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On Some Quasiconvex Functions with Linear Growth

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We establish (i) that the quasiconvexifcation of the distance function to any closed (possibly unbounded) subset of the space of conformal matrices E_∂ in $M^{2\times 2}$ is bounded from below by the distance function itself, that is, $Q\,\mathrm{dist}(\cdot, K) \geq c\,\mathrm{dist}(\cdot, K)$, where $c > 0$ is a constant independent of K; (ii) some estimates of quasiconvexifications of the distance function to a closed subset of $M^{2\times 2}$ which is 'supported' by E_∂; (iii) $Q\,\mathrm{dist}^p(\cdot, K) = Q\,\mathrm{dist}^p(\cdot, Q_p(K))$ for any $p \geq 1$ and any closed $K \subset M^{N\times n}$; (iv) for some nonconvex $K \subset M^{2\times 2}$, $Q\,\mathrm{dist}(\cdot, K)$ is homogeneous of degree one, conjugate invariant and convex, and $Q_1(K) = C(K)$.

1. Introduction

In this note we study some nonconvex, non-negative quasiconvex functions with linear growth at infinity obtained by using quasiconvex relaxations of the distance function to a closed set in $M^{2\times 2}$. The zero sets of these quasiconvex functions can be unbounded. We also give some conditions such that a homogeneous quasicovex function of degree one in $M^{2\times 2}$ is convex in some two dimensional subspaces.

More precisely, we show that for every closed subset K of E_∂ ($E_{\bar\partial}$, respectively)-the space of conformal (anti-conformal, respectively) matrices in $M^{2\times 2}$ - the quasiconvexification of the distance function $\mathrm{dist}(\cdot, K)$ is bounded below by itself, that is,

$$c\,\mathrm{dist}(P, K) \leq Q\,\mathrm{dist}(P, K), \tag{1.1}$$

and the constant $c > 0$ is independent of K. From the definition of quasiconvex relaxation (see Definition 1.1 below), we have

$$Q\,\mathrm{dist}(P, K) \leq \mathrm{dist}(P, K).$$

Therefore, $Q\,\mathrm{dist}(P, K)$ is not convex if K is not convex. If $K \subset E_\partial$ ($E_{\bar\partial}$, respectively) is closed and non-convex, we show that $\mathrm{dist}(\cdot, K)$ is not rank-1 convex in $M^{2\times 2}$, justifying the non-trivialness of (1.1). We also obtain an estimate of the lower bound for $Q\,\mathrm{dist}(\cdot, K)$ for any closed set $K \subset M^{2\times 2}$ which is supported (the precise definition of a supporting space will be given later) by E_∂ ($E_{\bar\partial}$, respectively). In the case where E_∂ is the supporting space of K, we have that

$$c\,\mathrm{dist}(P, K) - C|P_{E_{\bar\partial}}(P)| \leq Q\,\mathrm{dist}(P, K)$$

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for $P \in K$, where $P_{E_{\bar\partial}}$ is the orthogonal projection from $M^{2\times 2}$ to $E_{\bar\partial}$.

Motivated from [7], we also study the behaviour of a nonnegative homogeneous quasicovex function $f : M^{2\times 2} \mapsto \mathbb{R}$of degree 1 under the conjugate invariant condition (see [23]) which is a less restrictive condition than that of [7]. We show in Theorem 2.7 below that f must be convex in certain two dimensional subspaces of $M^{2\times 2}$ while f is not necessarily convex (Remark 2.8). This result seems only valid in $M^{2\times 2}$ because we need a lemma in [7] (see Prop. 1.6 below) which holds only in two dimensional spaces.

We focus on subsets of E_∂ and $E_{\bar\partial}$ in $M^{2\times 2}$ because of the following two reasons.

(1) The weak type (1,1) estimates for the projection $P_{E_{\bar\partial}}(D\phi)$ is classical and is readily available in [18]. Therefore we do not need too much harmonic analysis preparation. In fact, it is possible to establish a more general version of Theorem 2.2 for any subspace E of $M^{N\times n}$ under the assumption that E does not have rank-one matrices [13]. However we need to establish a more general weak type (1,1) estimate for a special class of singular integral operators.

(2) In [23, 25], the connected subsets of $M^{2\times 2}$ were characterized and used to construct nonconvex, nonnegative quasiconvex with p-the growth at infinity. It was proved in [25] that in $M^{2\times 2}$, a closed connected set K does not have rank-one connections if and only if K is a Lipschitz graph of a mapping f from a closed set of E_∂ to $E_{\bar\partial}$ (or from a closed set of $E_{\bar\partial}$ to E_∂ respectively), such that

$$|f(A) - f(B)| < |A - B|, \qquad A \neq B.$$

It was established in [23] that for any $p \in (1, \infty)$, there exists some $c(p) > 0$, if K is such a graph satisfying $|f(A) - f(B)| \le k|A - B|$ and $k^p < c(p)$, then the quasiconvex relaxation $Q\,\mathrm{dist}^p(\cdot, K)$ satisfies

$$\{P \in M^{2\times 2}, Q\,\mathrm{dist}^p(P, K) = 0\} = K.$$

It turns out that $c(p) \to 0$ as $p \to 1_+$. This motivated the study of the limiting case, that is, the graphs are reduced to closed subsets in E_∂ and $E_{\bar\partial}$ respectively.

The existence of nonconvex, nonnegative quasiconvex functions with subquadratic and linear growth were established in [19] and [22] respectively, where the zero sets of the functions are compact. A result of Müller [17] shows that there exists a nontrivial homogeneous quasiconvex function of degree one. Yan [21] proved that the p-quasiconvex hull of the set $\mathbb{R}_+SO(n)$ is larger than itself for $p < n/2$ and $n > 2$ (the p-quasiconvex hull of $K \subset M^{N\times n}$ can be defined by $Q_pK = (Qf)^{-1}(0)$, where Qf is the quasiconvexification (see Definition 1.1 below) of the function f, where $f(P) = \mathrm{dist}^p(P, K)$, $P \in M^{N\times n}$). This indicates that the quasiconvex relaxations of the distance function to an unbounded nonconvex set might be convex. It is known that the n-th quasiconvex hull of $\mathbb{R}_+SO(n)$ remains itself. Recently, Dacorogna [7] showed that if $f : M^{2\times 2} \to \mathbb{R}$is rank-one convex, positively homogeneous of degree one and in addition, f is $SO(2)$ rotationally invariant in the sense that $f(RAS) = f(A)$ for R, $S \in SO(2)$, $A \in M^{2\times 2}$, then f is necessarily convex. Under the less restricted condition that f is conjugating invariant, that is $f(RAR^T) = f(A)$ for $R \in SO(2)$ and $A \in M^{2\times 2}$, in [23] it was established the existence of p-homogeneous, conjugating invariant, quasiconvex functions for any $p > 1$ with their zero sets of the form

$$C_P = \{xE_1 + yE_2 + rP,\ (x, y) \in \mathbb{R}^2\},$$

where E_1, E_2 is a basis of $E_{\bar{\partial}}$, $r = \sqrt{x^2 + y^2}$ and $P \in E_\partial$ is a fixed matrix. More precisely, for any $p > 1$ there exists some $c(p) > 0$ ($\lim_{p\to 1_+} c(p) = 0$), whenever $|P| < c(p)$ then $Q\,\mathrm{dist}^p(\cdot, C_P)$ is p-homogeneous, conjugate invariant with C_P as its zero set.

These results imply that for an unbounded set $K \subset M^{2\times 2}$ the existence of a non-negative quasiconvex function

$$f : M^{2\times 2} \to \mathbb{R}_+, \quad f^{-1}(0) = K, \quad \text{and } 0 \le f(P) \le C|P|^p + C_1$$

does depend on the behaviour of the set K near infinity.

We will show in this note that for any C_P, the 1-quasiconvex hull of C_P equals its convex hull, that is, $Q_1(C_P) = C(C_P)$ and $Q\,\mathrm{dist}(\cdot, C_P) = \mathrm{dist}(\cdot, C(C_P))$, hence $Q\,\mathrm{dist}(\cdot, C_P)$ is convex. As a tool, though it stands on its own right, we establish the following identity for any $p \ge 1$ and any closed set $K \subset M^{N\times n}$:

$$Q\,\mathrm{dist}^p(\cdot, K) = Q\,\mathrm{dist}^p(\cdot, Q_p(K)).$$

Some results on lower semicontinuity of quasiconvex functionals in BV spaces have been established recently [1, 11, 12]. The integrands used in that approach are quasiconvex functions with linear growth at infinity. As far as I know, very few examples are known for such functions besides those with compact zero sets [19, 22]. [17] provided the first example of nonconvex quasiconvex functions of linear growth with unbounded zero sets.

Quasiconvex relaxation of certain distance functions to a given set in the space of matrices is an important subject in the study of martensitic phase transitions and optimal design problems (see [5, 3, 4, 10, 14, 15]). As far as I know, explicit relaxation formulas are hard to obtain and there are only a few known examples [6, 8, 14, 15]. Hence an estimate of the lower bound of the quasiconvex relaxation will provide us useful information on the set itself and on the relaxed function. A result in the same spirit as those in this note was established in [24] for $SO(n)$, that is

$$c(n)\,\mathrm{dist}^2(\cdot, SO(n)) \le Q\,\mathrm{dist}^2(P, SO(n)).$$

In order to state and prove our main results, we need some preparation.

We denote by $M^{N\times n}$ the space of all real $N \times n$ matrices, with $\mathbb{R}^{Nn}$ norm, $\mathrm{meas}(U)$ is the Lebesgue measure of a measurable subset $U \subset \mathbb{R}^n$ and let

$$\mathrm{dist}(Q, K) = \inf_{P\in K} |\, Q - P\,|$$

be the distance function from a point $Q \in M^{N\times n}$ to a set $K \subset M^{N\times n}$. From now on let Ω be a nonempty, open and bounded subset of $\mathbb{R}^n$. We denote by Du the gradient of a (vector-valued) function u and we define the space $C_0^k(\Omega, \mathbb{R}^N)$ in the usual way. If $K \subset M^{N\times n}$, let $C(K)$ be its convex hull. Define the spaces of conformal matrices E_∂ and anti-conformal matrices $E_{\bar{\partial}}$ as

$$E_\partial = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix},\ a, b \in \mathbb{R} \right\}, \qquad E_{\bar{\partial}} = \left\{ \begin{pmatrix} a & b \\ b & -a \end{pmatrix},\ a, b \in \mathbb{R} \right\}.$$

Let $f : M^{N\times n} \to \mathbb{R}$ be a continuous function. The following are some conditions related to weak lower semicontinuity of the integral $\int_\Omega f(Du(x))dx$ (c.f. [2, 16, 6]).

(1) f is rank-one convex if for each matrix $A \in M^{N\times n}$ and each rank-one matrix $B = a \otimes b \in M^{N\times n}$, the function $t \to f(A+tB)$ is convex.
(2) f is quasiconvex at $A \in M^{N\times n}$ on Ω, if for any smooth function $\phi : \Omega \to \mathbb{R}^N$ compactly supported in Ω,

$$\int_\Omega f(A + D\phi(x))dx \geq \int_\Omega f(A)dx$$

holds. f is quasiconvex if it is quasiconvex at every $A \in M^{N\times n}$. The class of quasiconvex functions is independent of the choice of Ω.

It is well-known that quasiconvexity implies rank-one convexity (cf. [2, 16, 6]) while rank-one convexity does not, in general, imply quasiconvexity [20].

To construct quasiconvex functions, we need the following

Definition 1.1 ([6]). Suppose that $f : M^{N\times n} \to \mathbb{R}$ is a continuous function. The quasiconvexification of f is defined by

$$\sup\{g \leq f;\, g \text{ quasiconvex }\}$$

and will be denoted by Qf.

Proposition 1.2 ([6]). *Suppose that $f : M^{N\times n} \to R$ is continuous, then*

$$Qf(P) = \inf_{\phi\in C_0^\infty(\Omega;R^N)} \frac{1}{\text{meas}(\Omega)} \int_\Omega f(P + D\phi(x))\, dx, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. In particular the infimum in (1.2) is independent of the choice of Ω.

Definition 1.3. For a closed subset $K \subset M^{N\times n}$, we define the p-quasiconvex hull $Q_p(K)$ $(1 \leq p < \infty)$ as follows:

$$Q_p(K) = \{P \in M^{N\times n},\, Q\,\text{dist}^p(P, K) = 0\},$$

where $Q\,\text{dist}^p(\cdot, K)$ is the quasiconvexification of $\text{dist}^p(\cdot, K)$.

If K is compact, $Q_p(K)$ is independent of $p \geq 1$ [22]. However, this claim is not necessarily true if K is unbounded (see [21]).

The following result is a special case of a more general theorem (see [9, pages 234, 236]).

Proposition 1.4 (The measurable selection theorem). *Let B be a compact subset of $\mathbb{R}^p$ and g a continuous function of $\overline{\Omega} \times B$. Then, there exists a Lebesgue measurable mapping $\tilde{u} : \Omega \to B$ such that for all $x \in \Omega$:*

$$g(x, \tilde{u}(x)) = \min_{a\in B}\{g(x, a)\}.$$

A direct consequence of Proposition 1.4 is the following:

Proposition 1.5. *Let $B \subset \mathbb{R}^p$ be a compact subset and let $u : \overline{\Omega} \to \mathbb{R}^p$ be a continuous mapping. Then there exists a measurable mapping $\tilde{u} : \Omega \to B$ such that for all $x \in \Omega$*

$$|u(x) - \tilde{u}(x)| = \text{dist}(u(x), B).$$

We need the following result established in [7],

Proposition 1.6. *Let* $g : \mathbb{R}^2 \mapsto \mathbb{R}$ *be such that*

(1) $g(tx, ty) = tg(x, y)$ *for every* $t \geq 0$ *and* $x, y \in \mathbb{R}$;
(2) g *is separately convex (i.e.,* $g(x, \cdot)$ *and* $g(\cdot, y)$ *are convex for fixed* x *and fixed* y, *respectively).*

Then, g *is convex in* $\mathbb{R}^2$.

We conclude our preliminaries by giving a technical condition:

Definition 1.7. A non-empty, closed subset K of $M^{2\times 2}$ is supported by E_∂ ($E_{\bar\partial}$, respectively), if there exists an orthonormal basis of $E_{\bar\partial}$ (E_∂, respectively) $\{e_1, e_2\}$, such that $e_i \cdot P \geq 0$ for all $P \in K$ and $i = 1, 2$, '$\cdot$' being the inner product of 2×2 matrices.

We call E_∂ ($E_{\bar\partial}$, respectively) the supporting space of K.

2. Statement of results

Lemma 2.1. *Suppose that* $K \subset E_\partial$ *(*$E_{\bar\partial}$*, respectively) is closed and non-convex. Then* $\operatorname{dist}(\cdot, K)$ *is not rank-1 convex.*

Theorem 2.2. *Suppose that* $K \subset E_\partial$ *(*$E_{\bar\partial}$*, respectively) is closed (possibly unbounded). Then, there exists a constant* $c > 0$ *independent of* K*, such that*

$$c\operatorname{dist}(P, K) \leq Q\operatorname{dist}(P, K) \leq \operatorname{dist}(P, K), \tag{2.1}$$

for every $P \in M^{2\times 2}$.

If we denote by

$$K_\epsilon = \{P \in M^{2\times 2}, \operatorname{dist}(P, K) \leq \epsilon\},$$

the ϵ-neighbourhood of K, we have the following simple consequence of Theorem 2.2.

Corollary 2.3. *Under the assumption of* Theorem 2.2,

$$Q_1(K_\epsilon) \subset K_{\epsilon/c}, \tag{2.2}$$

for every $\epsilon > 0$*, where* $c > 0$ *is the constant given by* Theorem 2.2.

Theorem 2.4. *Suppose that* $K \subset M^{2\times 2}$ *is closed and is supported by* E_∂ *(*$E_{\bar\partial}$*, respectively). Then, there exists a constant* $c > 0$ *independent of* K*, such that*

$$\begin{aligned} c\operatorname{dist}(P, K) - |P_{E_{\bar\partial}}(P)| &\leq Q\operatorname{dist}(P, K), \\ (c\operatorname{dist}(P, K) - |P_{E_\partial}(P)| &\leq Q\operatorname{dist}(P, K), \quad \textit{respectively}). \end{aligned}$$

In particular,

$$c\operatorname{dist}(P, K) \leq Q\operatorname{dist}(P, K)$$

whenever $P \in E_\partial$ *(*$P \in E_{\bar\partial}$ *respectively), which implies* $(Q_1K) \cap E_\partial = K \cap E_\partial$ *(*$(Q_1K) \cap E_{\bar\partial} = K \cap E_{\bar\partial}$*, respectively).*

Remark 2.5. A special case of [23, Th. 4.1] is that for every closed subset $K \subset E_\partial$ ($E_{\bar\partial}$ respectively), $Q_p(K) = K$ for all $p > 1$. Combining Theorem 2.2 and that result, we see that $Q_p(K) = K$ for all $p \geq 1$. The second statement in Theorem 2.4 implies that the inequality given by Theorem 2.2 holds on the supporting spaces, and the intersection of the 1-quasiconvex hull with the supporting space does not enlarge the original intersection.

The following is a general result relating the p-quasiconvex hull of a closed set in $M^{N\times n}$ and the quasiconvexification of the distance function. It might be a useful tool in the study of quasiconvexification of distance functions. We need this result here for the proof of Theorem 2.7 below.

Theorem 2.6. *Let $K \subset M^{N\times n}$ be non-empty and closed. Then*

$$Q\,\mathrm{dist}^p(\cdot, K) = Q\,\mathrm{dist}^p(\cdot, Q_p(K)),$$

for every $1 \leq p < \infty$.

Theorem 2.7. *Let $f : M^{2\times 2} \to \mathbb{R}$ be a nonnegative, 1-homogeneous, conjugate invariant rank-one convex function. Let $A \in E_\partial$, $B \in E_{\bar\partial}$ be any fixed matrices and $S(A,B) = \mathrm{span}[A,B]$ be the subspace in $M^{2\times 2}$ spanned by A, B. Then the restriction of f on $S(A,B)$, $f|_{S(A,B)}$ is convex.*

Remark 2.8. In [17], the existence of a nonnegative homogeneous quasiconvex function of degree 1 was constructed which vanishes on the union of two one dimensional subspaces of E_∂. From Theorem 2.2, and the fact that every conformal matrix is conjugating invariant in the sense that $RAR^T = A$, for $R \in SO(2)$, $A \in E_\partial$, we see that for every $K \subset E_\partial$, K is conjugate invariant. Therefore functions satisfying the assumptions of Theorem 2.7 are not necessarily convex on E_∂. This result is nearly optimal for the convexity of functions covered by Theorem 2.7.

Corollary 2.9. *Let A, B and* $\mathrm{span}[A,B]$ *be as in* Theorem 2.7. *Suppose that $K \subset$* $\mathrm{span}[A,B]$ *is scaling invariant, that is, $P \in K$ implies $tP \in K$ for all $t \geq 0$. Then $Q_1(K) = C(K)$ and $Q\,\mathrm{dist}(\cdot, K)$ is convex.*

Corollary 2.10. *Let C_P be the cone based on $E_{\bar\partial}$:*

$$C_P = \{xE_1 + yE_2 + rP,\ (x,y) \in \mathbb{R}^2\},$$

where E_1, E_2 is the basis of $E_{\bar\partial}$ defined by

$$E_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$r = \sqrt{x^2 + y^2}$ and $P \in E_\partial$ is a fixed matrix. Then $Q_1(C_P) = C(C_P)$ and $Q\,\mathrm{dist}(\cdot, C_P)$ is homogeneous of degree one, conjugating invariant and convex.

Remark 2.11. If we assume that $|P| < 1$ is sufficiently small, we have, (see [23]) that $Q_2(C_P) = C_P$, hence $Q_2(C_P) \neq Q_1(C_P)$. Corollary 2.10 provides another class of closed sets other than that given by Yan [21] such that the p-quasiconvex hull for an unbounded set may depend on p.

3. Proofs of results

Proof of Lemma 2.1. Since $K \subset E_\partial$ is closed and not convex, we may find A, $B \in K$, such that the line segment $L = \{P = tA + (1-t)B,\ 0 < t < 1\}$ does not intersect K. Since A and B are conformal matrices, there exist $Q \in SO(2)$ and $a > 0$, such that $B - A = aQ$. Let

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we see that $aQJ \in E_{\bar\partial}$ and $aQJ + aQ$ is a rank-1 matrix. If $\operatorname{dist}(\cdot, K)$ was rank-1 convex, we would have

$$\operatorname{dist}(A + \frac{1}{2}[aQ + aQJ], K) \le \frac{1}{2}\operatorname{dist}(A, K) + \frac{1}{2}\operatorname{dist}(A + [aQ + aQJ], K).$$

Notice that $\operatorname{dist}(A, K) = 0$. Since $A + \frac{1}{2}aQ \notin K$, we have

$$\operatorname{dist}(A + \frac{1}{2}[aQ + aQJ], K) > |P_{E_{\bar\partial}}(A + \frac{1}{2}[aQ + aQJ])| = \frac{\sqrt{2}}{2}a,$$

while because $P_{E_\partial}(A + [aQ + aQJ]) = B$, we have

$$\operatorname{dist}(A + [aQ + aQJ], K) = |P_{E_{\bar\partial}}(A + [aQ + aQJ])| = \sqrt{2}a.$$

Combining the above three inequalities, we see that $\frac{\sqrt{2}}{2}a < \frac{\sqrt{2}}{2}a$. This contradiction implies that $\operatorname{dist}(\cdot, K)$ is not rank-one convex. $\square$

Proof of Theorem 2.2. We prove the result for $K \subset E_\partial$ only. The case for $E_{\bar\partial}$ is similar. Notice that the upper bound in (2.1) is trivial because $Qf \le f$ is always true (see (1.2)).

We use the weak type $(1,1)$ estimate for singular integral operators [18] as in [17]. For a fixed $P \in M^{2\times 2}$, we have, from Proposition 1.2, that there exists a sequence (ϕ_j) in $C_0^\infty(D, \mathbb{R}^2)$ such that

$$\lim_{j\to\infty} \int_D \operatorname{dist}(P + D\phi_j(x), K)dx = Q\operatorname{dist}(P, K) := a \ge 0, \tag{3.1}$$

where $D \subset \mathbb{R}^2$ is the unit square.

Let $P_{E_{\bar\partial}}$ be the orthogonal projection from $M^{2\times 2}$ to $E_{\bar\partial}$. Notice that $E_{\bar\partial}$ is the orthogonal complement of E_∂. Now, since $K \subset E_\partial$, we have

$$|P_{E_{\bar\partial}}(A)| \le \operatorname{dist}(A, K)$$

for every $A \in M^{2\times 2}$.

If $Q\operatorname{dist}(P, K) = a > 0$, we have, up to a subsequence,

$$\lim_{j\to\infty} \int_D |P_{E_{\bar\partial}}(P + D\phi_j(x))|dx \le a.$$

Since $|P_{E_{\bar{\delta}}}(Q)|$ is a convex function in Q, we have $|P_{E_{\bar{\delta}}}(P)| \leq a$ so that

$$\int_D |P_{E_{\bar{\delta}}}(D\phi_j(x))|dx \leq 2a + \delta_j, \tag{3.2}$$

where $\delta_j > 0$ and $\delta_j \to 0$ as $j \to \infty$. Notice that (3.2) implies

$$\int_D \left(|\frac{\partial \phi_j^{(1)}(x)}{\partial x_1} - \frac{\partial \phi_j^{(2)}(x)}{\partial x_2}| + |\frac{\partial \phi_j^{(1)}(x)}{\partial x_2} + \frac{\partial \phi_j^{(2)}(x)}{\partial x_1}| \right) \leq 2a + \delta_j.$$

Extending ϕ_j outside D by zero and setting $\psi_j = (\phi_j^{(1)}, -\phi_j^{(2)})$, we have

$$\int_D [|\operatorname{div} \psi_j| + |\operatorname{curl} \psi_j|]dx \leq 2a + \delta_j.$$

From the weak (1,1) type estimates in the singularity operator theory (see [18, Ch.2 and pp. 60]), we have

$$\text{meas}\left(\{x \in \mathbb{R}^2,\ |D\psi_j(x)| > \lambda\}\right) \leq \frac{C}{\lambda} \int_D [|\operatorname{div} \psi_j| + |\operatorname{curl} \psi_j|]dx \leq \frac{(2a + \delta_j)C}{\lambda},$$

for every $\lambda > 0$, where $C > 0$ is a constant depending only on the operators div and curl. Therefore, we have

$$\text{meas}\left(\{x \in \mathbb{R}^2,\ |D\phi_j(x)| > \lambda\}\right) \leq \frac{(2a + \delta_j)C}{\lambda}.$$

Since the distance function $\operatorname{dist}(\cdot, K)$ satisfies

$$|\operatorname{dist}(A, K) - \operatorname{dist}(B, K)| \leq |A - B|$$

for A, $B \in M^{2\times 2}$, we see that

$$\operatorname{dist}(P, K) > \operatorname{dist}(P + D\phi_j(x), K) + \lambda \ \text{ implies } \ |D\phi_j(x)| > \lambda.$$

In other words,

$$D_\lambda := \{x \in \Omega,\ \operatorname{dist}(P, K) > \operatorname{dist}(P + D\phi_j(x), K) + \lambda\} \subset \{x \in \Omega,\ |D\phi_j(x)| > \lambda\},$$

so that $\text{meas}(D_\lambda) \leq \frac{(2a+\delta_j)C}{\lambda}$. Choosing $\lambda = 2Ca + \sqrt{2C}a$, we see that $(2a + \delta_j)C/\lambda < 1$ for sufficiently large j. Hence,

$$\begin{aligned} \int_D \operatorname{dist}(P + D\phi_j(x), K)dx &\geq \int_{D\setminus D_\lambda} \operatorname{dist}(P + D\phi_j(x), K)dx \\ &\geq [\operatorname{dist}(P, K) - \lambda]\left(1 - \frac{(2a + \delta_j)C}{\lambda}\right), \end{aligned}$$

for sufficiently large $j > 0$. Passing to the limit in the above inequality, and noticing that $\lim_{j\to\infty} \int_D \operatorname{dist}(P + D\phi_j(x))dx = a$, we obtain

$$a \geq [\operatorname{dist}(P, K) - \lambda]\left(1 - \frac{2aC}{\lambda}\right), \tag{3.3}$$

which implies $\mathrm{dist}(P,K) \le (2C+1+\sqrt{2C})a$. Letting $c = (2C+1+\sqrt{2C})^{-1}$, we conclude that

$$c\,\mathrm{dist}(P,K) \le a = Q\,\mathrm{dist}(P,K).$$

If $a = 0$, we let $\lambda > 0$ be any fixed number. For sufficiently large $j > 0$, we have $C\delta_j < \lambda$. We then proceed as in the first case to obtain (3.3) with $a = 0$. Hence

$$\mathrm{dist}(P,K) \le \lambda,$$

for every $\lambda > 0$, thus $\mathrm{dist}(P,K) = 0$. The proof is complete. □

Proof of Corollary 2.3. Let $P \in Q_1(K_\epsilon)$. Then $Q\,\mathrm{dist}(P,K_\epsilon) = 0$, and since

$$\mathrm{dist}(P,K) \le \mathrm{dist}(P,K_\epsilon) + \epsilon, \tag{3.4}$$

from Theorem 2.2 and inequality (3.4), we obtain

$$c\,\mathrm{dist}(P,K) \le Q\,\mathrm{dist}(P,K) \le Q\,\mathrm{dist}(P,K_\epsilon) + \epsilon = \epsilon$$

which implies $P \in K_{\epsilon/c}$. □

Proof of Theorem 2.4. Similar to the proof of Theorem 2.2, we prove the theorem only in the case where K is supported by E_∂. The proof for the other case is similar. Let $P \in M^{2\times 2}$ be fixed and let (ϕ_j) be a sequence in $C_0^\infty(D,\mathbb{R}^2)$ such that

$$\lim_{j\to\infty} \int_D \mathrm{dist}(P + D\phi_j, K)dx = Q\,\mathrm{dist}(P,K) = a \ge 0.$$

For each fixed $j > 0$, since $\phi_j \in C_0^\infty(D,\mathbb{R}^2)$, we have for some large $R_j > 0$,

$$\mathrm{dist}(P + D\phi_j(x), K) = \mathrm{dist}(P + D\phi_j(x), K \cap \overline{B(0,R_j)}),$$

$\overline{B(0,R_j)}$ being the closed ball in $M^{2\times 2}$, centred at the origin with radius R_j.

Now we apply Proposition 1.5 to the function $F(x,Q) = |P + D\phi_j(x) - Q|$ for $x \in \bar{D}$ and $Q \in K \cap \overline{B(0,R_j)}$. There exists a measurable mapping $X_j : \Omega \to K \cap \overline{B(0,R_j)}$, such that

$$|P + D\phi_j(x) - X_j(x)| = \mathrm{dist}(P + D\phi_j(x), K \cap \overline{B(0,R_j)}) = \mathrm{dist}(P + D\phi_j(x), K),$$

almost everywhere in Ω. Setting

$$Y_j(x) = P_{E_{\bar\partial}}(X_j(x)),$$

we see from the assumption that E_∂ is the supporting space of K that the components of Y_j do not change signs in Ω. Let

$$\int_D \mathrm{dist}(P + D\phi_j, K)dx = a + \delta_j,$$

where $\delta_j \geq 0$ and $\lim_{j\to\infty} \delta_j = 0$. Since ϕ_j is zero on the boundary of D, we have that

$$\begin{aligned}
a+\delta_j &= \int_D |P + D\phi_j(x) - X_j(x)|dx \\
&\geq \int_D |P_{E_{\bar{\delta}}}(P + D\phi_j(x)) - Y_j(x)|dx \\
&\geq \int_D \left(\sum_{i=1}^{2}[e_i \cdot (P + D\phi_j(x) - Y_j(x))]^2\right)^{1/2} dx \\
&\geq \frac{1}{\sqrt{2}}\int_D \sum_{i=1}^{2} |e_i \cdot (P + D\phi_j(x) - Y_j(x))|dx \\
&\geq \frac{1}{\sqrt{2}}\sum_{i=1}^{2} |\int_D e_i \cdot (P - Y_j(x))dx| \\
&\geq \frac{1}{\sqrt{2}}(\sum_{i=1}^{2} |\int_D e_i \cdot Y_j(x)dx|) - |P_{E_{\bar{\delta}}}(P)| \\
&= \frac{1}{\sqrt{2}}(\sum_{i=1}^{2} \int_D |e_i \cdot Y_j(x)|dx) - |P_{E_{\bar{\delta}}}(P)| \\
&\geq \frac{1}{2}\int_D |Y_j(x)|dx - |P_{E_{\bar{\delta}}}(P)|.
\end{aligned}$$

The last inequality holds because the components of $Y_j(x)$ do not change signs. Hence we have

$$\int_D |Y_j(x)|dx \leq 2(a + \delta_j + |P_{E_{\bar{\delta}}}(P)|).$$

We also have

$$\begin{aligned}
a+\delta_j &\geq \int_D |[P_{E_{\bar{\delta}}}(P + D\phi_j(x)) - Y_j(x)|dx \\
&\geq \int_D |P_{E_{\bar{\delta}}}(D\phi_j(x))|dx - \int_D |Y_j(x)|dx - |P_{E_{\bar{\delta}}}(P)|.
\end{aligned}$$

Combining this inequality and the previous one, we see that

$$\int_D |P_{E_{\bar{\delta}}}(D\phi_j(x))|dx \leq 3(a + \delta_j + |P_{E_{\bar{\delta}}}(P)|).$$

Similar to the argument as in the proof of Theorem 2.2, we have

$$\operatorname{meas}(\{x \in D,\ |D\phi_j(x)| > \lambda\}) \leq \frac{C}{\lambda}\int_D |P_{E_{\bar{\delta}}}(D\phi_j(x))|dx \leq \frac{3C}{\lambda}(a + \delta_j + |P_{E_{\bar{\delta}}}(P)|),$$

for every $\lambda > 0$. If $a > 0$ or $|P_{E_{\bar{\delta}}}(P)| > 0$, we choose

$$\lambda = 6C(a + |P_{E_{\bar{\delta}}}(P)|)$$

and, applying the same method as for Theorem 2.2, we have

$$a + \delta_j \geq (\operatorname{dist}(P, K) - \lambda)\left(1 - \frac{6C}{\lambda}(a + \delta_j + |P_{E_{\bar{\delta}}}(P)|)\right).$$

Passing to the limit $j \to \infty$ we obtain

$$\operatorname{dist}(P, K) \leq 6(C+1)(Q \operatorname{dist}(P, K) + |P_{E_{\bar{\delta}}}(P)|).$$

The proof is finished if we take $c = [6(C+1)]^{-1}$.

If $a = 0$ and $|P_{E_{\bar{\delta}}}(P)| = 0$, we see that $\int_D |Y_j| dx \to 0$. We may choose any fixed number $\lambda > 0$ and, following the proof for the case $a > 0$, we deduce that $\operatorname{dist}(P, K) \leq \lambda$. The conclusion follows by letting $\lambda \to 0$. $\square$

Notice that if $e_i \cdot P \leq 0$, $i = 1, 2$, we may drop the term $|P_{E_{\bar{\delta}}}(P)|$ in the proof of Theorem 2.4 to obtain a better estimate $c \operatorname{dist}(P, K) \leq 6(C+1) Q \operatorname{dist}(P, K)$.

Proof of Theorem 2.6. Let $P \in M^{N\times n}$ be fixed and $P_0 \in Q_p(K)$ be such that

$$\operatorname{dist}(P, Q_p(K)) = |P - P_0|.$$

Since $P_0 \in Q_p(K)$, by definition, $Q \operatorname{dist}^p(P_0, Q_p(K)) = 0$. From Proposition 1.2, there exists a sequence (ϕ_j) in $C_0^\infty(D_n, \mathbb{R}^N)$, such that

$$\lim_{j\to\infty} \int_{D_n} \operatorname{dist}(P_0 + D\phi_j(x), K) dx = 0, \tag{3.5}$$

where D_n is the unit cube in $\mathbb{R}^n$. Similar to the proof of Theorem 2.4, we have, because $D\phi_j$ is bounded for each fixed j that we may apply Proposition 1.6 to find a sequence of measurable mappings $P_j : D_n \to K$, such that for each fixed j, P_j is a bounded mapping, and

$$\operatorname{dist}^p(P_0 + D\phi_j(x), K) = |P_0 + D\phi_j(x) - P_j(x)|^p$$

almost everywhere in D_n. Now, by the definition of quasiconvexification, for any given $\epsilon > 0$, we have

$$\begin{aligned} Q \operatorname{dist}^p(P, K) &\leq \int_{D_n} Q \operatorname{dist}^p(P + D\phi_j(x), K) dx \\ &\leq \int_{D_n} \operatorname{dist}^p(P + D\phi_j(x), K) dx \\ &\leq \int_{D_n} |P + D\phi_j(x) - P_j(x)|^p dx \\ &\leq (1+\epsilon) \int_{D_n} |P - P_0|^p dx + C(\epsilon, p) \int_{D_n} |P_0 + D\phi_j(x) - P_j(x)|^p dx \\ &= (1+\epsilon) \operatorname{dist}^p(P, Q_p(K)) + C(\epsilon, p) \int_{D_n} \operatorname{dist}^p(P_0 + D\phi_j(x), K) dx, \end{aligned}$$

where $C(\epsilon, p) > 0$ is a constant depending only on ϵ and p. Passing to the limit $j \to \infty$ in the above inequality, and taking into account of (3.5), we have

$$Q \operatorname{dist}^p(P, K) \leq (1+\epsilon) \operatorname{dist}^p(P, Q_p(K))$$

for each fixed $\epsilon > 0$. Hence

$$Q \operatorname{dist}^p(P, K) \leq \operatorname{dist}^p(P, Q_p(K)),$$

for every $P \in M^{N\times n}$. From Definition 1.1, we see that

$$Q\operatorname{dist}^p(P,K) \le Q\operatorname{dist}^p(P,Q_p(K)). \tag{3.6}$$

From $K \subset Q_p(K)$, we always have

$$Q\operatorname{dist}^p(P,Q_p(K)) \le \operatorname{dist}^p(P,Q_p(K)) \le \operatorname{dist}^p(P,K),$$

which again, by Definition 1.1, implies

$$Q\operatorname{dist}^p(P,Q_p(K)) \le Q\operatorname{dist}^p(P,K). \tag{3.7}$$

Combining (3.6) and (3.7), the conclusion follows. □

Proof of Theorem 2.7. We may assume that $|A|=1$, $|B|=1$. Then we may find Q_0, $Q \in SO(2)$, such that $A = Q_0E$, $B = QQ_0E_1Q^T = Q_0Q^2E_1$, where $E = \frac{1}{\sqrt{2}}I$, I being the identity matrix, and E_1 is defined in Corollary 2.10. We seek to prove that $f(xQ_0E + yQQ_0E_1Q^T)$ is convex in (x,y). Since f is conjugating invariant and $RQ_0ER^T = Q_0E$ for all $R \in SO(2)$, we only need to prove that $f(Q_0[xE+yE_1])$ is convex. Since

$$xE + yE_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix},$$

we let $x+y=u$, $x-y=v$ and define

$$g(u,v) = f\left(\frac{1}{\sqrt{2}}Q_0\begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix}\right).$$

Since f is homogeneous of degree 1, so is g. f is rank-1 convex, hence g is separately convex. Apply Proposition 1.6, we see that g is convex in (u,v), so is $f(Q_0[xE+yE_1])$ in (x,y). The proof is finished. □

Proof of Corollary 2.9. Since K is scaling invariant, we see that $\operatorname{dist}(\cdot,K)$ and $Q\operatorname{dist}(\cdot,K)$ are both homogeneous of degree 1. Because $Q\operatorname{dist}(\cdot,K)$ is also rank-1 convex and $Q\operatorname{dist}(\cdot,K) \ge \operatorname{dist}(\cdot,C(K))$, Theorem 2.7 implies that $Q\operatorname{dist}(\cdot,K)$ is a convex function on $S(A,B)$. Therefore, $Q_1(K) = C(K)$. Finally, since $\operatorname{dist}(\cdot,C(K))$ is a convex function, hence is quasiconvex. We then have, from Theorem 2.6 that

$$Q\operatorname{dist}(\cdot,K) = Q\operatorname{dist}(\cdot,Q_1(K)) = Q\operatorname{dist}(\cdot,C(K)) = \operatorname{dist}(\cdot,C(K)).$$

Thus, $Q\operatorname{dist}(\cdot,K)$ is convex. □

Proof of Corollary 2.10. Using a similar method as in the proof of Corollary 2.9, we see that $Q_1(C_P \cap \operatorname{span}[A,P]) = C(C_P \cap \operatorname{span}[A,P])$ for every $A \in E_{\bar{\delta}}$. Since

$$Q\operatorname{dist}(\cdot, C_P\cap\operatorname{span}[A,P]) \ge Q\operatorname{dist}(\cdot,C_P),$$

we have $C(C_P\cap\operatorname{span}[A,P]) \subset Q_1(C_P)$. Since we also have

$$C(C_P) = \cup_{A\in E_{\bar{\delta}}} C(C_P\cap\operatorname{span}[A,P]),$$

we see that $Q_1(C_P) = C(C_P)$. A similar argument as in the proof of Corollary 2.9 gives

$$Q\operatorname{dist}(\cdot,C_P) = \operatorname{dist}(\cdot,C(C_P)).$$

The proof is complete. □

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