi^t$ yield, cf. Lemma 2.3,

$$\begin{aligned} |\, v_{\lambda,\mu}(t) - \mathrm{proj}(v_{\lambda,\mu}(t), C(t))|^2 &= 2\lambda\varphi^t_\lambda(v_{\lambda,\mu}(t)) = 2\psi(t) \\ &\le 2T\lambda L^2(1+c\mu)/(4c) \le R_1\,\lambda \end{aligned} \tag{16}$$

for some $R_1 > 0$ independent of $\mu \in\,]0, 3/2c]$ and $\lambda > 0$. For such μ we also find $\alpha > 0$ with $c/(1+c\mu) - 1/2\alpha = c/4$. Therefore, since $L|w| \le \alpha L^2/2 + w^2/2\alpha$ for $w \in \mathbb{R}$, (15) implies a.e. on $[0,T]$

$$\begin{aligned} \frac{c}{4}\,|u'_{\lambda,\mu}(t)|^2 + \lambda^{-1}\,\psi'(t) &= \Big(\frac{c}{1+c\mu} - \frac{1}{2\alpha}\Big)\,|u'_{\lambda,\mu}(t)|^2 + \lambda^{-1}\,\psi'(t) \\ &\le \frac{\alpha}{2}\,L^2 = \Big(\frac{1+c\mu}{c\,(3-c\mu)}\Big)\,L^2 \le 5L^2/3c\,. \end{aligned}$$

Because $\psi \ge 0$ we may integrate this inequality and use $\psi(0) = 0$ to obtain a uniform L^2-bound for $u'_{\lambda,\mu}$ with $\mu \in\,]0, 3/2c]$ and $\lambda > 0$. Hence in particular the L^1-norms of those $u'_{\lambda,\mu}$ are uniformly bounded, and thus by (13) we also obtain a uniform bound for $|u_{\lambda,\mu}|_\infty$, so that the proof is complete by (16). □

Lemma 3.3. *There exists a constant $c_3 > 0$ (again dependent only on T, L, and c) such that for all $\mu \in\,]0, 3/2c]$ and $\lambda \in\,]0, 1]$*

$$|v_{\lambda,\mu}|_\infty + |w_{\lambda,\mu}|_\infty \le c_3,$$

with $w_{\lambda,\mu} = \mathrm{proj}(v_{\lambda,\mu}(\cdot), C(\cdot)) \in C([0,T]; H)$.

Proof. Since $u_0 \in D(A)$ we have

$$|J_\mu u_0| = |u_0 - \mu\, A_\mu u_0| \le |u_0| + \mu\, |A_\mu u_0| \le |u_0| + 3\, |A^0 u_0|/2c\,,$$

so that by Lemma 3.2 for $t \in [0,T]$

$$\begin{aligned} |J_\mu u_{\lambda,\mu}(t)| &\le |J_\mu u_{\lambda,\mu}(t) - J_\mu u_0| + |J_\mu u_0| \le |u_{\lambda,\mu}(t) - u_0| + |J_\mu u_0| \\ &\le c_2 + |u_0| + |u_0| + 3|A^0 u_0|/2c =: R_2\,. \end{aligned}$$

By assumption (H2), $M([0,R_2]) \subset [0,R_3]$ for some $R_3 > 0$. Thus we obtain

$$|v_{\lambda,\mu}(t)| = |A_\mu u_{\lambda,\mu}(t)| \le \|A(J_\mu u_{\lambda,\mu}(t))\| \le M(|J_\mu u_{\lambda,\mu}(t)|) \le R_3$$

for $t \in [0,T]$. Consequently, by Lemma 3.2, we may choose c_3 appropriately. $\square$

To obtain a solution of (5) in the limit, we fix sequences $(\lambda_n) \subset\,]0,1]$ and $(\mu_n) \subset\,]0,3/2c]$ with $\lambda_n \to 0^+$ and $\mu_n \to 0^+$, and denote by u_n resp. v_n resp. w_n the functions u_{λ_n,μ_n} resp. v_{λ_n,μ_n} resp. w_{λ_n,μ_n}. The following lemma corresponds to (3.12)–(3.14) and (4.1)–(4.3) in [9].

Lemma 3.4. *Let* (H4) *be satisfied. Then there exists a subsequence, again indexed with $n \in \mathbb{N}$, and functions $u \in W^{1,2}([0,T];H)$ and $v \in L^\infty([0,T];H)$ such that $u(0) = u_0$, $v(0) = v_0$, $u_n \to u$ in $C([0,T];H)$ and $v_n \rightharpoonup v$ in $L^2([0,T];H)$. In particular, $J_{\mu_n} u_n \to u$ in $C([0,T];H)$.*

Proof. By Lemma 3.2 and Lemma 3.3 we find a subsequence (w.l.o.g. the whole sequence), an a.c. $u \in W^{1,2}([0,T];H)$ and $v \in L^\infty([0,T];H)$ such that $u_n \rightharpoonup u$ in $L^2([0,T];H)$, $u_n(t) \rightharpoonup u(t)$ in H for all $t \in [0,T]$, $u_n' \rightharpoonup u'$ in $L^2([0,T];H)$ and $v_n \rightharpoonup^* v$ in $L^\infty([0,T];H)$, hence in particular $v_n \rightharpoonup v$ in $L^2([0,T];H)$. Moreover, $\{w_n(t) : n \in \mathbb{N}\} \subset C(t) \cap \overline{B}_{c_3}(0)$ for every $t \in [0,T]$ by Lemma 3.3. So by (H4), a diagonal argument and Lemma 3.2, cf. (16), we may also assume that $v_n(t) \to v(t)$ for every $t \in [0,T] \cap \mathbb{Q}$. Thus $v_n(0) = A_{\mu_n} u^0_{\mu_n} = v_0$ yields $v(0) = v_0$. Because A^{-1} is $1/c$-Lipschitz by (10), and because Lemma 2.2 (a) implies $\mu z + A^{-1} z = A_\mu^{-1} z$ for $\mu > 0$ and $z \in H$, hence $\mu_n v_n(t) + A^{-1} v_n(t) = u_n(t)$, we also obtain $u_n(t) \to u(t)$ for $t \in [0,T] \cap \mathbb{Q}$ by Lemma 3.3. Therefore $u_n \to u$ in $C([0,T];H)$ by Arzelà-Ascoli's theorem, since Lemma 3.2 implies $\sup_{n \in \mathbb{N}} |u_n(t) - u_n(s)|^2 \le c_2\, |t-s|$ for $s,t \in [0,T]$. Consequently, in particular $J_{\mu_n} u_n = u_n - \mu_n v_n \to u$ uniformly. Finally, by Lemma 2.2 (b) and since $u_0 \in D(A)$,

$$|u_n(0) - u_0| = |u^0_{\mu_n} - u_0| \le \left(\frac{1 + c\mu_n}{c}\right) |A_{\mu_n} u^0_{\mu_n} - A_{\mu_n} u_0| \to |v_0 - A^0 u_0|/c. \tag{17}$$

Because of $v_0 = A^0 u_0$ we consequently obtain $u(0) = u_0$. $\square$

Lemma 3.5. *Let* (H3a) *or* (H3b) *and* (H4) *be satisfied. Then* (u,v) *from* Lemma 3.4 *is a solution of* (5).

Proof. Let $\mathcal{A}\bar{u} = \{\bar{v} \in L^2([0,T];H) : \bar{v}(t) \in A\bar{u}(t)$ a.e. in $[0,T]\}$ for $\bar{u} \in L^2([0,T];H)$. Then $\mathcal{A}$ is a mmop by [3, Ex. 2.3.3], and we have $(J_{\mu_n}u_n, v_n) \in \mathcal{A}$ by definition of v_n. Therefore $(u,v) \in \mathcal{A}$ by Lemma 3.4 and [3, Prop. 2.5], i.e. $v(t) \in Au(t)$ a.e. in $[0,T]$. To show that $v(t) \in C(t)$ for a.e. $t \in [0,T]$, we fix $\delta > 0$ and define

$$M_\delta = \{z \in L^2([0,T];H) : |z(t) - \mathrm{proj}(z(t), C(t))|^2 \leq \delta \text{ for a.e. } t \in [0,T]\,\}.$$

Since φ_λ^t (cf. Lemma 2.3) is convex and $M_\delta = \{z \in L^2([0,T];H) : \varphi_\lambda^t(z(t)) \leq \delta/2\lambda$ for a.e. $t \in [0,T]\}$ for all $\lambda > 0$, M_δ is convex. Because M_δ is also closed, it is weakly closed in $L^2([0,T];H)$, and $v_n \in M_\delta$ for all sufficiently large $n \in \mathbb{N}$ by (16). Therefore Lemma 3.4 yields $v \in M_\delta$ for all $\delta > 0$. Thus by definition of M_δ we obtain $v(t) = \mathrm{proj}(v(t), C(t)) \in C(t)$ on the complement of some null set. Hence it remains to verify the differential inclusion from (5). For that, we will use (H3a) or (H3b). By the defining property of a projection we have $u_n'(t) \cdot [z - v_n(t)] \geq 0$ for all $n \in \mathbb{N}$, $t \in [0,T]$ and $z \in C(t)$ from (12). Thus $0 \geq \delta^*(-u_n'(t), C(t)) + \langle u_n'(t),\, v_n(t)\rangle$ for $n \in \mathbb{N}$ and $t \in [0,T]$, and therefore

$$\begin{aligned} 0 &\geq \int_0^T \delta^*(-u_n'(t), C(t))\,dt + \int_0^T \langle u_n'(t),\, v_n(t)\rangle\,dt \\ &= \int_0^T \delta^*(-u_n'(t), C(t))\,dt + \langle u_n',\, v_n\rangle_{L^2([0,T];H)} \end{aligned} \tag{18}$$

for $n \in \mathbb{N}$. To take $\liminf_{n\to\infty}$ of this inequality, first note that as a consequence of [13, Corollary p. 227] and of Lemma 3.4

$$\liminf_{n\to\infty} \int_0^T \delta^*(-u_n'(t), C(t))\,dt \geq \int_0^T \delta^*(-u'(t), C(t))\,dt. \tag{19}$$

In case that (H3b) is satisfied, this condition and Lemma 3.4 imply $v_n \to v$ in $L^2([0,T];H)$, and therefore $\langle u_n',\, v_n\rangle_{L^2([0,T];H)} \to \langle u',\, v\rangle_{L^2([0,T];H)}$. But this also holds, if (H3a) is satisfied. Indeed, because of $A = \partial\psi$, Lemma 2.5 (a) applied to $u = u_n$ and $z = v_n(t) = A_{\mu_n}u_n(t) = \partial\psi_{\mu_n}(u_n(t))$ gives

$$\int_0^T \langle u_n'(t),\, v_n(t)\rangle\,dt = \psi_{\mu_n}(u_n(T)) - \psi_{\mu_n}(u_n(0)) \to \psi(u(T)) - \psi(u_0) = \int_0^T \langle u'(t),\, v(t)\rangle\,dt,$$

since the convergence follows from (11), Lemma 3.2 and Lemma 2.5 (b), and since the last equality is obtained again by means of Lemma 2.5 (a), because we already know that u is a.c., hence differentiable a.e. in $[0,T]$, and $v(t) \in \partial\psi(u(t))$ a.e. in $[0,T]$. Therefore we may take $\liminf_{n\to\infty}$ of (18) to find in both cases (H3a) or (H3b)

$$0 \geq \int_0^T [\delta^*(-u'(t), C(t)) + \langle u'(t),\, v(t)\rangle]\,dt.$$

But the integrand is ≥ 0 a.e. on $[0,T]$, since $v(t) \in C(t)$ a.e. Hence we obtain $\delta^*(-u'(t), C(t)) + \langle u'(t),\, v(t)\rangle = 0$ a.e., and this completes the proof by definition of the normal cone. □

4. Existence of solutions of (1)

In this section we want to investigate (1). We start with an example showing that in some situations no solution can exist.

Example 4.1. Let $H = \mathbb{R}$, $\Gamma = [-1, 1]$ and $g(t, x) = 1 + t - x$ for $(t, x) \in \mathbb{R}^2$. Then with $q_0 = 0$ the necessary conditions $g(0, q_0) = 1 \in \Gamma$ and $\emptyset \neq g(t, \cdot)^{-1}(\Gamma) = [t, 2 + t]$ are satisfied, but (1) with $q_0 = 0$ has no (local) solution. Indeed, because of $N_\Gamma(1) = [0, \infty[$ and $N_\Gamma(x) = \{0\}$ for $|x| < 1$, the continuity of $t \mapsto g(t, q(t)) \in \Gamma$ would imply $q'(t) \leq 0$ a.e. in some $[0, \delta]$, contradicting $q(t) \in [t, 2 + t]$ for $t \in [0, \delta]$.

Let us also remark that if $g = g(t, x)$ is t-independent, then $q(t) = q_0$ is a stationary solution of (1) if the necessary condition $g(q_0) \in \Gamma$ is satisfied.
To prove existence of solutions of (1), we want to apply Theorem 3.1 with (H3b) and (H4) in $H = \mathbb{R}^n$ by using Lemma 1.1. In this way we obtain

Theorem 4.2. *Let $g \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ and $\Gamma \subset \mathbb{R}^n$ be closed and convex. Suppose that for some $\alpha > 0$ and $\beta \in\,]-\infty, 1[$*

$$\langle g(t, x) - g(s, y),\, x - y\rangle \geq \alpha\, |x - y|^2 - \beta\, (t - s)^2 \quad \textit{for} \quad t, s \in \mathbb{R},\ x, y \in \mathbb{R}^n. \tag{20}$$

Then (1) *has a solution in the sense of* Definition 2.1 *on every $[0, T]$ with $T > 0$.*

Proof. We define $\mathcal{H}$, $C(\cdot)$ and the single-valued A with $D(A) = \mathbb{R} \times \mathbb{R}^n = \mathcal{H}$ as in Lemma 1.1, i.e. $A(t, x) = (t, g(t, x))$. Then for all $\lambda \geq 0$, $t, s \in [0, T]$ and $x, y \in \mathbb{R}^n$ by (20)

$$\begin{aligned}
&\langle (A + \lambda I_{\mathcal{H}})(t, x) - (A + \lambda I_{\mathcal{H}})(s, y),\, (t, x) - (s, y)\rangle_{\mathcal{H}} \\
&= (1 + \lambda)(t - s)^2 + \lambda |x - y|^2 + \langle g(t, x) - g(s, y),\, x - y\rangle \\
&\geq (1 + \lambda - \beta)(t - s)^2 + (\lambda + \alpha)|x - y|^2 \\
&\geq (1 - \beta)(t - s)^2 + \alpha |x - y|^2 \geq c\, |(t, x) - (s, y)|_{\mathcal{H}}^2
\end{aligned}$$

with $c = (1 - \beta) \wedge \alpha > 0$. Therefore (10) is satisfied, and clearly (11) holds with $M(r) = r + \sup\{|g(t, x)| : (t, x) \in \mathbb{R} \times \mathbb{R}^n, |(t, x)| \leq r\}$. Moreover, $A + I_{\mathcal{H}}$ is onto (cf. [6, Thm. 11.6]), and thus A is a mmop. Since $C(\cdot)$ is Lipschitz with $L = 1$ and (H4) holds, we only have to verify (H3b). For that, we will first show that for $\mu \in\,]0, 1]$

$$\sup_{|z| \leq r} |A(z) - A_\mu(z)| \leq \sqrt{\mu^2 r^2 + \omega_{2M(\sqrt{2}r)}\Big(\mu M(\sqrt{2}r)\Big)^2} =: \tilde{\omega}(r, \mu)\,, \tag{21}$$

where for $R > 0$ and $\delta > 0$

$$\omega_R(\delta) = \sup\{|g(t, x) - g(s, y)| :\ |t|, |s|, |x|, |y| \leq R,\ |t - s| \leq \delta,\ |x - y| \leq \delta\}$$

is a restricted modulus of continuity of g, which is continuous at $\delta = 0$, since g is uniformly continuous on $[-R, R] \times \overline{B}_R(0)$. Thus $\tilde{\omega}(r, \mu) \to 0$ as $\mu \to 0^+$ for fixed $r \geq 0$.
To prove (21), first note that by solving the equation $(t, x) = (I_{\mathcal{H}} + \mu A)(s, y)$ for (s, y), we find for $\mu > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$

$$J_\mu(t, x) = \Big(\frac{t}{1 + \mu}\,,\ \Big[I + \mu g(\frac{t}{1 + \mu}, \cdot)\Big]^{-1}(x)\Big),$$

and hence

$$A_\mu(t,x) = \Big(\frac{t}{1+\mu}\,,\ \frac{1}{\mu}\Big(x - \Big[I + \mu g(\frac{t}{1+\mu},\cdot)\Big]^{-1}(x)\Big)\Big).$$

To estimate $|A(t,x) - A_\mu(t,x)|$, let $y = \mu^{-1}(x - [\ldots]^{-1}(x))$. Then with $\tau = t/(1+\mu)$ one arrives at $g(\tau, x - \mu y) = y$, and consequently in case that $|z| = |(t,x)| \le r$ and $\mu \in\ ]0,1]$

$$|A(t,x) - A_\mu(t,x)|^2 = |(t, g(t,x)) - (\tau, y)|^2 = \Big|\Big(\frac{\mu t}{1+\mu}\,,\ g(t,x) - g(\tau, x-\mu y)\Big)\Big|^2$$
$$\le \mu^2 t^2 + |g(t,x) - g(\tau, x - \mu y)|^2 \le \mu^2 r^2 + \omega_{2M(\sqrt{2}r)}\Big(\mu M(\sqrt{2}r)\Big)^2.$$

For the last estimate we have used $|\tau| \le |t| \le r$, hence $|(\tau,x)| \le \sqrt{2}r$, and so $|g(\tau,x)| \le M(\sqrt{2}r)$. Thus also $|y| = |g(\tau,\cdot)_\mu(x)| \le |g(\tau,\cdot)^0 x| = |g(\tau,x)| \le M(\sqrt{2}r)$. Hence we have shown (21).

To verify (H3b) from this, let $\mu_n \to 0^+$, $u_n \to u$ in $C([0,T];\mathcal{H})$ and $v_n = A_{\mu_n} u_n \rightharpoonup v$ in $L^2([0,T];\mathcal{H})$. Since every A_μ is Lipschitz with constant $1/\mu$, in fact $v_n \in C([0,T];\mathcal{H})$. Because $A : \mathcal{H} \to \mathcal{H}$ is single-valued and continuous, $(v_n) \subset C([0,T];\mathcal{H})$ is a Cauchy-sequence. To see this, choose $r > 0$ such that $|u|_{C([0,T];\mathcal{H})} \le r$ and $|u_n|_{C([0,T];\mathcal{H})} \le r$ for all $n \in \mathbb{N}$. Then for $t \in [0,T]$ and $n, m \in \mathbb{N}$ due to (21)

$$|v_n(t) - v_m(t)| \le |A_{\mu_n} u_n(t) - A u_n(t)| + |A u_n(t) - A u_m(t)| + |A u_m(t) - A_{\mu_m} u_m(t)|$$
$$\le \tilde{w}(r, \mu_n) + \tilde{w}(r, \mu_m) + |A u_n - A u_m|_{C([0,T];\mathcal{H})} \to 0 \quad \text{as} \quad n, m \to \infty$$

uniformly in $t \in [0,T]$. Thus (21) even implies $v_n \to v$ in $C([0,T];\mathcal{H})$, so that the proof of Theorem 4.2 is complete. □

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