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# Rank-one-convex and Quasiconvex Envelopes for Functions Depending on Quadratic Forms

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In this paper we are interested in functions defined, on a set of matrices, by the mean of quadratic forms and we compute the rank-one-convex, quasiconvex, polyconvex and convex envelopes of these functions. For that, and for a given quadratic form, we prove, in a first part, some general decomposition results for matrices, with a rank-one-compatibility condition. We also study the James-Ericksen stored energy function.

*Keywords:* rank-one-convex, quasiconvex, envelope, quadratic form, James-Ericksen function, Pipkin's formula.

## 1. Introduction

Let us denote by $\mathbb{M}^{m\times n}$ the set of $m \times n$ real matrices and by $W$ a function defined on $\mathbb{M}^{m\times n}$ with values in $\mathbb{R}$. Moreover, let $\Omega$ be a bounded domain in $\mathbb{R}^n$. The Calculus of Variations in the vectorial case addresses problems of the type : minimize

$$I_1(u) = \int_\Omega W(\nabla u(x))\, dx \tag{1.1}$$

over some class of functions. Here $\nabla u$ denotes the Jacobian matrix of $u$ -i.e. the matrix defined by

$$\nabla u = \left(\frac{\partial u_i}{\partial x_j}\right), \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

where $u_1, \dots, u_m$ denote the components of $u$. In general $I_1(u)$ is not lower semicontinuous and the direct method of the Calculus of Variations fails for the minimization of (1.1) (see [8]). One way to overcome the situation is to consider the so-called relaxed problem, that is to minimize

$$I_2(u) = \int_\Omega QW(\nabla u(x))\, dx \tag{1.2}$$

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where $QW$ denotes the quasiconvex envelope of $W$. We refer the reader to [8] for the relationship between (1.1) and (1.2). Before to go on, let us recall the definition of quasiconvexity and related notions.

• $W$ is said to be *polyconvex* if there exists a convex function $\hat{W}$ such that

$$W(F) = \hat{W}(T(F))$$

where $T(F)$ stands for the vector of all minors of $F$ (see [8]).

• $W$ is said to be *quasiconvex* if

$$W(F) \leq \frac{1}{|D|} \int_D W(F + \nabla v(x))\; dx \tag{1.3}$$

for any bounded domain $D$ and any smooth function $v : D \longrightarrow \mathbb{R}^m$, vanishing on the boundary of $D$.

• $W$ is said to be *rank-one-convex* if

$$W(\lambda F + (1-\lambda)G) \leq \lambda W(F) + (1-\lambda)W(G)$$

for any couple $F$, $G$ such that

$$\text{rank}(F - G) \leq 1$$

and any $\lambda \in [0, 1]$.

The notion of polyconvexity has been introduced by J. Ball (see [1]) to address problems of nonlinear elasticity (see also [5], [6]). Quasiconvexity goes back to Morrey (see [11]) and insures weak lower semi continuity of $I_1(u)$ in some spaces (see [12], [8], [2]). Of course, condition (1.3) is not easy to test.

It is now well known that

$$W \;\text{ convex } \Longrightarrow W \;\text{ polyconvex } \Longrightarrow W \;\text{ quasiconvex } \Longrightarrow W \;\text{ rank-one-convex.} \tag{1.4}$$

These implications are one way in the sense that the converse implication does not hold in general. It has been an outstanding challenge to decide that

$$W \;\text{ rank-one-convex } \not\Longrightarrow W \;\text{ quasiconvex.}$$

This has been established recently by V. Šverák (see [16]) for dimensions $m \geq 3$ and $n \geq 2$. Of course, in the case $m = 1$ or $n = 1$ all these notions are the same (see [8]).

This terminology being precised, one can define the following convex, polyconvex, quasiconvex, rank-one-convex envelopes by setting

$$CW = \sup\{f \; ; \; f \text{ convex and } f \leq W\}$$

$$PW = \sup\{f \; ; \; f \text{ polyconvex and } f \leq W\}$$

$$QW = \sup\{f \; ; \; f \text{ quasiconvex and } f \leq W\}$$

$$RW = \sup\{f \; ; \; f \text{ rank-one-convex and } f \leq W\}.$$

Clearly by (1.4) one has

$$CW \le PW \le QW \le RW \tag{1.5}$$

and these four envelopes coincide in the case $m = 1$ or $n = 1$, but also, in the general case, when $RW$ is convex.

The goal of this paper is to compute some of these envelopes for functions $W$ defined on the set of $m \times n$ matrices through quadratic forms.

In the last section, we will consider a function used, for instance in [7], to study a two-dimensional crystal. This energy density, proposed by Ericksen and James, is given by

$$\phi(F) = \tilde{\phi}(C) = \kappa_1(\mathrm{tr}(C) - 2)^2 + \kappa_2 c_{12}^2 + \kappa_3 \left( \left( \frac{c_{11} - c_{22}}{2} \right)^2 - \varepsilon^2 \right)^2 \tag{1.6}$$

where

$$C = F^T F = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

is the Cauchy-Green strain tensor, where the nonnegative constants $\kappa_1$, $\kappa_2$, $\kappa_3$ are elastic moduli, and where $\varepsilon$ is the transformation strain.

In the case where $\kappa_3 = 0$, the function $\tilde{\phi}$ is convex and thus the rank-one-convex envelope of $\phi$ is convex and can be compute by using the Pipkin formula (see [13], [14], [15] and [10]).

See also [9] for a numerical approach of minimization problems associated to the functionnal $\phi$.

Finally, let us recall that, for $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$, we denote by $a \otimes b$ the rank-one-matrix defined by $(a \otimes b)_{ij} = a_i b_j$.

## 2. Decomposition results for matrices

In this section, we denote by $q$ a quadratic form defined on $\mathbb{M}^{m\times n}$:

$$q : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$$

and by $\beta$ the symmetric bilinear form associated to $q$, that is the function defined on $\mathbb{M}^{m\times n} \times \mathbb{M}^{m\times n}$ by

$$\forall F, G \in \mathbb{M}^{m\times n}\ , \quad \beta(F,G) = \frac{1}{2}\Big( q(F+G) - q(F) - q(G) \Big).$$

We will assume that $q \not\equiv 0$, and thus either the range of $q$ is $\mathbb{R}$, or $q$ is nonnegative, or $q$ is nonpositive.

We have the following decomposition result:

**Proposition 2.1.** *Let us consider $F \in \mathbb{M}^{m\times n}$ and $\alpha \in \mathbb{R}$ such that $q(F) \le \alpha$. Assume there exist $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ satisfying $q(a \otimes b) > 0$. Then, there exists $\lambda \in [0,1]$ and $t \in \mathbb{R}_+$ such that, if $E = t a \otimes b$, one has*

$$q(F + \lambda E) = q(F - (1-\lambda)E) = \alpha. \tag{2.1}$$

**Proof.** First, let us remark that, for $F, E \in \mathbb{M}^{m\times n}$ and $\lambda \in [0,1]$, one has

$$q(F+\lambda E) = q(F) + 2\lambda\beta(F,E) + \lambda^2 q(E) \tag{2.2}$$

$$q(F-(1-\lambda)E) = q(F) - 2(1-\lambda)\beta(F,E) + (1-\lambda)^2 q(E) \tag{2.3}$$

If $q(F) = \alpha$, then (2.1) holds with $t = 0$. Now, let us assume that $q(F) < \alpha$. Since $q(a\otimes b) > 0$, we have

$$\frac{\beta(F,a\otimes b)^2}{q(a\otimes b)^2} - \frac{q(F)-\alpha}{q(a\otimes b)} > 0$$

and, if we set

$$t = 2\left(\frac{\beta(F,a\otimes b)^2}{q(a\otimes b)^2} - \frac{q(F)-\alpha}{q(a\otimes b)}\right)^{\frac{1}{2}} \qquad \text{and} \qquad E = ta\otimes b$$

then

$$\frac{\beta(F,E)^2}{q(E)^2} - \frac{q(F)-\alpha}{q(E)} = \frac{1}{t^2}\left(\frac{\beta(F,a\otimes b)^2}{q(a\otimes b)^2} - \frac{q(F)-\alpha}{q(a\otimes b)}\right) = \frac{1}{4}\,. \tag{2.4}$$

Consequently, by choosing

$$\lambda = \frac{1}{2}\left(1 - \frac{2\beta(F,E)}{q(E)}\right) \tag{2.5}$$

we obtain, with (2.4)

$$\lambda + (\lambda - 1) = \frac{-2\beta(F,E)}{q(E)}$$

and

$$\lambda(\lambda-1) = \frac{\beta(F,E)^2}{q(E)^2} - \frac{1}{4} = \frac{q(F)-\alpha}{q(E)}\,.$$

Therefore, $\lambda$ and $\lambda - 1$ are the solutions of the following equation

$$q(E)X^2 + 2\beta(F,E)X + q(F) - \alpha = 0.$$

Then (2.2) and (2.3) give (2.1). Moreover, (2.4) and (2.5) imply that $\lambda \in [0,1]$. □

Now, let us consider $\tilde{\Theta} : \mathbb{R}^m \times \ldots \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ an antisymmetric $n$-linear function, and denote by $\Theta$ the function defined on $\mathbb{M}^{m\times n}$ by $\Theta(F) = \tilde{\Theta}(F_1,\ldots,F_n)$, where $F_j$ is the $j^{\text{th}}$ column of the matrix $F$.

**Proposition 2.2.** *Let us consider $F \in \mathbb{M}^{m\times n}$ and $\alpha \in \mathbb{R}$ such that $q(F) \le \alpha$. Assume there exist $j \in \{1,\ldots,n\}$ and $b \in \mathbb{R}^n$ satisfying*

$$q(F_j \otimes b) > 0 \qquad \textit{and} \qquad b_j = 0$$

*where $b_j$ is the $j^{th}$ entry of $b$.*

*Then, there exist $\lambda \in [0,1]$, $A, B \in \mathbb{M}^{m\times n}$ such that,*

$$F = (1-\lambda)A + \lambda B \ , \quad rank(A-B) \le 1 \tag{2.6}$$

$$q(A) = q(B) = \alpha \tag{2.7}$$

$$\Theta(A) = \Theta(B) = \Theta(F) \tag{2.8}$$

**Proof.** First, we use the previous proposition and so there exists a real $t$ such that, if we set

$$A = F + \lambda t F_j \otimes b \qquad \text{and} \qquad B = F - (1-\lambda) t F_j \otimes b$$

one has (2.6) and (2.7).
Next, since $b_j = 0$ and $\tilde{\Theta}$ is antisymmetric,

$$\begin{aligned}\Theta(A) &= \Theta(F + \lambda t F_j \otimes b)\\ &= \tilde{\Theta}(F_1 + \lambda t b_1 F_j, \dots, F_{j-1} + \lambda t b_{j-1} F_j, F_j, F_{j+1} + \lambda t b_{j+1} F_j, \dots, F_n + \lambda t b_n F_j)\\ &= \tilde{\Theta}(F_1, \dots, F_n).\end{aligned}$$

By same way, we compute $\Theta(B)$ and (2.8) holds. □

**Remark 2.3.** This last proposition gives, in the case where $q$ is positive definite, some results already obtained in [3] (lemme 3.2 p. 31, lemme 1.2 p. 41) and [4] (theorem 2.1).

## 3. Rank-one-convex envelope of function depending on a quadratic form

In this section, we still denote by $q$ a quadratic form defined on $\mathbb{M}^{m\times n}$ ($q \not\equiv 0$), by $I$ an interval of $\mathbb{R}$ and by $\varphi : I \longrightarrow \mathbb{R}$ a function satisfying

$$\inf_{t\in I} \varphi(t) = \mu > -\infty. \tag{3.1}$$

Thanks to (3.1), there exist $\alpha \in \bar{I}$ and a sequence $t_k \in I$ such that

$$\lim_{k\to+\infty} t_k = \alpha \qquad \text{and} \qquad \lim_{k\to+\infty} \varphi(t_k) = \mu. \tag{3.2}$$

For instance, if $\varphi^{-1}(\{\mu\}) \neq \emptyset$, we can choose $\alpha \in \varphi^{-1}(\{\mu\})$ and $\forall k$, $t_k = \alpha$.
We have the following result:

**Lemma 3.1.** *Let us assume that either $I = \mathbb{R}$ or $I = \mathbb{R}_+$, and consider the function $W$ defined on $\mathbb{M}^{m\times n}$ by*

$$W(F) = \varphi(q(F)).$$

*If there exists a rank-one-matrix $a \otimes b$ such that*

$$q(a \otimes b) > 0 \tag{3.3}$$

*then, for $F \in \mathbb{M}^{m\times n}$, one has*

$$q(F) \leq \alpha \implies RW(F) = QW(F) = PW(F) = CW(F) = \mu. \tag{3.4}$$

**Proof.** Let us consider $F \in \mathbb{M}^{m\times n}$ such that $q(F) < \alpha$. Then, by (3.2), there exists $k_0 \in \mathbb{N}$ such that

$$\forall k \geq k_0 \ , \quad q(F) \leq t_k.$$

So, using proposition 2.1, there exist a rank-one-matrix $E_k$ and $\lambda_k \in [0,1]$ such that

$$q(F + \lambda_k E_k) = q(F - (1-\lambda_k)E_k) = t_k$$

and if we set $A_k = F + \lambda_k E_k$ and $B_k = F - (1-\lambda_k)E_k$ then

$$\begin{aligned} &F = (1-\lambda_k)A_k + \lambda_k B_k \\ &\operatorname{rank}(A_k - B_k) \leq 1 \\ &q(A_k) = q(B_k) = t_k \end{aligned}$$

and thus

$$\begin{aligned} RW(F) &\leq (1-\lambda_k)RW(A_k) + \lambda_k RW(B_k) \\ &\leq (1-\lambda_k)W(A_k) + \lambda_k W(B_k) \\ &= (1-\lambda_k)\varphi(q(A_k)) + \lambda_k \varphi(q(B_k)) \\ &= \varphi(t_k). \end{aligned}$$

Therefore, using (3.2) we obtain

$$q(F) < \alpha \implies RW(F) = \mu.$$

Finally, by continuity of $q$ and $RW$, and thanks to (1.5), (3.4) holds for all the matrices $F$ such that $q(F) \leq \alpha$. □

**Theorem 3.2.** *Let us assume that $I = \mathbb{R}_+$, $q$ is nonnegative, and consider the function $W$ defined on $\mathbb{M}^{m\times n}$ by*

$$W(F) = \varphi(q(F)).$$

*Then, for $F \in \mathbb{M}^{m\times n}$, one has*

$$q(F) \leq \alpha \implies RW(F) = QW(F) = PW(F) = CW(F) = \mu.$$

**Proof.** In order to apply the previous lemma, we are going to prove that the condition (3.3) is always true.
Since $q$ is nonnegative and $q \not\equiv 0$ then, thanks to the Gauss-decomposition theorem, there exists a linear form $l \not\equiv 0$ on $\mathbb{M}^{m\times n}$ such that

$$\forall F \in \mathbb{M}^{m\times n} \ , \quad q(F) \geq (l(F))^2.$$

Next, $l^{-1}(\{0\})$ is a hyperplane of $\mathbb{M}^{m\times n}$, but the vectorial space spanned by the rank-one-matrices is the whole space $\mathbb{M}^{m\times n}$. Therefore, there exists a rank-one-matrix $a \otimes b$ such that $q(a \otimes b) > 0$.
So, we can apply lemma 3.1 and the proof is complete. □

**Theorem 3.3.** *Let us assume that* $I = \mathbb{R}$, $q : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$ *is onto, and consider the function* $W$ *defined on* $\mathbb{M}^{m\times n}$ *by*

$$W(F) = \varphi(q(F)).$$

*If there exist two rank-one-matrices* $a \otimes b$ *and* $c \otimes d$ *such that*

$$q(a \otimes b) > 0 \quad \text{and} \quad q(c \otimes d) < 0 \tag{3.5}$$

*then,* $RW = QW = PW = CW = \mu$.

**Proof.** Let $F \in \mathbb{M}^{m\times n}$.
First, assume that $q(F) \le \alpha$; then, thanks to (3.5) and lemma 3.1, we obtain

$$RW(F) = QW(F) = PW(F) = CW(F) = \mu.$$

Next, assume that $q(F) \ge \alpha$. Let us consider the function $\check{\varphi} : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\check{\varphi}(t) = \varphi(-t)$. Then

$$W(F) = \check{\varphi}(-q(F))$$

$$-\,q(c \otimes d) > 0$$

and

$$\inf_{t\in I} \check{\varphi}(t) = \mu$$

thus, since $-q(F) \le -\alpha$, we can apply lemma 3.1 and obtain

$$RW(F) = QW(F) = PW(F) = CW(F) = \mu.$$

The proof is now complete. □

**Remark 3.4.** For a quadratic form with a range equal to $\mathbb{R}$, it is not always possible to have (3.5); indeed, when $m = n = 2$, the quadratic form $F \longmapsto \det F$ is onto and for every $a, b \in \mathbb{R}^2$ one has $\det(a \otimes b) = 0$.

## 4. Some applications

**Example 4.1.** Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be convex and such that $\inf_{t\in\mathbb{R}} \psi(t) = \psi(0)$.
If $q$ is a nonnegative quadratic form on $\mathbb{M}^{m\times n}$, $\alpha$ a positive real number and $W$ the function defined on $\mathbb{M}^{m\times n}$ by

$$W(F) = \psi(q(F) - \alpha)$$

then, for $F \in \mathbb{M}^{m\times n}$, one has

$$RW(F) = \begin{cases} \psi(0) & \text{if} \quad q(F) \le \alpha \\ W(F) & \text{if} \quad q(F) \ge \alpha \end{cases} \tag{4.1}$$

Indeed, if we set $\varphi(t) = \psi(t - \alpha)$, then

$$\mu = \inf_{t \in \mathbb{R}_+} \varphi(t) = \psi(0) = \varphi(\alpha)$$

and thus, by theorem 3.2, one has

$$q(F) \leq \alpha \implies RW(F) = \mu = \psi(0).$$

Moreover, the function $\overline{W}$ defined by

$$\overline{W}(F) = \begin{cases} \psi(0) & \text{if} \quad q(F) \leq \alpha \\ W(F) & \text{if} \quad q(F) \geq \alpha \end{cases}$$

is convex (since $q$ is convex, $\psi$ is convex and non decreasing on $\mathbb{R}_+$) and $\leq W$; therefore $\overline{W} \leq RW$. So, if $q(F) \geq \alpha$ one has

$$W(F) = \overline{W}(F) \leq RW(F) \leq W(F).$$

Thus (4.1) holds, and since $RW$ is convex, we have

$$RW = QW = PW = CW.$$

**Example 4.2.** Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying

$$\inf_{t \in \mathbb{R}} \varphi(t) = \mu > -\infty.$$

Let us consider the following quadratic form on $\mathbb{M}^{m \times n}$

$$q(F) = \sum_{(i,j) \in \mathcal{I}} f_{ij}^2 - \sum_{(i,j) \in \mathcal{J}} f_{ij}^2$$

where $\mathcal{I}$ and $\mathcal{J}$ are two disjoint nonempty subsets of $\{1, \ldots, m\} \times \{1, \ldots, n\}$.
Clearly, the range of $q$ is $\mathbb{R}$ and the conditions (3.5) occur; so, we can apply theorem 3.3, and, if $W : \mathbb{M}^{m \times n} \longrightarrow \mathbb{R}$ is defined by $W(F) = \varphi(q(F))$, then $RW = QW = PW = CW = \mu$.

**Example 4.3.** Let us consider the quadratic form defined on $\mathbb{M}^{m \times n}$ by

$$q(F) = \sum_{i=1}^{s} |F_i|^2 - \sum_{i=s+1}^{n} |F_i|^2$$

where $1 \leq s \leq n - 1$ and $F_1, \ldots, F_n$ denote the columns of the matrix $F$.
Now, let $\tilde{\Theta} : \mathbb{R}^m \times \ldots \times \mathbb{R}^m \longrightarrow \mathbb{R}^p$ be an antisymmetric $n$-linear function, and denote by $\Theta$ the function defined on $\mathbb{M}^{m \times n}$ by $\Theta(F) = \tilde{\Theta}(F_1, \ldots, F_n)$. Moreover, assume that $\Theta$ is polyaffine (i.e. $\Theta$ and $-\Theta$ are polyconvex); for instance, if $m = n$ we can consider $\Theta(F) = \det F$, and, if $m = n + 1$, $\Theta(F) = \mathrm{adj}_n F$, see [8]. Next, let $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be

such that $\psi(\alpha) = 0$ ($\alpha \in \mathbb{R}_+^*$), $g : \mathbb{R}^p \longrightarrow \mathbb{R}$ be a convex function and $W : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$ defined by

$$W(F) = \psi(q(F)) + g(\Theta(F)).$$

Then,

$$RW = QW = PW = g \circ \Theta. \tag{4.2}$$

To prove (4.2), it is sufficient, since $g \circ \Theta$ is polyconvex, to show that

$$RW = g \circ \Theta. \tag{4.3}$$

First, since $\psi \geq 0$, one has $W \geq g \circ \Theta$ and thus $RW \geq g \circ \Theta$. Next, let $F \in \mathbb{M}^{m\times n}$ be such that $F_1 \neq 0$ and $F_n \neq 0$; thus, if $b = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, 1)$, then

$$q(F_n \otimes b) > 0 \quad \text{and} \quad q(F_1 \otimes c) < 0. \tag{4.4}$$

• Assume that $q(F) \leq \alpha$; by (4.4) and proposition 2.2 there exist $\lambda \in [0, 1]$, $A, B \in \mathbb{M}^{m\times n}$ such that,

$$\begin{aligned} &F = (1-\lambda)A + \lambda B\ , \quad \operatorname{rank}(A - B) \leq 1 \\ &q(A) = q(B) = \alpha \\ &\Theta(A) = \Theta(B) = \Theta(F). \end{aligned}$$

Therefore,

$$\begin{aligned} RW(F) &\leq (1-\lambda)RW(A) + \lambda RW(B) \\ &\leq (1-\lambda)W(A) + \lambda W(B) \\ &= (1-\lambda)g(\Theta(A)) + \lambda g(\Theta(B)) \\ &= g(\Theta(F)). \end{aligned}$$

• Assume that $q(F) \geq \alpha$; then $-q(F) \leq -\alpha$ and, since $-q(F_1 \otimes c) > 0$, we can proceed as above to obtain

$$RW(F) \leq g(\Theta(F)).$$

So, for $F \in \mathbb{M}^{m\times n}$ such that $F_1 \neq 0$ and $F_n \neq 0$ we have $RW(F) = g(\Theta(F))$. Finally, by continuity of $RW$ and $g \circ \Theta$, the equality (4.3) occurs.

**Example 4.4.** Let us consider the function $W$ defined on $\mathbb{M}^{m\times n}$ by

$$W(F) = \varphi(|q(F)|^{\frac{1}{2}})$$

where $q$ is a quadratic form on $\mathbb{M}^{m\times n}$ and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is such that

$$\inf_{t \in \mathbb{R}_+} \varphi(t) = \varphi(0).$$

Then, if $q$ is either nonnegative or nonpositive, $PW > CW$ in general (see [8], theorem 1.3 (iii) p. 217, 218). But, if $q$ is onto and if (3.5) holds, then by theorem 3.3, one has $RW = PW = QW = CW = \varphi(0)$.

## 5. The case of Ericksen-James stored energy function

In this last section, we would like to consider the function $\phi : \mathbb{M}^{2\times 2} \longrightarrow \mathbb{R}$ defined by (1.6).

For $F = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ one has

$$\begin{aligned} \phi(F) &= \kappa_1(f_{11}^2 + f_{21}^2 + f_{12}^2 + f_{22}^2 - 2)^2 + \kappa_2(f_{11}f_{12} + f_{21}f_{22})^2 \\ &\qquad + \kappa_3\left(\left(\frac{f_{11}^2 + f_{21}^2 - f_{12}^2 - f_{22}^2}{2}\right)^2 - \varepsilon^2\right)^2 \\ &= \phi_1(F) + \phi_2(F) + \phi_3(F). \end{aligned}$$

If we set

$$F_1 = \begin{pmatrix} f_{11} \\ f_{21} \end{pmatrix} \quad \text{and} \quad F_2 = \begin{pmatrix} f_{12} \\ f_{22} \end{pmatrix}$$

then

$$\begin{aligned} \phi_1(F) &= \kappa_1(|F_1|^2 + |F_2|^2 - 2)^2 \\ \phi_2(F) &= \kappa_2(F_1.F_2)^2 \\ \phi_3(F) &= \kappa_3\left(\left(\frac{|F_1|^2 - |F_2|^2}{2}\right)^2 - \varepsilon^2\right)^2 \end{aligned}$$

Now, let us denote by $q_1$, $q_2$ and $q_3$ the following quadratic forms

$$\begin{aligned} q_1(F) &= |F_1|^2 + |F_2|^2 \\ q_2(F) &= F_1.F_2 \\ q_3(F) &= |F_1|^2 - |F_2|^2. \end{aligned}$$

Therefore, thanks to theorems 3.2 and 3.3 (see also examples 4.1 and 4.2), it is easy to obtain

$$\forall F \in \mathbb{M}^{2\times 2}, \;\; R\phi_1(F) = \begin{cases} 0 & \text{if} \quad q_1(F) \leq 2 \\ \phi_1(F) & \text{if} \quad q_1(F) \geq 2 \end{cases} \tag{5.1}$$

$$\forall F \in \mathbb{M}^{2\times 2}, \;\; R\phi_2(F) = 0 \tag{5.2}$$

$$\forall F \in \mathbb{M}^{2\times 2}, \;\; R\phi_3(F) = 0 \tag{5.3}$$

**Remark 5.1.** The equality (5.2) can also be obtained by using the Pipkin formula; see [10] and below.
We have the following result:

**Theorem 5.2.** *If $\kappa_1 = 0$, then*

$$R\phi = Q\phi = P\phi = C\phi = 0. \tag{5.4}$$

**Proof.** Let $F \in \mathbb{M}^{2\times 2}$.
First, assume that $q_3(F) \le 2\varepsilon$. Let $a \in \{F_2\}^\perp$, $a \neq 0$ and $b = (1,0)$; then

$$q_3(a \otimes b) = a_1^2 + a_2^2 > 0.$$

So, by proposition 2.1, there exist $t \in \mathbb{R}_+$ and $\lambda \in [0,1]$ such that, if we set

$$A = F + \lambda t a \otimes b \qquad \text{and} \qquad B = F - (1-\lambda) t a \otimes b$$

then

$$F = (1-\lambda)A + \lambda B\ , \quad \operatorname{rank}(A-B) \le 1$$
$$q_3(A) = q_3(B) = 2\varepsilon.$$

Next

$$q_2(A) = A_1.A_2 = (F_1 + \lambda t a).F_2 = F_1.F_2 = q_2(F).$$

The same computation gives $q_2(B) = q_2(F)$.
Therefore, for $F \in \mathbb{M}^{2\times 2}$ such that $q_3(F) \le 2\varepsilon$, one has

$$\begin{aligned} R\phi(F) &\le (1-\lambda)R\phi(A) + \lambda R\phi(B) \\ &\le (1-\lambda)\phi(A) + \lambda\phi(B) \\ &= (1-\lambda)\phi_2(A) + \lambda\phi_2(B) \\ &= \phi_2(F). \end{aligned} \tag{5.5}$$

Next, assume that $q_3(F) \ge 2\varepsilon$. Let $a \in \{F_1\}^\perp$, $a \neq 0$ and $b = (0,1)$; then

$$q_3(a \otimes b) = -a_1^2 - a_2^2 < 0.$$

Applying proposition 2.1 for the quadratic form $-q_3$, we see there exists $t \in \mathbb{R}_+$ and $\lambda \in [0,1]$ such that, if we set

$$A = F + \lambda t a \otimes b \qquad \text{and} \qquad B = F - (1-\lambda) t a \otimes b$$

then

$$F = (1-\lambda)A + \lambda B\ , \quad \operatorname{rank}(A-B) \le 1$$
$$-q_3(A) = -q_3(B) = -2\varepsilon.$$

Now, as before, we can prove that $q_2(A) = q_2(B) = q_2(F)$, and for $F \in \mathbb{M}^{2\times 2}$ such that $q_3(F) \ge 2\varepsilon$, one has

$$R\phi(F) \le \phi_2(F). \tag{5.6}$$

Thus, (5.5) and (5.6) give $R\phi \leq \phi_2$ , which implies $R\phi \leq R\phi_2$. Finally (5.2) gives $R\phi = 0$ and (5.4). □

After having obtained this first result, we were hoping to be able to prove that $R\phi = R\phi_1$; unfortunately this is not true as we will see in the next theorem. Before that, let us recall the Pipkin formula; when a function $W : \mathbb{M}^{m\times n} \longrightarrow \mathbb{R}$ (with $m \geq n$) satisfies

$$\forall F \in \mathbb{M}^{m\times n}\ , \quad W(F) = \tilde{W}(C) \quad \text{where} \quad C = F^T F$$

and, if $\tilde{W}$ is convex, then

$$\forall F \in \mathbb{M}^{m\times n}\ , \quad RW(F) = QW(F) = PW(F) = CW(F) = \inf_{S\in \S_n^+} \tilde{W}(F^T F + S) \quad (5.7)$$

where $\S_n^+$ denote the set of real $n \times n$ symmetric positive semidefinite matrices. See [10] (theorem 2 and comment (i) following this theorem). One has:

**Theorem 5.3.** *If $\kappa_3 = 0$, then $R\phi = Q\phi = P\phi = C\phi$ and for $F \in \mathbb{M}^{2\times 2}$ and $C = F^T F$, one has*

- $R\phi(F) = 0$ *if $tr(C) \leq 2$ and $2|c_{12}| \leq 2 - tr(C)$*
- $R\phi(F) = \kappa_1(tr(C) - 2)^2 + \kappa_2 c_{12}^2$ *if $tr(C) \geq 2$ and $\kappa_2|c_{12}| \leq 2\kappa_1(tr(C) - 2)$*
- $R\phi(F) = \kappa_1(tr(C) - 2)^2 + \kappa_2 c_{12}^2 - \dfrac{(2\kappa_1(tr(C) - 2) - \kappa_2|c_{12}|)^2}{4\kappa_1 + \kappa_2}$

$$\text{if } \begin{cases} tr(C) \geq 2 \text{ and } \kappa_2|c_{12}| \geq 2\kappa_1(tr(C) - 2) \\ \text{or} \\ tr(C) \leq 2 \text{ and } 2|c_{12}| \geq 2 - tr(C) \end{cases}$$

**Proof.** Since $\kappa_3 = 0$, then for $F \in \mathbb{M}^{2\times 2}$ and $C = F^T F$, one has

$$\phi(F) = \tilde{\phi}(C) = \kappa_1(\mathrm{tr}(C) - 2)^2 + \kappa_2 c_{12}^2.$$

Clearly, the function $\tilde{\phi}$ is convex, and using (5.7) we can write $R\phi = Q\phi = P\phi = C\phi$ and

$$\forall F \in \mathbb{M}^{2\times 2}\ , \quad R\phi(F) = \inf_{S\in \S_2^+} \tilde{\phi}(F^T F + S). \quad (5.8)$$

Let us remark that, if $S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$, then

$$S \in \S_2^+ \iff s_{12} = s_{21},\ s_{11} \geq 0,\ s_{22} \geq 0 \ \text{ and } \ s_{12}^2 \leq s_{11}s_{22}. \quad (5.9)$$

Now, let us consider $F \in \mathbb{M}^{2\times 2}$, $C = F^T F$ and set $p = \mathrm{tr}(C) - 2$ and $r = c_{12}$. Thanks to (5.8) and (5.9) we have

$$R\phi(F) = \inf_{(x,y,z)\in D} h(x, y, z)$$

where

$$h(x,y,z) = \tilde{\phi}\left(C + \begin{pmatrix} x^2 & z \\ z & y^2 \end{pmatrix}\right) = \kappa_1(x^2+y^2+p)^2 + \kappa_2(z+r)^2$$

and $D = \{(x,y,z) \in \mathbb{R}^3 \ ; \ z^2 \le x^2y^2\}$.
Since $h(x,y,z) \longrightarrow +\infty$ when $x^2+y^2+z^2 \longrightarrow +\infty$, it follows that $\inf_{(x,y,z)\in D} h(x,y,z)$ is attained by a certain $(x_0,y_0,z_0) \in D$.

• <u>*Case 1*</u> : Let us assume that $p \le 0$ and $|2r| \le -p$; then there exists $(x_0,y_0,z_0) \in D$ such that

$$x_0^2 + y_0^2 = -p \quad \text{and} \quad z_0 = -r$$

and thus

$$h(x_0,y_0,z_0) = 0 = \inf_{(x,y,z)\in D} h(x,y,z).$$

• <u>*Case 2*</u> : Let us assume that either $p > 0$ or $|2r| > -p$; then

$$\forall (x,y,z) \in D \ , \quad (x^2+y^2+p, z+r) \neq (0,0). \tag{5.10}$$

Next,

$$\frac{\partial h}{\partial x}(x,y,z) = 2\kappa_1(x^2+y^2+p)x$$

$$\frac{\partial h}{\partial y}(x,y,z) = 2\kappa_1(x^2+y^2+p)y$$

$$\frac{\partial h}{\partial z}(x,y,z) = 2\kappa_2(z+r)$$

and therefore, thanks to (5.10), it is easy to see that

$$\forall (x,y,z) \in \overset{\circ}{D}, \quad \nabla h(x,y,z) \neq 0$$

which implies

$$\inf_{(x,y,z)\in D} h(x,y,z) = \inf_{(x,y,z)\in \partial D} h(x,y,z) = \inf_{(x,y)\in \mathbb{R}^2} g(x,y)$$

with $g(x,y) = \kappa_1(x^2+y^2+p)^2 + \kappa_2(xy+r)^2$. Now, to obtain this last infimum, let us compute $\nabla g(x,y)$:

$$\frac{\partial g}{\partial x}(x,y) = 2\kappa_1(x^2+y^2+p)x + 2\kappa_2(xy+r)y$$

$$\frac{\partial g}{\partial y}(x,y) = 2\kappa_1(x^2+y^2+p)y + 2\kappa_2(xy+r)x$$

So, if $\nabla g(x_0,y_0) = 0$ then

$$\begin{cases} (x_0^2+y_0^2+p)(x_0^2-y_0^2) = 0 \\ (x_0y_0+r)(x_0^2-y_0^2) = 0 \end{cases}$$

which gives, with (5.10), $x_0^2 = y_0^2$. Therefore

$$\inf_{(x,y)\in\mathbb{R}^2} g(x,y) = \min_{\varepsilon\in\{-1,1\}} \Big( \inf_{x\in\mathbb{R}} l_\varepsilon(x) \Big) \tag{5.11}$$

where

$$\begin{aligned} l_\varepsilon(x) &= \kappa_1(2x^2+p)^2 + \kappa_2(\varepsilon x^2 + r)^2 \\ &= (4\kappa_1+\kappa_2)x^4 + 2(2\kappa_1 p + \varepsilon\kappa_2 r)x^2 + \kappa_1 p^2 + \kappa_2 r^2. \end{aligned}$$

Now, if we look for the infimum of the function $x \longmapsto \alpha x^4 + 2\beta x^2 + \gamma$, we obtain immediatly

$$\inf_{x\in\mathbb{R}} l_\varepsilon(x) = \begin{cases} \kappa_1 p^2 + \kappa_2 r^2 & \text{if} \quad 2\kappa_1 p + \varepsilon\kappa_2 r \geq 0 \\ \kappa_1 p^2 + \kappa_2 r^2 - \dfrac{(2\kappa_1 p + \varepsilon\kappa_2 r)^2}{4\kappa_1+\kappa_2} & \text{if} \quad 2\kappa_1 p + \varepsilon\kappa_2 r \leq 0 \end{cases}$$

and to conclude, it is enough to replace $p$ and $r$ by their values and use (5.11). □

**Remark 5.4.** When $\kappa_3 \neq 0$, the function $\tilde{\phi}$ is not convex and we can not apply the Pipkin formula.

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