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On the Minimal Extension of Increasing $*$-Weakly Semicontinuous Sublinear Functionals from L^∞_+ [1]

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Properties of increasing $*$-weakly lower semicontinuous (LSC) sublinear functionals on the positive cone L^∞_+ are of importance for study miscellaneous non-controlled factors from a unified viewpoint based on the notion of sublinear expectation [3, 4 and 5]. For every such functional N there are defined the class $\mathcal{A}_N$ of closed convex subsets $A \subset L^1_+$ satisfying the condition

$$N(\varphi) = \sup\{\langle \varphi, f\rangle : f \in A\} \quad \forall \varphi \in L^\infty_+$$

and the class $\mathcal{G}_N$ of increasing $*$-weakly LSC sublinear extentions of N from L^∞_+ to L^∞. $\mathcal{A}_N$ is ordered for inclusion and $\mathcal{G}_N$ is ordered in a natural way:

$$Q_1 \le Q_2 \Leftrightarrow Q_1(\varphi) \le Q_2(\varphi) \quad \forall \varphi \in L^\infty,$$

where $Q_1, Q_2 \in \mathcal{G}_N$. The existence of the minimal elements in $\mathcal{A}_N$ and in $\mathcal{G}_N$ is proved and their description is given. The orders induced in $L^{\infty *}$ by convex cones conjugate to $K_\varepsilon = \{\varphi \in L^\infty : \varphi \ge \varepsilon\|\varphi\|\}$, $\varepsilon > 0$, are of substantial use in proving the theorem.

Keywords: sublinear functionals, closed in measure convex sets in L^1, lower semicontinuity, sublinear expectation, quasi-extremal elements of convex sets.

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1. Introduction

Let $(\Omega, \mathcal{F}, \nu)$ be a finite positive measure space, L^1 be the Banach lattice of (classes of ν-equivalent) integrable functions with the norm

$$\|f\| = \int |f| d\nu \quad \forall f \in L^1, \tag{1}$$

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and L^∞ be the Banach lattice of (classes of ν-equivalent) bounded measurable functions with the norm

$$\|\varphi\| = \operatorname*{vrai}_{\nu} \max_{\omega\in\Omega}\{|\varphi(\omega)|\} \quad \forall\varphi \in L^\infty. \tag{2}$$

Let us recall that the Banach lattice L^∞ is conjugate to L^1 and L^1 is a sublattice of $L^{\infty *}$. In situations when an element $f \in L^1$ is regarded as a functional in $L^{\infty *}$ we shall use the notation $\langle\varphi, f\rangle = \int \varphi f d\nu \;\; \forall\varphi \in L^\infty$ and permit writings of the form $g = f + h$, where $g, h \in L^{\infty *}$. For the elements of L^∞_+ containing the indicators of sets $F \in \mathcal{F}$ we use the symbol χ_F.
The following proposition establishes the equivalence of different conditions of lower semicontinuity (LSC) for increasing sublinear functionals defined on L^∞_+.

Proposition 1.1. *Let $N: L^\infty_+ \to \mathbb{R}^1$ be an increasing sublinear functional.*
The following assertions are equivalent:
(a) *N is LSC in measure ν;*
(b) *N is LSC with respect to the family of prenorms*

$$\mathcal{P} = \{\|\cdot\|_f = \int |\cdot| f d\nu: f \in L^1_+\}; \tag{3}$$

(c) *N is continuous on increasing sequences, that is,*

$$N(\sup\{\varphi_k\}) = \sup\{N(\varphi_k)\}$$

if $\varphi_{k+1} \geq \varphi_k$ $(k = 1, 2, \ldots)$ and $\sup\{\varphi_k\} \in L^\infty_+$;
(d) *N is LSC in the $*$-weak topology $\sigma(L^\infty, L^1)$.*

We shall study increasing sublinear functionals $N: L^\infty_+ \to \mathbb{R}^1$ satisfying any of the semicontinuity conditions (a)–(d). Without loss of generality of our considerations we shall assume that $N(\mathbf{1}) = 1$. Such functionals will be called *sublinear expectations* (SE). They have interesting application in operation research [3, 4 and 5], see the last section of the paper.
With every SE N we connect the class $\mathcal{A}_N$ of closed convex subsets $A \subset L^1_+$ satisfying the condition

$$N(\varphi) = \sup\{\int \varphi f d\nu: f \in A\} \quad \forall\varphi \in L^\infty_+ \tag{4}$$

and the class $\mathcal{G}_N$ of increasing $*$-weakly LSC sublinear extentions of N from L^∞_+ to L^∞. $\mathcal{A}_N$ is ordered for inclusion and $\mathcal{G}_N$ is ordered in a natural way: $Q_1 \leq Q_2 \Leftrightarrow Q_1(\varphi) \leq Q_2(\varphi) \quad \forall\varphi \in L^\infty$, where $Q_1, Q_2 \in \mathcal{G}_N$. The main result of this paper is the following theorem.

Theorem 1.2. *For any SE N in the classes $\mathcal{A}_N$ and $\mathcal{G}_N$ there exist minimal elements.*

2. Lower semicontinuity of increasing sublinear functionals on L^∞_+

At first let us prove the equivalence of conditions (a), (b) and (c) in Proposition 1.1. The topology $\mathcal{P}$ is stronger than the topology of convergence in measure ν, hence (a) implies (b). Let $\varphi \in L^\infty_+$, $\varphi_k \in L^\infty_+$, $k = 1, 2 \ldots$. If $\varphi_k \uparrow \varphi$ then $\varphi_k \to \varphi$ in any space $L^1(\Omega, \mathcal{F}, \mu)$, where $d\mu = f d\nu$, $f \in L^1_+$. Therefore the condition (b) and the increase of N imply

$$N(\varphi) \leq \lim_{k\to\infty} \inf N(\varphi_k) \leq \lim_{k\to\infty} \sup N(\varphi_k) \leq N(\varphi),$$

that is, the condition (c) holds. Let $\varphi_k \to \varphi$ in measure ν; $N(\varphi_k) \leq a < \infty \quad \forall k \geq 1$. To prove (a) it suffices to obtain the inequality $N(\varphi) \leq a$. Choose a subsequence $\varphi_{k_n} \to \varphi$ $\nu-$a.e. and put $\psi_m = \inf\{\varphi_{k_n} \colon n \geq m\}$. Then $\psi_m \uparrow \varphi$ $\nu-$a.e. Taking into account that $N(\psi_m) \leq N(\varphi_{k_m}) \leq a$ we obtain $N\varphi) = \lim N(\psi_m) \leq a$.
The equivalence of (b) and (d) follows from Proposition 2.1.

Proposition 2.1. *The space L^1 is topologically conjugate to $(L^\infty, \mathcal{P})$.*

Proof. By the definition of the locally convex lattice (LCL) $(L^\infty, \mathcal{P})$ all elements of L^1 are continuous linear functionals. Let g be an arbitrary continuous linear functional on $(L^\infty, \mathcal{P})$. Since g is continuous in the norm (2), by the Yosida-Hewitt theorem [6] there exist functionals $f \in L^1$, $h \in L^{\infty *}$, $S_n \in \Omega$ $(n = 1, 2, \ldots)$, satisfying the conditions $g = f + h$, $S_n \uparrow \Omega$, $\langle \varphi \chi_{S_n}, h\rangle = 0 \quad \forall \varphi \in L^\infty \quad \forall n = 1, 2, \ldots$. Suppose $h \neq \mathbf{0}$ and choose $\varphi \in L^\infty$ from the condition $\langle \varphi, h\rangle = 1$. Then the sequence $\varphi(\mathbf{1} - \chi_{S_n})$ converges to $\mathbf{0}$ in the topology $\mathcal{P}$ but the number sequence

$$\begin{aligned}\langle \varphi(\mathbf{1} - \chi_{S_n}), g\rangle &= \langle \varphi(\mathbf{1} - \chi_{S_n}), f\rangle + \langle \varphi(\mathbf{1} - \chi_{S_n}), h\rangle = \\ &= \langle \varphi(\mathbf{1} - \chi_{S_n}), f\rangle + \langle \varphi, h\rangle - \langle \varphi\chi_{S_n}, h\rangle = \langle \varphi(\mathbf{1} - \chi_{S_n}), f\rangle + 1\end{aligned}$$

converges to 1 which contradicts the choice of g. Thus $g = f \in L^1$. □

Let us prove the equivalence of (b) and (d) in Proposition 1.1. Since the LCLs $(L^\infty, \mathcal{P})$ and $(L^\infty, \sigma(L^\infty, L^1))$ have the same set of continuous linear functionals, see Proposition 2.1, the classes of closed convex sets in them also coincide. Therefore, if a convex set $\{\varphi \in K \colon N(\varphi) \leq a\}$ is closed in one of these topologies it is closed in the other one. Proposition 1.1 has been proved.

3. Decompositions of sublinear expectations

Throughout we fix $(\Omega, \mathcal{F}, \nu)$ and a SE $N \colon L^\infty_+ \to \mathbb{R}^1$.

Definition 3.1. A subset $A \subset L^1_+$ satisfying the condition (4) is called a *decomposition* of the SE N.

The $\mathcal{P}$-subdifferential $\partial Q(\mathbf{0})$ of the extension

$$Q(\varphi) = N(\sup\{\varphi, \mathbf{0}\}) \quad \forall \varphi \in L^\infty \tag{5}$$

is the maximal for inclusion decomposition $A_{max}(N)$. Theorem 0.29 and Corollary 3 of Theorem 0.31 [6] imply that $A_{max}(N)$ is convex, $\sigma(L^1, L^\infty)$-weakly closed and hence closed in L^1. Since $N(\mathbf{1}) < \infty$ the maximal decomposition is bounded in L^1.

Proposition 3.2. *The maximal decomposition $A_{max}(N)$ is closed in measure ν.*

Proof. Let $f_k \in A_{max}(N) \quad \forall k = 1, 2, \dots, f_k \to f \in L^1_+$ in measure ν. Let $f_{k_m} \to f$ $\nu-$a.e. Set an arbitrary $\varphi \in L^\infty_+$. By the Fatou lemma applied to f_{k_m} and the measure μ with $d\mu = \varphi d\nu$ we obtain $\int f\varphi d\nu \le \lim_{k\to\infty} \inf \int (f_{k_m})\varphi d\nu \le N(\varphi)$. Thus $f \in A_{max}(N)$.

□

For any subset $D \subset L^1_+$ we put

$$\mathcal{R}(D) = \{f \in L^1_+ \colon \exists g \in D, g \ge f\}, \tag{6}$$

$$\overline{\mathcal{R}}(D) \quad \text{is the closure of} \quad \mathcal{R}(D) \quad \text{in} \quad L^1. \tag{7}$$

If A is an arbitrary decomposition of N then [3]

$$A_{max}(N) = \overline{\mathcal{R}}(\mathrm{conv}(A)), \tag{8}$$

where $\mathrm{conv}(A)$ is the convex hull of A.

In applications, see the last section, for generalization of some concepts, for example, completeness of statistical structures [1] it is desirable to operate with minimal sets generating by (4) the given sublinear expectation. Since in general the maximal decomposition is not compact extremal and extreme elements [8] need not exist (the maximal decomposition is compact with respect to $\sigma(L^1, L^\infty)$ if and only if [3] we have the implication $\varphi_{k+1} \le \varphi_k$ $(k = 1, 2, \dots) \Rightarrow N(\inf\{\varphi_k\}) = \inf\{N(\varphi_k)\})$. The attempt to define the "minimal" decomposition as the set of maximal elements of $A_{max}(N)$ is not fruitful: the example in Section 5 shows that among maximal elements of $\overline{\mathcal{R}}(\mathrm{conv}(A))$, where $\int f d\nu = 1 \quad \forall f \in A$, there exist elements with norm less than one. The problem is solved by the proof of the existence of the minimal closed in L^1 convex decomposition $A_{min}(N)$, which implies the assertion of Theorem 1.2 about the class $\mathcal{A}_N$.

4. Thin decompositions and quasi-extremal elements

Being a bounded sublinear functional on the normed space L^∞ a SE is continuous in the norm on L^∞. Therefore it is determined uniquely by its values on the open positive cone

$$L^\infty_{++} = \{\varphi \in L^\infty \colon \exists c \in \mathbb{R}^1, \quad c > 0, \quad c \cdot \mathbf{1} \le \varphi\} = \mathrm{int} L^\infty_+. \tag{9}$$

Definition 4.1. Let $\varepsilon \in \mathbb{R}^1$, $0 < \varepsilon < 1$. The *ε-thin decomposition* of a SE N is the set in L^1 of the form

$$A_\varepsilon(N) = \bigcup_{\varphi \in L^\infty_{++}} \{f \in A_{max}(N) \colon \int f\varphi d\nu > N(\varphi) - \varepsilon \cdot \mathrm{vrai\,min}(\varphi)\} \tag{10}$$

where $\mathrm{vrai\,min}(\varphi) = \sup\{\alpha \in \mathbb{R}^1 \colon \alpha \mathbf{1} \le \varphi\}$ is the essential minimum of φ.

Proposition 4.2. *Let A be an arbitrary closed convex decomposition of a SE N, and let $G_\varepsilon(A) = \bigcup_{f \in A} \{g \colon \|g - f\| < \varepsilon\}$ be the ε-neighbourhood of A. Then* $\mathrm{conv}(A_\varepsilon(N)) \subset G_\varepsilon(A)$ $\forall \varepsilon > 0$.

Proof. Take an arbitrary positive ε. Let $f \in \mathcal{R}(A)\backslash G_\varepsilon(A)$, where $\mathcal{R}$ is defined by (6). Choose $g \in A$, $g \geq f$. The inequality $\|g - f\| \geq \varepsilon$ implies $\int f\varphi d\nu \leq \int g\varphi d\nu - \varepsilon \cdot \operatorname{vrai\,min}(\varphi) \leq N(\varphi) - \varepsilon \cdot \operatorname{vrai\,min}(\varphi) \quad \forall\varphi \in L^\infty_{++}$, that is, $f \notin A_\varepsilon(N)$. Hence $\mathcal{R}(A) \cap (G_\varepsilon(A))^c \subset (A_\varepsilon(N))^c$. Moreover since $(A_\varepsilon(N))^c$ is closed in the restriction of the topology of L^1 to $A_{max}(N)$ we have $\overline{\mathcal{R}}(A) \cap (G_\varepsilon(A))^c \subset (A_\varepsilon(N))^c$. Taking into account the inclusion $A_\varepsilon(N) \subset A_{max}(N)$ and the formula (8) we obtain $A_\varepsilon(N) \subset G_\varepsilon(A)$. Now the assertion of Proposition 4.2 follows from convexity of the set $G_\varepsilon(A)$. □

Corollary 4.3. *The intersection of convex closures of all ε-thin decompositions is a subset of any closed convex decomposition.*

Corollary 4.4. *If a decomposition T of N is a subset of any ε-thin decomposition then the closed convex hull of T is $A_{min}(N)$.*

Let X, X^* be a normed vector space and its conjugate space.

Definition 4.5. An element x_0 of a set $A \subset X$ is called *quasi-extremal with respect to a cone $C \subset X^*$* if $\forall\delta > 0\ \exists f \in C$: $\langle x_0, f\rangle > \sup\{\langle x, f\rangle : x \in A\} - \delta\|f\|$.

Quasi-extremal elements are useful in the work with bounded sets whose weak closures are not compact in weak topologies, when there exist insufficiently many extremal elements or they do not exist at all.

For every $\varepsilon \in \mathbb{R}^1$, $0 < \varepsilon < 1$, we define the cone of *ε-separated from zero* positive elements of L^∞ by the formula

$$K_\varepsilon = \{\varphi \in L^\infty : \varphi \geq \varepsilon\|\varphi\| \cdot \mathbf{1}\}. \tag{11}$$

The family of cones $\{K_\varepsilon : \varepsilon \in (0, 1)\}$ increases monotonically as $\varepsilon \downarrow 0$, and then $\bigcup_{\varepsilon>0} K_\varepsilon = L^\infty_{++} \bigcup \{\mathbf{0}\}$.

Proposition 4.6. *Let $D_\varepsilon(N)$ be the set of quasi-extremal with respect to K_ε elements of $A_{max}(N)$. Then*

$$\bigcup_{0<\varepsilon<1} D_\varepsilon(N) \subset \bigcap_{0<\varepsilon<1} A_\varepsilon(N). \tag{12}$$

Proof. Take an arbitrary $\varepsilon \in (0, 1)$. Let $f \in A_{max}(N)\backslash A_\varepsilon(N)$. Then $\int f\varphi d\nu \leq N(\varphi) - \varepsilon \cdot \operatorname{vrai\,min}(\varphi) \quad \forall\varphi \in L^\infty_{++}$ and in particular $\int f\varphi d\nu \leq N(\varphi) - \varepsilon\|\varphi\|\alpha^{-1} \quad \forall\varphi \in K_\alpha$ $\forall\alpha \in (0, 1)$, that is, $f \notin \bigcup_{0<\alpha<1} D_\alpha(N)$. Hence $\bigcup_{0<\alpha<1} D_\alpha(N) \subset A_\varepsilon(N) \quad \forall\varepsilon \in (0, 1)$. □

For the proof of existence of $A_{min}(N)$ it is sufficient to construct for an arbitrary SE a decomposition consisting of K_ε-quasi-extremal elements. Such a decomposition will be constructed with the help of the Zorn lemma and the orders in $L^{\infty*}$ induced by the cones K_ε.

5. Order induced by the cone of ε-separated from zero positive measurable functions

First let us study the set of maximal elements in $A_{max}(N)$ for the usual order in the Banach lattice L^1. Since the maximal decomposition is bounded and closed in measure ν in L^1, it contains the suprema of all included chains. The Zorn lemma implies that the set of maximal elements in $A_{max}(N)$ is a decomposition for the SE N. The following example shows that this decomposition contains elements which do not belong to $A_{min}(N)$.

Example 5.1. Let $\Omega = \{1, 2, \ldots\}$, $\mathcal{F} = 2^\Omega$ and $\nu(\{\omega\}) = 2^{-\omega} \quad \forall \omega \in \Omega$. We define the SE N by the decomposition $A = \{f_n : n = 1, 2, \ldots\}$ with $f_n = g_n + h_n$, $g_n = (1 - 2^{-n})\delta(1, \omega)$, $h_n = (1 + 2^n)\delta(n + 1, \omega)$, where $\delta(n, m)$ is the Kronecker symbol. Note that $\|f_n\| = 1 \quad \forall n$. The element $g = \delta(1, \omega)$ is the limit of the increasing sequence $g_n \in A_{max}(N)$. It is maximal in $A_{max}(N)$ and besides $\|g\| = 2^{-1}$. By many reasons, it is inadmissible that the "minimal decomposition" would contain such elements. If $A_{min}(N)$ exists it does not contain g because all elements of the convex closure of A have the unit norm. Let us show that the natural order in L^1 can be "corrected" so that the element g should not be maximal. Fix $\varepsilon \in (0, 1)$ and choose an arbitrary nonzero element φ in the cone (11). Then

$$\int \varphi f_n d\nu - \int \varphi g d\nu = -2^{-n-1}\varphi(1) + 2^{-1}(1 + 2^{-n})\varphi(n + 1) \geq$$
$$\geq -2^{-n-1}\|\varphi\| + 2^{-1}(1 + 2^{-n})\varepsilon\|\varphi\| > 0 \quad \forall n > \log_2(\varepsilon^{-1} - 1).$$

Hence for the order, defined in L^1 by the rule

$$f_1 \leq f_2 \Leftrightarrow \int \varphi f_1 d\nu \leq \int \varphi f_2 d\nu \quad \forall \varphi \in K_\varepsilon,$$

the element g is not maximal.
For every $\varepsilon \in \mathbb{R}^1$, $0 < \varepsilon < 1$, we define a cone in $L^{\infty *}$ by the formula

$$K_\varepsilon^* = \{g \in L^{\infty *} : \langle \varphi, g\rangle \geq 0 \quad \forall \varphi \in K_\varepsilon\}, \tag{13}$$

where K_ε is defined by (11). Since K_ε is a closed convex cone it follows from (13) that

$$K_\varepsilon = \{\varphi \in L^\infty : \langle \varphi, g\rangle \geq 0 \quad \forall g \in K_\varepsilon^*\} \quad \forall \varepsilon \in (0, 1). \tag{14}$$

Obviously $K_\varepsilon - K_\varepsilon = L^\infty$ and hence $(-K_\varepsilon^*) \cap K_\varepsilon^* = \{\mathbf{0}\}$, that is, the cone K_ε^* is proper. Therefore the space $L^{\infty *}$ with the order relation

$$g_1 \leq g_2 \Leftrightarrow g_2 - g_1 \in K_\varepsilon^* \tag{15}$$

is an ordered vector space with the positive cone K_ε^* [8].
To avoid confusion with the natural order in $L^{\infty *}$ we use terms: K_ε^*-*chain*, K_ε^*-*majorant*, K_ε^*-*maximal element*, and so on.

Theorem 5.2. *Let B be a bounded closed convex subset of L^1_+, $\varepsilon \in \mathbb{R}^1$, $0 < \varepsilon < 1$. Then every element of B is K_ε^*-majorized by some K_ε^*-maximal element of B. If besides*

B is closed in measure ν then every K^*_ε-maximal element of B remains K^*_ε-maximal in the closure of B in the $(*)$-weak topology $\sigma(L^{\infty *}, L^\infty)$ and is quasi-extremal with respect to the cone K_ε.

Proof. Let $\{g_\alpha\}$ be a K^*_ε-chain in B, and let $\overline{B}$ be the closure of B in the $(*)$-weak topology $\sigma(L^{\infty *}, L^\infty)$. The $(*)$-weak compactness of $\overline{B}$ implies existence of the $(*)$-weak limit $g = \lim_\alpha g_\alpha \in \overline{B}$. Let us prove that $g \in B$, and then the first assertion of the theorem will follow from the Zorn lemma.

According to the Yosida-Hewitt theorem [6] g can be represented uniquely in the form $g = f + h$, where $f \in L^1_+$, $h \in L^{\infty *}_+$, and there exists a sequence of sets $S_n \uparrow \Omega$ whose indicators satisfy $\langle \chi_{S_n}, h\rangle = 0 \quad \forall n = 1, 2, \ldots$. Take arbitrary $\delta \in \mathbb{R}^1$, $\varepsilon < \delta < 1$, and $\varphi \in K_\delta$. Put $\varphi_n = \varphi\chi_{S_n} + \varepsilon\|\varphi\|(\mathbf{1} - \chi_{S_n}) \in K_\varepsilon$. Note that $\varphi_n \uparrow \varphi$, $\langle\varphi, g\rangle = \langle\varphi, f\rangle + \langle\varphi, h\rangle \geq \langle\varphi, f\rangle + \delta\|\varphi\|\|h\|$, $\langle\varphi_n, g\rangle = \langle\varphi_n, f\rangle + \langle\varepsilon\|\varphi\|(\mathbf{1} - \chi_{S_n}), h\rangle \leq \langle\varphi_n, f\rangle + \varepsilon\|\varphi\|\|h\|$.

By uniting these inequalities we obtain

$$\langle\varphi, g\rangle - \langle\varphi_n, g\rangle \geq \langle(\varphi - \varphi_n), f\rangle + (\delta - \varepsilon)\|\varphi\|\|h\| \geq (\delta - \varepsilon)\|\varphi\|\|h\|. \tag{16}$$

Suppose $\|h\| > 0$. Continuity of the functional $\sup\limits_\alpha\langle(\cdot), g_\alpha\rangle$ on increasing sequences in L^∞ implies $\sup\limits_\alpha\langle\varphi_n, g_\alpha\rangle \uparrow \sup_\alpha\langle\varphi, g_\alpha\rangle = \langle\varphi, g\rangle$. Therefore there exist α, n for which $\langle\varphi_n, g_\alpha\rangle > \langle\varphi, g\rangle - (\delta - \varepsilon)\|\varphi\|\|h\|/2$. The relation $\varphi_n \in K_\varepsilon$ implies $\langle\varphi_n, g\rangle \geq \langle\varphi_n, g_\alpha\rangle$. By uniting the last two inequalities we obtain $\langle\varphi_n, g\rangle - \langle\varphi, g\rangle > -(\delta - \varepsilon)\|\varphi\|\|h\|/2$ which contradicts (16). So $\|h\| = 0$ and $g \in L^1_+$. Since the restriction of the $(*)$-weak topology in $L^{\infty *}$ to L^1 coincides with the weak topology $\sigma(L^1, L^\infty)$ and closed convex sets in L^1 are weakly closed we have $g \in B$.

Assume that B is closed in measure ν. Let f be an arbitrary K^*_ε-maximal element of B. Since the cone K^*_ε is proper we have $B \cap (f + K^*_\varepsilon) = \{f\}$. By Theorem 7.1 [6], see also [2 and 7], $B = P_1(\overline{B})$, where P_1 is the positive operator of projection of $L^{\infty *}$ onto L. Therefore any element of the intersection $\overline{B} \cap (f + K^*_\varepsilon)$ can be represented as the sum $f + h$, where h is a positive element of the algebraic complement of the subspace L in $L^{\infty *}$. Suppose that some element $f + h$ of $\overline{B} \cap (f + K^*_\varepsilon)$ K^*_ε-majorizes f and besides $\|h\| > 0$. We exclude from the analysis the trivial case $B = \{f\}$. Choose $q \in B$, $q \neq f$, $\beta \in \mathbb{R}^1$, $0 < \beta < \varepsilon\|h\| \cdot (\varepsilon\|h\| + \|q - f\|)^{-1}$ and put $g = \beta q + (1 - \beta)(f + h) \in \overline{B}$. Since $P_1(g) = \beta q + (1 - \beta)f \neq f$ the element g cannot majorize f in $\overline{B}$. This contradicts the inequality

$$\begin{aligned}\langle\varphi, g\rangle - \langle\varphi, f\rangle = (1 - \beta)\langle\varphi, h\rangle + \beta\langle\varphi, q - f\rangle \geq (1 - \beta)\varepsilon\|\varphi\|\|h\| - \\ - \beta\|q - f\|\|\varphi\| \geq 0 \quad \forall\varphi \in K_\varepsilon.\end{aligned}$$

Hence $\|h\| = 0$ and the element f is K^*_ε-maximal in $\overline{B}$.

It remains to prove that the element f is quasi-extremal with respect to the cone K_ε. Note that $\|f\| > 0$ because otherwise we would have the trivial situation $B = \{\mathbf{0}\}$ excluded from the analysis. Take a positive $\delta \in \mathbb{R}^1$ and choose $\gamma \in \mathbb{R}^1$, $0 < \gamma < \delta\|f\|^{-1}$. Since f is K^*_ε-maximal in $\overline{B}$ and the cone K^*_ε is proper we have $\overline{B} \cap ((1 + \gamma)f + K^*_\varepsilon) = \emptyset$. The

$(*)$-weakly compact set $\overline{B}$ and the $(*)$-weakly closed set $((1+\gamma)f + K_\varepsilon^*)$ can be separated strictly by some element of L^∞ and so there exists $\alpha, \beta \in \mathbb{R}^1$, and $\varphi \in L^\infty$ such that

$$\sup\{\langle\varphi, g\rangle : g \in \overline{B}\} \le \alpha < \beta \le \inf\{\langle\varphi, g\rangle : g \in ((1+\gamma)f + K_\varepsilon^*)\}.$$

The right-hand inequality implies $\langle\varphi, g\rangle \ge 0 \quad \forall g \in K_\varepsilon^*$. Therefore, see (14), $\varphi \in K_\varepsilon$. Finally we obtain

$$\begin{aligned}\langle\varphi, f\rangle \ge \beta - \gamma\langle\varphi, f\rangle > \sup\{\langle\varphi, g\rangle : g \in \overline{B}\} - \gamma\|\varphi\|\|f\| > \\ > \sup\{\langle\varphi, g\rangle : g \in B\} - \delta\|\varphi\|,\end{aligned}$$

that is, f is quasi-extremal with respect to the cone K_ε. □

6. Theorem on the minimal closed convex decomposition and minimal extensions of sublinear functionals

Definition 6.1. *The north-eastern boundary (NEB)* of the maximal decomposition of a SE N is the set

$$neb(N) = \bigcup_{0<\varepsilon<1} T_\varepsilon(N), \tag{17}$$

where $T_\varepsilon(N)$ is the subset of K_ε^*-maximal elements in the maximal decomposition $A_{max}(N)$.

Theorem 6.2. *For any sublinear expectation N the NEB of its maximal decomposition is a decomposition the closure of whose convex hull is $A_{min}(N)$.*

Proof. Since a SE is determined uniquely by its values on the cone (9) and the maximal decomposition is norm bounded, closed in measure and convex, by Theorem 5.2 the NEB of the maximal decomposition is a decomposition consisting of quasi-extremal elements with respect to the cones K_ε $(0 < \varepsilon < 1)$. By Proposition 4.6 the NEB of the maximal decomposition is included in any ε-thin decomposition. It remains to apply Corollary 4.4 of Proposition 4.2. □

The assertion of Theorem 1.2 about the class $\mathcal{A}_N$ is a simple corollary from Theorem 6.2. Let Q be any element of $\mathcal{G}_N$. Then $A_{min}(N) \subset \partial Q(\mathbf{0}) \subset A_{max}(N)$, where $\partial Q(\mathbf{0})$ is the $\mathcal{P}$-subdifferential of Q at $\mathbf{0}$. Hence $Q(\varphi) \ge Q_0(\varphi) \quad \forall\varphi \in L^\infty$, where $Q_0(\varphi) = \sup\{\int \varphi f d\nu : f \in A_{min}(N)\} \quad \forall\varphi \in L^\infty$. Theorem 1.2 has been proved.

7. Sublinear expectations in operation research

In some applied problems of decision making for adequate description of non-controlled factors it is necessary to unite several models of different types: it is natural to consider some non-controlled factors to be random, others to be indefinite, the third to be fuzzy. There are also indefinite factors with a random set of possible values, random factors with an indefinite distribution and other more complicated compositions. It turns out that the most of similar situations can be described from unified positions with the help of sublinear expectations.

Let Ω be a set of a priori unknown possible values ω of a non-controlled factor, $(\Omega, \mathcal{F})$ be the measurable space of elementary events (not certainly random). We introduce the denotations: $\overline{\mathcal{L}}^0(\Omega, \mathcal{F})$ — for the lattice of all measurable functions $\varphi: \Omega \to \mathbb{R}^1 \cup \{+\infty, -\infty\}$; $\mathcal{L}^0(\Omega, \mathcal{F})$ — for the vector lattice of all measurable functions $\varphi: \Omega \to \mathbb{R}^1$; $\mathcal{L}^\infty(\Omega, \mathcal{F})$ — for its vector sublattice of bounded functions.

Definition 7.1. *A sublinear expectation (on* $\overline{\mathcal{L}}^0$*)* is any functional $N: \overline{\mathcal{L}}^0_+(\Omega, \mathcal{F}) \to \mathbb{R}^1 \cup \{+\infty\}$ with properties:
(a) $N(\mathbf{1}) = 1$;
(b) $N(\varphi_1) \leq N(\varphi_2) \quad$ if $\varphi_1 \leq \varphi_2$;
(c) $N(\alpha\varphi) = \alpha N(\varphi) \quad \forall \alpha \in \mathbb{R}^1_+$;
(d) $N(\varphi_1 + \varphi_2) \leq N(\varphi_1) + N(\varphi_2)$;
(e) $N(\varphi_k) \uparrow N(\varphi) \quad$ if $\varphi_k \uparrow \varphi$;
(f) there exists a finite positive measure ν defined on $(\Omega, \mathcal{F})$ and such that $\nu(F) = 0 \Rightarrow N(\chi_F) = 0 \quad \forall F \in \mathcal{F}$.

The triplet $(\Omega, \mathcal{F}, N)$ will be called a *space with a sublinear expectation*, a measure ν will be called a *dominating measure* for N.

Interpretation. Let be given $(\Omega, \mathcal{F}, N)$. A connected optimization problem is defined as a pair (U, Φ), where U is some set of strategies $u, \Phi(u, \omega): U \times \Omega \to \mathbb{R}^1_+ \cup \{+\infty\}$ is a non-negative measurable at each fixed $u \in U$ purpose function constructed in a scale of negative utility, for example damage function. We shall believe that the space with a SE $(\Omega, \mathcal{F}, N)$ gives a correct description of an non-controlled factor if for any connected optimization problem the rule

$$u_1 \text{ is not worse than } u_2 \Leftrightarrow N(\Phi(u_1, \omega)) \leq N(\Phi(u_2, \omega)) \tag{18}$$

establishes a prefence relation on U which conform with our intuition.

Example 7.2. Random factor. Let be given a probability space $(\Omega, \mathcal{F}, P)$. The SE of the form

$$N(\varphi) = \int \varphi \, dP \tag{19}$$

(expectation) corresponds to the model of a random factor.

Example 7.3. Indefinite factor. To construct the model of an indefinite factor we set a space of elementary events $(\Omega, \mathcal{F})$ with a finite positive measure ν. This measure will be needed only to factorize the lattice $\overline{\mathcal{L}}^0_+$ and so it will be called a *factorizing measure*. We define the SE on $\overline{\mathcal{L}}^0_+$ as the essential maximum:

$$N(\varphi) = \operatorname*{vrai}_{\nu} \max_{\omega \in \Omega} \varphi(\omega) \overset{\text{(def)}}{=} \sup\{\alpha: \nu\{\omega: \varphi(\omega) > \alpha\} > 0\}. \tag{20}$$

For a proper choice of the factorizing measure the essential maximum will not differ from the traditional supremum for a wide class of functions sufficient for the practical application.

Example 7.4. Indefinite factor with a fuzzy set of possible values. We set a space of elementary events $(\Omega, \mathcal{F})$ with a factorizing measure ν and a measurable non-negative function $\lambda(\omega)$ for which $\operatorname{vraimax}_{\nu\ \omega\in\Omega}\{\lambda(\omega)\} = 1$. The SE is defined by the formula

$$N(\varphi) = \operatorname*{vrai}_{\nu} \max_{\omega\in\Omega}\{\lambda(\omega)\varphi(\omega)\}. \tag{21}$$

If $\lambda(\omega)$ is the indicator of some set $F \in \mathcal{F}$ then we obtain the model of an indefinite factor taking values in the set F. In the general case the function $\lambda(\omega)$ can be interpreted as the function of belonging to a fuzzy set of possible values ω.

Example 7.5. Indefinite factor with a random set of possible values. At the same time with a space of elementary events $(\Omega, \mathcal{F})$ let be given a probability space $(X, \mathcal{X}, P)$. We assume that the set of possible values ω of an indefinite factor is given by its indicator $\lambda_x(\omega)$ with a random parameter $x \in X$. The function $\lambda_x(\omega)$ is believed to be measurable in the joint arguments. We set a factorizing measure μ on $(\Omega, \mathcal{F})$. The construction of the model is completed by defining the SE by the formula

$$N(\varphi) = \int_X (\operatorname*{vrai}_{\mu} \max_{\omega\in\Omega}\{\lambda_x(\omega)\varphi(\omega)\})dP(x). \tag{22}$$

The proper measurability of the integrand follows from Theorem 6 [3]. The dominating measure is $\nu = \mu \times P$.

Example 7.6. Sublinear expectation associated with a statistical structure. Let $(\Xi, \mathcal{W})$ be a measurable sample space, $(\Theta, \mathcal{V}, \mu)$ be a measurable (parameter) space with a factorizing measure μ and P_θ be a transition probability acting from $(\Theta, \mathcal{V})$ to $(\Xi, \mathcal{W})$. The SE defined on $\overline{\mathcal{L}}^0_+(\Xi \times \Theta, \mathcal{W} \otimes \mathcal{V})$ by the formula

$$N(\varphi) = \operatorname*{vrai}_{\mu} \max_{\theta\in\Theta} \int \varphi(\xi, \theta)dP_\theta(\xi) \tag{23}$$

will be called *the sublinear expectation associated with the statistical structure*

$$(\Xi, \mathcal{W}, \{P_\theta: \theta \in (\Theta, \mathcal{V}, \mu)\}). \tag{24}$$

The dominating measure is $\nu(A \times B) = \int \chi_A(\theta)P_\theta(B)d\mu(\theta)$, where $A \in \mathcal{V}$ and $B \in \mathcal{W}$. Definition 7.1 implies trivially that a SE is constant on every class of ν-equivalent functions in $\overline{\mathcal{L}}^0_+(\Omega, \mathcal{F})$, where ν is a dominating measure. By applying the standard extension procedure for functionals which are increasing and continuous on increasing sequences it is not difficult to show that a SE is extended uniquely from $\mathcal{L}^\infty_+(\Omega, \mathcal{F})$ to $\overline{\mathcal{L}}^0_+(\Omega, \mathcal{F})$. Therefore $L^\infty_+(\Omega, \mathcal{F}, \nu)$ is a convenient domain for a SE. This was used at the beginning of the paper.

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