Journal of Convex Analysis
Volume 9 (2002), No. 2, 415–428

Critical Point Theory for Vector Valued Functions

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Received December 12, 2000
Revised manuscript received September 18, 2001

We consider a continuous function defined on a metric space with values in a Banach space endowed with an order cone. In this setting, we provide an extension of min − max techniques, such as the Mountain pass theorem and Ljusternik-Schnirelman theory, without assuming the order cone to have nonempty interior.

Keywords: Vector optimization, nonsmooth critical point theory

1991 Mathematics Subject Classification: 49J40, 58E05

1. Introduction

It is well known that the Calculus of variations started with the problem of recognizing the solutions of some meaningful problems to be extrema of suitable real valued functionals. An important development, since the seminal papers by Ljusternik, Schnirelman and Morse, has been Critical point theory, where one tackles the more general problem of finding critical points, not necessarily extrema, of a given real valued functional (see e.g. [4, 12, 13, 24] for systematic expositions of the theory).

A further development, motivated mainly by applications to Economics, has concerned the case in which one considers functionals with values in a normed space endowed with an order cone. This has led to the notions of Pareto extremum and Pareto equilibrium, which correspond to those of extremum and critical point, respectively. While much attention has been devoted to Pareto extrema (see e.g. [10]), the study of Pareto equilibria is mainly connected with a series of papers by Smale [16, 17, 18, 19, 20, 21, 22, 23], who proposed

*The research of the first two authors was partially supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (40% – 1999) and by Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni.
†The research of the third author was partially supported by the Ministry of Education, Science and Technology of Bulgaria under contract MM-506/95.

ISSN 0944-6532 / $ 2.50

an extension of Morse theory to vector valued functionals, with possible applications to Economics. Then Malivert [11] provided an extension of Ljusternik-Schnirelman theory to vector valued functionals. He also considered the wider, and important in Optimization, class of locally Lipschitz functionals, thus following the corresponding extension made by Chang [3] for real valued functionals.

A common feature of all these studies on Pareto equilibria is the fact that the order cone is supposed to have nonempty interior. In this way one can consider the target space $\mathbb{R}^n$ with the standard cone $P = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for any } i\}$ and also the Lebesgue space L^p with $P = \{f \in L^p : f(x) \geq 0 \text{ a.e.}\}$, provided that $p = \infty$. On the other hand, the important case of L^p with $1 \leq p < \infty$, in particular L^2, is excluded.

The main purpose of this paper is to propose an extension of min – max techniques, such as the Mountain pass theorem [1] and Ljusternik-Schnirelman theory, to vector valued functionals, without assuming the order cone to have nonempty interior. Our approach also falls within the subject of nonsmooth analysis, as we consider continuous functionals defined on a metric space. With regard to this aspect, it also represents a development of [6, 7, 8, 9] to the vector valued case.

The paper is organized as follows: Section 2 contains the preliminaries, the main definitions and the first examples. Section 3 is devoted to the proof of a Quantitative deformation lemma, in the line of [5, 14, 15] for the scalar case. In Section 4 we prove a version of the Mountain pass theorem and, finally, Section 5 provides a result on the Ljusternik-Schnirelman category.

2. Preliminaries

Let (X, d) be a metric space, and, for $x \in X$ and $\delta > 0$, denote by $\mathrm{B}_\delta(x)$ the open ball centered at x, with radius δ. Let Y be a Banach space and $P \subset Y$ a closed convex cone, which will be called the *cone of the positive elements* of Y. A subset F of Y is said to be *(negatively) invariant* if $F = F - P$. Let also $P_0 \subset P$ be a closed convex set not containing zero.

Two typical examples are:

$$Y = \mathbb{R}^n\,, \quad P = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for any } i\}\,, \quad P_0 = \{x \in \mathbb{R}^n : x_i \geq 1 \text{ for any } i\}\,,$$
$$Y = L^p\,, \quad P = \{f \in L^p : f(x) \geq 0 \text{ a.e.}\}\,, \quad P_0 = \{f \in L^p : f(x) \geq 1 \text{ a.e.}\}\,.$$

Observe that, if $\tau, t \in \mathbb{R}$ and $\tau \leq t$, then $tP_0 + P \subset \tau P_0 + P$. In fact, for every $y_0 \in P_0$ and $y \in P$, we have

$$ty_0 + y = \tau y_0 + ((t - \tau)y_0 + y)$$

and the assertion follows, as P is a convex cone containing P_0.

Now we consider a continuous function $f : X \to Y$.

Definition 2.1. For every $x \in X$, we denote by $|d_{P_0} f|(x)$ the supremum of the $\sigma \in [0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : \mathrm{B}_\delta(x) \times [0, \delta] \to X$ such that

(*i*) $d(\mathcal{H}(\xi, t), \xi) \leq t$,

(*ii*) $f(\mathcal{H}(\xi, t)) \in f(\xi) - \sigma t P_0 - P$, whenever $\xi \in \mathrm{B}_\delta(x)$ and $t \in [0, \delta]$. The extended real number $|d_{P_0} f|(x)$ is called the *weak slope* of f at x, with respect to the set P_0.

A *critical point* (w.r.t. P_0) for f is a point x such that $|d_{P_0} f|(x) = 0$.

It is readily seen that the function $\{x \mapsto |d_{P_0} f|(x)\}$ is lower semicontinuous. In the scalar case $Y = \mathbb{R}$, we simply denote by $|df|(x)$ the weak slope obtained with the standard choice $P = [0, +\infty[$ and $P_0 = [1, +\infty[$. The reader interested in familiarizing with this notion, in the scalar case, can consult [2, 7].

Remark 2.2. If $P_0 + P \subset P_0$, then (ii) in the definition above is equivalent to:

(ii') $f(\mathcal{H}(\xi, t)) \in f(\xi) - \sigma t P_0$.

Remark 2.3. If $P_0 \supset P_1$, then for every $x \in X$ one has $|d_{P_0} f|(x) \geq |d_{P_1} f|(x)$. The set of the critical points is antitone with respect to the set P_0.

The following example highlights the role of the set P_0 in the definition of weak slope.

Example 2.4. Let $X = [1, +\infty[$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, $P_0 = \{(x, y) \in \mathbb{R}^2 : xy = 1, \, x > 0\} + P$ and $f(x) = (x^3, -\frac{1}{x})$. Then $|d_{P_0} f|(x) \geq 1$ for x large enough. On the other hand, if we take $P_0 = (1, 1) + P$, then $|d_{P_0} f|(x) \to 0$ as $x \to +\infty$.

If $Y = \mathbb{R}^n$, $P = \mathbb{R}^n_+$ and $P_0 = (1, \ldots, 1) + P$, we have that

$$|d_{P_0} f|(x) \leq \min_{1 \leq i \leq n} |df_i|(x).$$

In particular, if X is an open subset of a normed space and f is of class C^1, we have that

$$|d_{P_0} f|(x) \leq \min_{1 \leq i \leq n} |f_i'(x)|.$$

Remark 2.5. Suppose that $\mathrm{int}\, P \neq \emptyset$ and take $e \in \mathrm{int}\, P$. Then the critical points of a function f are the same with the choice of $P_0 = \{e\}$, $P_0 = e + P$ (obvious by definition) and $P_0 = v + P$ for any $v \in \mathrm{int}\, P$. For, it can be shown that, if $v \in \mathrm{int}\, P$ and $\varepsilon > 0$ is such that $v - \varepsilon e \in P$, then $\left|d_{\{v\}} f\right|(x) \geq \varepsilon \left|d_{\{e\}} f\right|(x)$.

Definition 2.6. Given two subsets V, S of X, an f-*deformation* of the set S into the set V is a continuous map $\eta : X \times [0, 1] \to X$ such that:

(i) $\eta(x, 0) = x$, for any $x \in X$;
(ii) $f(\eta(x, t)) \in f(x) - P$, for any $x \in X$;
(iii) $\eta(x, 1) \in V$, for any $x \in S$.

In the scalar case the minimax methods connect the behavior of the weak slope to the possibility of f–deforming a set into another set. The next two examples want to emphasize some of the problems arising in the vector case.

Example 2.7. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, $P_0 = (1, 1) + P$ and $f(x) = (x, x^2)$. Then $|d_{P_0} f|(x) = 0$ if (and only if) $x \leq 0$. Take $u < w < z < v \leq 0$ and set $F = (v, u^2) - P$ and $A = (z, w^2) - \mathrm{int}\, P$. The set $f^{-1}(F) = [u, v]$ cannot be f-deformed into the the set $f^{-1}(A) =]w, z[$. In fact, to move the point v toward z the second component of f must grow, and to move the point u toward w the first component of f must grow. Observe that the set $\{x : x \leq 0\}$ is the set of the Pareto minima of f.

Example 2.8. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$, $P_0 = (1,1) + P$ and $f(x) = (x, 0)$. Then every $x \in X$ is a critical point. So in this case no Deformation Lemma (like) result could be applied. On the other hand we can change the set P_0 and take $P_0 = (1,0) + P$. In this case $|d_{P_0} f|(x) = 1$ for any $x \in \mathbb{R}$. However, it is not true that we can make an f-deformation of $f^{-1}(F)$ into $f^{-1}(A)$, for every closed set F and open set A contained in Y. Consider for instance $F = (1,1) - P$ and $A = (-1,-1) - \operatorname{int} P$. This shows that, in order to be able to make an f-deformation of $f^{-1}(F)$ into $f^{-1}(A)$, even in absence of critical points we must impose some extra condition on the sets F and A. The following definition provides such a condition.

Definition 2.9. Given two subsets $F \subset Y$ and $A \subset Y$, F is said to be *reachable* with respect to P_0 from A at time $t \geq 0$ (or just reachable, if such a $t \geq 0$ exists) if

$$F - tP_0 \subset A\,,$$

or also, equivalently,

$$F \subset \bigcap_{y \in P_0} (A + ty)\,.$$

When there is no danger of confusion, we shall avoid to refer to the set P_0, when speaking about reachability.

Coming back to the previous examples, we see that in Example 2.7 it is not possible to f–deform the set $f^{-1}(F)$ into $f^{-1}(A)$, due to the presence of critical points, even if F is reachable from A; in Example 2.8 f does not have critical points, but it is not possible to f–deform the set $f^{-1}(F)$ into $f^{-1}(A)$, due to the fact that F is not reachable from A.

For every $t \geq 0$, let us denote by

$$A^{+t} := \bigcap_{y \in P_0} (A + ty)$$

the set reachable from A at time t. Observe that, whenever $t, \tau \geq 0$, we have

$$(A^{+t})^{+\tau} = A^{+(t+\tau)}\,.$$

Indeed, "$\subset$" is obvious. To prove "$\supset$", it is sufficient to treat the case in which $t, \tau > 0$. Let us fix an arbitrary point x in $A^{+(t+\tau)}$ and arbitrary elements y and z of P_0. We want to show that $x \in A + tz + \tau y$. It is the case because of the convexity of P_0:

$$x \in A^{+(t+\tau)} \subset A + (t+\tau)\left(\frac{t}{t+\tau} z + \frac{\tau}{t+\tau} y\right) = A + tz + \tau y\,.$$

Remark 2.10. Suppose $P_0 = p_0 + P$, for some $p_0 \in P \setminus \{0\}$. Then, if A is an invariant set, we have

$$A^{+t} = A + tp_0\,.$$

In this case, if A is an open set, then also A^{+t} is open for every $t \geq 0$. More generally, if $p_0 \in P$ is such that $P_0 \subset p_0 + P$, we have:

$$A^{+t} \supset A + tp_0 \,.$$

Thus, if there is $t \geq 0$ such that $F \subset A + tp_0$, then F is reachable from A. In particular, with the choice of $p_0 = 0$, we see that $A \subset A^{+t}$ for any $t \geq 0$ and that each set F contained in A is reachable from A.

Remark 2.11. If the cone P has an interior point e, and if $P_0 = e + P$, then every upper bounded set F is reachable from every A, invariant subset of Y. For, if $F \subset b - P$, fix $x \in A$ and take $t > 0$ so large that $e + \frac{x-b}{t} \in P$. Then, for some $p \in P$,

$$b = x - tp + te \,,$$

so that $b \in A + te$, and

$$F \subset b - P \subset A + te = A^{+t} \,.$$

3. Quantitative Deformation Lemma

In this section we are given again a continuous function $f : X \to Y$. Let us point out that, in the next result, the completeness of X is not assumed.

Theorem 3.1. *Let $\sigma : X \to [0, +\infty[$ be a continuous function such that*

$$\forall x \in X : \quad |d_{P_0} f|\,(x) \neq 0 \implies |d_{P_0} f|\,(x) > \sigma(x) \,.$$

Then there exist a lower semicontinuous function $T : X \to [0, +\infty]$ and a continuous map $\eta : \Lambda \to X$, where

$$\Lambda = \{(x, t) \in X \times [0, +\infty[:\ t < T(x)\} \,,$$

such that, for every $(x, t) \in \Lambda$, one has

$$\begin{aligned} &d(\eta(x,t), x) \leq t \,, \\ &f(\eta(x,t)) \in f(x) - \left(\inf_{0 \leq s \leq t} \sigma(\eta(x,s))\right) tP_0 - P \,, \\ &|d_{P_0} f|\,(x) \neq 0 \implies T(x) > 0 \,. \end{aligned}$$

Moreover, one also has that:

(a) if $0 < T(x) < +\infty$, then $\eta(x, \cdot)$ is a Cauchy function as $t \to T(x)$;

(b) if $0 < T(x) < +\infty$ and there exists $\overline{x} := \lim\limits_{t \to T(x)} \eta(x,t)$, then $|d_{P_0} f|\,(\overline{x}) = 0$.

Proof. First of all, there exist two continuous maps $\eta_1 : X \times [0, +\infty[\to X$ and $\tau_1 : X \to [0, +\infty[$ such that, for every $x \in X$ and $t \in [0, +\infty[$, we have

$$\begin{aligned} &d(\eta_1(x,t), u) \leq t \,, \\ &f(\eta_1(x,t)) \in f(x) - P \,, \\ &t \leq \tau_1(x) \implies f(\eta_1(x,t)) \in f(x) - \sigma(x) t P_0 - P \,, \\ &|d_{P_0} f|\,(x) \neq 0 \implies \tau_1(x) > 0 \,. \end{aligned}$$

This can be proved by adapting step-by-step the proof of [6, Theorem 2.8] (see also [2, Theorem 1.1.10]).

For $h \geq 2$, we define recursively two continuous maps $\eta_h : X \times [0, +\infty[\to X$ and $\tau_h : X \to [0, +\infty[$ as

$$\eta_h(x,t) = \begin{cases} \eta_{h-1}(x,t) & \text{if } 0 \leq t \leq \tau_{h-1}(x)\,, \\ \eta_1(\eta_{h-1}(x,\tau_{h-1}(x)), t - \tau_{h-1}(x)) & \text{if } t \geq \tau_{h-1}(x)\,, \end{cases}$$

$$\tau_h(x) = \tau_{h-1}(x) + \tau_1(\eta_{h-1}(x, \tau_{h-1}(x)))\,.$$

Then we define $T : X \to [0, +\infty]$ by

$$\forall x \in X : \quad T(x) = \lim_h \tau_h(x)\,.$$

Being an increasing limit of continuous functions, T is lower semicontinuous. Moreover, $T(x) > 0$ whenever $|d_{P_0} f|\,(x) \neq 0$. If

$$(x,t) \in \Lambda := \{(x,t) \in X \times [0, +\infty[:\ t < T(x)\}\ ,$$

we have $t < \tau_h(x)$ for some $h \geq 1$. Then there exists a neighborhood W of (x,t) such that $s < \tau_h(\xi)$ for any $(\xi, s) \in W$. It follows

$$\forall (\xi, s) \in W, \forall k \geq h : \quad \eta_k(\xi, s) = \eta_h(\xi, s)\,.$$

Therefore, we can define a continuous map $\eta : \Lambda \to X$ by

$$\forall (x,t) \in \Lambda : \quad \eta(x,t) = \lim_h \eta_h(x,t)\,.$$

If $0 \leq t \leq \tau_1(x)$, we have

$$d(\eta(x,t), x) = d(\eta_1(x,t), x) \leq t\,,$$

$$f(\eta(x,t)) = f(\eta_1(x,t)) \in f(x) - \sigma(x) t P_0 - P \subset f(x) - \left(\inf_{0 \leq s \leq t} \sigma(\eta(x,s))\right) t P_0 - P\,.$$

Now, assume that these properties are true whenever $\tau_{h-2}(x) \leq t \leq \tau_{h-1}(x)$. If $\tau_{h-1}(x) \leq t \leq \tau_h(x)$, we have

$$\begin{aligned} d(\eta(x,t), x) &= d(\eta_1(\eta_{h-1}(x, \tau_{h-1}(x)), t - \tau_{h-1}(x)), x) \leq \\ &\leq d(\eta_1(\eta_{h-1}(x, \tau_{h-1}(x)), t - \tau_{h-1}(x)), \eta_{h-1}(x, \tau_{h-1}(x))) + \\ &\quad + d(\eta_{h-1}(x, \tau_{h-1}(x)), x) \leq \\ &\leq t - \tau_{h-1}(x) + \tau_{h-1}(x) = t\,, \end{aligned}$$

$$\begin{aligned} f(\eta(x,t)) &= f(\eta_1(\eta_{h-1}(x, \tau_{h-1}(x)), t - \tau_{h-1}(x)) \in \\ &\in f(\eta_{h-1}(x, \tau_{h-1}(x))) - \sigma(\eta_{h-1}(x, \tau_{h-1}(x)))(t - \tau_{h-1}(x)) P_0 - P \subset \\ &\subset f(x) - \left(\inf_{0 \leq s \leq \tau_{h-1}(x)} \sigma(\eta(x,s))\right) \tau_{h-1}(x) P_0 - P + \\ &\quad - \sigma(\eta_{h-1}(x, \tau_{h-1}(x)))(t - \tau_{h-1}(x)) P_0 - P \subset \\ &\subset f(x) - \left(\inf_{0 \leq s \leq t} \sigma(\eta(x,s))\right) t P_0 - P\,. \end{aligned}$$

Therefore, for every $t \in [0, T(x)[$, we have

$$\begin{aligned} & d(\eta(x,t),x) \le t\,, \\ & f(\eta(x,t)) \in f(x) - \left(\inf_{0\le s\le t} \sigma(\eta(x,s))\right) tP_0 - P\,. \end{aligned}$$

In a similar way, one can also show that

$$\forall h \ge 1,\, \forall t \in [\tau_h(x), T(x)[: \quad d(\eta(x,t), \eta(x,\tau_h(x))) \le t - \tau_h(x)\,. \tag{1}$$

Now assume that $0 < T(x) < +\infty$. From (1) it follows that

$$\forall h \ge 1,\, \forall t_1, t_2 \in [\tau_h(x), T(x)[: \quad d(\eta(x,t_2), \eta(x,t_1)) \le 2(T(x) - \tau_h(x))\,.$$

This implies that $\eta(x,\cdot)$ is a Cauchy function as $t \to T(x)$.

Finally, let $0 < T(x) < +\infty$ and assume there exists $\overline{x} := \lim\limits_{t\to T(x)} \eta(x,t)$.

Since

$$\tau_1(\eta(x,\tau_h(x))) = \tau_1(\eta_h(x,\tau_h(x))) = \tau_{h+1}(x) - \tau_h(x)\,,$$

we have $\tau_1(\overline{x}) = 0$. It follows $|d_{P_0} f|\,(\overline{x}) = 0$. □

Theorem 3.2. (Quantitative deformation lemma) *Assume that X is complete. Let S be a closed subset of X and U be an open neighborhood of S. Let $\sigma > 0$ be such that $|d_{P_0} f|\,(x) > \sigma$ for every $x \in S$.*

Then there exists a continuous map $\eta : X \times [0, +\infty[\to X$ such that, for every $x \in X$ and $t \ge 0$, one has:

(i) $d(\eta(x,t),x) \le t$;
(ii) $x \notin U \implies \eta(x,t) = x$;
(iii) $f(\eta(x,t)) \in f(x) - \sigma d(\eta(x,t),x)P_0 - P$;
(iv) $\eta(x,[0,t]) \subset S \implies f(\eta(x,t)) \in f(x) - \sigma t P_0 - P$.

Proof. Since $|d_{P_0} f|$ is lower semicontinuous, we may assume, without loss of generality, that $|d_{P_0} f|\,(x) > \sigma$ for each $x \in \mathrm{cl}\, U$. The function $\hat{\sigma} : X \to \mathbb{R}$ defined as

$$\hat{\sigma}(x) = \begin{cases} \sigma & \text{if } x \in \mathrm{cl}\, U \\ 0 & \text{if } x \in X \setminus \mathrm{cl}\, U \end{cases}$$

is upper semicontinuous and satisfies $\hat{\sigma}(x) \le |d_{P_0} f|\,(x)$ for each $x \in X$. Therefore, as an easy consequence of Michael's selection theorem, there exists a continuous function $\sigma : X \to \mathbb{R}$ satisfying, for each $x \in X$,

$$\begin{aligned} & \hat{\sigma}(x) \le \sigma(x) \le |d_{P_0} f|\,(x)\,, \\ & \hat{\sigma}(x) < |d_{P_0} f|\,(x) \implies \hat{\sigma}(x) < \sigma(x) < |d_{P_0} f|\,(x)\,. \end{aligned}$$

In particular, we have

$$\begin{aligned} & x \in \mathrm{cl}\, U \implies \sigma < \sigma(x) < |d_{P_0} f|\,(x)\,, \\ & |d_{P_0} f|\,(x) \ne 0 \implies \sigma(x) < |d_{P_0} f|\,(x)\,. \end{aligned}$$

Let T, Λ and $\hat{\eta}$ be as in Theorem 3.1. If we define $\lambda, \mu : X \to [0, +\infty]$ as

$$\lambda(x) = \begin{cases} \sup\{t \in [0, T(x)[: \hat{\eta}(x, [0, t]) \in S\} & \text{if } x \in S\,, \\ 0 & \text{if } x \in X \setminus S\,, \end{cases}$$

$$\mu(x) = \begin{cases} \sup\{t \in [0, T(x)[: \hat{\eta}(x, [0, t]) \in U\} & \text{if } x \in U\,, \\ 0 & \text{if } x \in X \setminus U\,, \end{cases}$$

it is clear that μ is lower semicontinuous and that $\lambda(x) \leq \mu(x)$ for each $x \in X$.

Observe also that, for every $x \in U$, if $T(x) < +\infty$, then $\lambda(x) < \mu(x) < T(x)$. In fact, since X is complete, there exists $\overline{x} = \lim\limits_{t \to T(x)} \hat{\eta}(x, t)$ and $|d_{P_0} f|\,(\overline{x}) = 0$. It follows that $\lambda(x) < \mu(x) < T(x)$, as $\mathrm{cl}\, U$ is closed and $\overline{x} \notin \mathrm{cl}\, U$. In particular, it is easy to verify that λ is upper semicontinuous.

Let $\tau : X \to [0, +\infty]$ be a continuous function satisfying, for every $x \in X$,

$$\begin{aligned} &\lambda(x) \leq \tau(x) \leq \mu(x)\,, \\ &\lambda(x) < \mu(x) \implies \lambda(x) < \tau(x) < \mu(x)\,. \end{aligned}$$

In particular, we have

$$\begin{aligned} &\forall x \in U : \quad \lambda(x) \leq \tau(x) \leq \mu(x) \leq T(x)\,, \\ &\forall x \in U : \quad T(x) < +\infty \implies \lambda(x) < \tau(x) < \mu(x) < T(x)\,, \\ &\forall x \in X \setminus U : \quad \tau(x) = 0\,. \end{aligned}$$

Define a continuous map $\eta : X \times [0, +\infty[\to X$ as

$$\eta(x, t) = \begin{cases} \hat{\eta}(x, \min\{t, \tau(x)\}) & \text{if } T(x) > 0\,, \\ x & \text{if } T(x) = 0\,. \end{cases}$$

Then assertions (i) and (ii) are easily verified. If $x \in S$, we have either $\lambda(x) < \tau(x)$ or $\lambda(x) = \tau(x) = +\infty$. In any case, if $\eta(x, [0, t]) \subset S$, it follows $t < \tau(x)$, whence

$$f(\eta(x, t)) = f(\hat{\eta}(x, t)) \in f(x) - \left(\inf_{0 \leq s \leq t} \sigma(\hat{\eta}(x, s))\right) t P_0 - P \subset f(x) - \sigma t P_0 - P\,.$$

Finally, if $x \in U$ and $t \leq \tau(x)$, we have $\hat{\eta}(x, [0, t]) \subset \mathrm{cl}\, U$, whence

$$\begin{aligned} f(\eta(x, t)) &= f(\hat{\eta}(x, t)) \in f(x) - \left(\inf_{0 \leq s \leq t} \sigma(\hat{\eta}(x, s))\right) t P_0 - P \\ &\subset f(x) - \sigma t P_0 - P \subset f(x) - \sigma d(\eta(x, t), x) P_0 - P\,. \end{aligned}$$

If $x \in U$ and $t > \tau(x)$, it follows

$$f(\eta(x, t)) = f(\eta(x, \tau(x))) \in f(x) - \sigma d(\eta(x, \tau(x)), x) P_0 - P = f(x) - \sigma d(\eta(x, t), x) P_0 - P\,.$$

If $x \notin U$, assertion (iii) is trivial. $\square$

Corollary 3.3. *Assume that X is complete. Let $F \subset Y$ be closed and invariant, and let $A \subset Y$ be open and invariant. Let F be T-reachable from A. Moreover, let $|d_{P_0} f|(x) > \sigma$ for every $x \in f^{-1}(F \setminus A)$.*

Then, for each closed subset C of $f^{-1}(A)$, there exists a continuous map η with properties (i) and (iii) of the Quantitative deformation lemma and satisfying also:

(v) $\eta(x, T/\sigma) \in f^{-1}(A)$, *for each* $x \in f^{-1}(F)$;
(vi) $\eta(x, t) = x$, *for each* $x \in C$ *and* $t \geq 0$.

Proof. Let $S = f^{-1}(F \setminus A)$ and let U be an open neighborhood of S such that $U \cap C = \emptyset$. Then the Quantitative deformation lemma provides us with a continuous map η satisfying the desired conditions. Indeed, assertion (vi) is evident. Moreover, if $x \in f^{-1}(F)$ and, for a contradiction, $\eta(x, T/\sigma) \notin f^{-1}(A)$, we have $\eta(x, [0, T/\sigma]) \subset f^{-1}(F \setminus A)$, as F and A are both invariant. It follows

$$f(\eta(x, T/\sigma)) \in f(x) - \sigma \frac{T}{\sigma} P_0 - P = f(x) - TP_0 - P \subset F - TP_0 \subset A\,,$$

whence a contradiction. □

4. The Mountain Pass Theorem

Aim of this section is to prove the vector version of the Mountain pass theorem. We are given a continuous function $f : X \to Y$ and we assume the metric space X to be complete. The next definition is the usual compactness notion of critical point theory, adapted to our setting.

Definition 4.1. Let F be a closed invariant subset of Y and A be an open invariant subset of Y such that F is reachable from A. A sequence (x_k) in $f^{-1}(F \setminus A)$ is said to be a $(PS)_{AF}$*-sequence*, if $|d_{P_0} f|(x_k) \to 0$. We say that f satisfies the $(PS)_{AF}$*-condition* if every $(PS)_{AF}$-sequence has a cluster point, which of course lies in $f^{-1}(F \setminus A)$ and is critical for f.

We set

$$K_{AF} := \{x \in X : \ f(x) \in F \setminus A \text{ and } |d_{P_0} f|(x) = 0\}\ .$$

Let us establish now some notation for the mountain pass theorem.

Let $x_0, x_1 \in X$ and set

$$\Gamma = \{p : [0,1] \to X : p \text{ is continuous}, p(0) = x_0 \text{ and } p(1) = x_1\}\ .$$

Let F be a closed invariant subset of Y and let A be an open invariant subset of Y.

Theorem 4.2. *With the above notations, let X be a complete metric space and $f : X \to Y$ a continuous function. Suppose that:*

(a) *F is reachable from A;*
(b) $f(x_0), f(x_1) \in A$;
(c) *there exists $p \in \Gamma$ such that $f(p(t)) \in F$ for every $t \in [0,1]$;*
(d) *for every $p \in \Gamma$ there exists $\bar{t}$ such that $f(p(\bar{t})) \notin A$;*
(e) *f satisfies condition $(PS)_{AF}$.*

Then $K_{AF} \neq \emptyset$.

Proof. Assume, for a contradiction, that $K_{AF} = \emptyset$. By $(PS)_{AF}$, there exists $\sigma > 0$ such that $|d_{P_0} f|(x) > \sigma$ for every $x \in f^{-1}(F \setminus A)$. From (a), (b), and Corollary 3.3, there exists a deformation $\eta : X \times [0,1] \to X$ such that

$$\begin{aligned} &\forall (x,t) \in X \times [0,1] : \quad f(\eta(x,t)) \in f(x) - P\,, \\ &\forall t \in [0,1] : \quad \eta(x_0,t) = x_0\,, \quad \eta(x_1,t) = x_1\,, \\ &\forall x \in f^{-1}(F) : \quad \eta(x,1) \in f^{-1}(A)\,. \end{aligned}$$

Let $p \in \Gamma$ such that $f(p([0,1])) \subset F$, according to (c). Setting $q(t) := \eta(p(t),1)$, $t \in [0,1]$, we have that $q \in \Gamma$ and $f(q([0,1])) \subset A$, in contradiction with (d). □

5. Ljusternik-Schnirelman category

This section is devoted to prove a Ljusternik-Schnirelman category result. Let $f : X \to Y$ be continuous and X complete. Let us be given an open interval $I \subset \mathbb{R}$, a family of open invariant sets $A_\alpha \subset Y$, $\alpha \in I$, and a family of closed invariant sets $C_\alpha \subset Y$, $\alpha \in I$, fulfilling the following monotonicity properties:

$$\alpha < \beta \implies A_\alpha \subset A_\beta\,,\, C_\alpha \subset C_\beta\,.$$

Definition 5.1. We say that condition (*) holds at level α if

$$(\alpha < \beta \implies C_\alpha \subset A_\beta) \ \wedge \ (\forall \vartheta > 0,\, \exists \varepsilon > 0 : \quad C_{\alpha+\varepsilon} - \vartheta P_0 \subset A_{\alpha-\varepsilon})\,.$$

The second condition can be equivalently written

$$\forall \vartheta > 0,\, \exists \varepsilon > 0 : \quad C_{\alpha+\varepsilon} \subset (A_{\alpha-\varepsilon})^{+\vartheta}\,.$$

We observe that in the scalar case (with $P = [0,+\infty[$ and $P_0 = [1,+\infty[$), we can choose $A_\alpha =]-\infty,\alpha[$ and $C_\alpha =]-\infty,\alpha]$. Then condition (*) is fulfilled at every level $\alpha \in \mathbb{R}$.

Now we give another Palais-Smale condition.

Definition 5.2. Given $\alpha \in I$, we say that (x_k) is a $(PS)_\alpha$-*sequence* if $|d_{P_0} f|(x_k) \to 0$ and, for every $\delta > 0$, it is $f(x_k) \in C_{\alpha+\delta} \setminus A_{\alpha-\delta}$ eventually as $k \to \infty$.

We say that f satisfies the $(PS)_\alpha$-*condition* if every such a sequence has a cluster point.

Observe that every cluster point x of a $(PS)_\alpha$-sequence is critical and satisfies

$$f(x) \in \left(\bigcap_{\delta>0} C_{\alpha+\delta}\right) \setminus \left(\bigcup_{\delta>0} A_{\alpha-\delta}\right).$$

We set

$$K_\alpha := \left\{ x \in X : \ |d_{P_0} f|(x) = 0 \text{ and } f(x) \in \left(\bigcap_{\delta>0} C_{\alpha+\delta}\right) \setminus \left(\bigcup_{\delta>0} A_{\alpha-\delta}\right) \right\}.$$

Observe that, if $\bigcap_{\delta>0} C_{\alpha+\delta} = C_\alpha$ and $\bigcup_{\delta>0} A_{\alpha-\delta} = A_\alpha$, then

$$K_\alpha = \{x \in X : |d_{P_0} f|(x) = 0 \text{ and } f(x) \in C_\alpha \setminus A_\alpha\}\,.$$

Proposition 5.3. *If $\alpha < \beta$ and condition (*) holds at some $\alpha' \in]\alpha, \beta[$, then $K_\alpha \cap K_\beta = \emptyset$.*

Proof. Consider $\alpha < \alpha' < \beta' < \beta$ such that condition (*) holds at level α'. Then we have $K_\alpha \subset f^{-1}(C_{\alpha'})$ and $K_\beta \cap f^{-1}(A_{\beta'}) = \emptyset$. Since $C_{\alpha'} \subset A_{\beta'}$, the assertion follows. $\square$

We now prove another corollary to the Quantitative deformation lemma.

Corollary 5.4. *Let $\alpha \in I$ be such that condition (*) holds at level α and f satisfies $(PS)_\alpha$.*

Then, for each neighborhood $\mathcal{N}$ of K_α, there exist $\varepsilon > 0$ and an f-deformation $\eta : X \times [0,1] \to X$ of $f^{-1}(C_{\alpha+\varepsilon}) \setminus \mathcal{N}$ into $f^{-1}(A_{\alpha-\varepsilon})$.

Proof. As K_α is compact, we can suppose, without loss of generality, that $\mathcal{N} = \mathcal{N}_{2\varrho}(K_\alpha)$ for some $\varrho > 0$. By the condition $(PS)_\alpha$, there are $\beta > \alpha > \gamma$ and $\sigma > 0$ such that $|d_{P_0} f|(x) > \sigma$ if $x \in f^{-1}(C_\beta \setminus A_\gamma) \setminus \mathcal{N}_\varrho(K_\alpha)$. Take $\vartheta = \varrho\sigma$ in condition (*) to get $\varepsilon > 0$ such that $\beta > \alpha + \varepsilon > \alpha - \varepsilon > \gamma$ and $C_{\alpha+\varepsilon} - \vartheta P_0 \subset A_{\alpha-\varepsilon}$. Now, set

$$S = f^{-1}(C_{\alpha+\varepsilon} \setminus A_{\alpha-\varepsilon}) \setminus \mathcal{N}_\varrho(K_\alpha) .$$

Call $\hat\eta : X \times [0, +\infty[\to X$ a deformation provided by the Quantitative deformation lemma, and finally define $\eta : X \times [0,1] \to X$ by

$$\eta(x,t) = \hat\eta(x, \varrho t) .$$

Let us verify that this choice of η provides the conclusions of the statement. From assertions (i) and (iii) we see that η is an f-deformation. Now, suppose that $x \in f^{-1}(C_{\alpha+\varepsilon}) \setminus \mathcal{N}$ and, for a contradiction, that $f(\eta(x,1)) \notin A_{\alpha-\varepsilon}$. Since $C_{\alpha+\varepsilon}$ and $A_{\alpha-\varepsilon}$ are both invariant, it follows that $\hat\eta(x, [0,\varrho]) \subset f^{-1}(C_{\alpha+\varepsilon} \setminus A_{\alpha-\varepsilon})$. On the other hand, from $x \notin \mathcal{N} = \mathcal{N}_{2\varrho}(K_\alpha)$ and assertion (i) of the Quantitative deformation lemma it follows that $\hat\eta(x, [0,\varrho]) \cap \mathcal{N}_\varrho(K_\alpha) = \emptyset$. From assertion (iv) we deduce that

$$f(\eta(x,1)) = f(\hat\eta(x,\varrho)) \in f(x) - \sigma\varrho P_0 - P \subset C_{\alpha+\varepsilon} - \vartheta P_0 \subset A_{\alpha-\varepsilon} ,$$

whence a contradiction. $\square$

Definition 5.5. Let $A \subset X$ and let C be a closed subset of X. We denote by $\operatorname{cat}_{(X,A)} C$ the least integer $k \geq 0$ such that C can be covered by $k+1$ open subsets $U_0, \ldots, U_k$ of X such that:

(a) there exists a deformation $\eta : X \times [0,1] \to X$ with $\eta(A \times [0,1]) \subset A$ and $\eta(U_0, 1) \subset A$;
(b) each $U_1, \ldots, U_k$ is contractible in X.

If no such integer k exists, we set $\operatorname{cat}_{(X,A)} C = +\infty$. We also set $\operatorname{cat}_X C = \operatorname{cat}_{(X,\emptyset)} C$.

For the main properties of the relative category index, we refer e.g. to [2].

Theorem 5.6. *Let A be an open invariant subset of Y. For every integer j with*

$$1 \leq j \leq \sup\left\{\operatorname{cat}_{(X,f^{-1}(A))} f^{-1}(C_\alpha) : \alpha \in I\right\} ,$$

define

$$\alpha_j = \inf\left\{\alpha \in I : \operatorname{cat}_{(X,f^{-1}(A))} f^{-1}(C_\alpha) \geq j\right\} .$$

Suppose that conditions $(PS)_\alpha$ and (*) hold for all $\alpha \in I$.

Then, if $\alpha_j = \alpha_{j+m-1} \in I$ for integers j, m with $m \geq 1$, one has $\operatorname{cat}_X K_{\alpha_j} \geq m$.

Proof. Set $\alpha = \alpha_j$ and denote by $\mathcal{N}$ an open neighborhood of K_α such that $\mathrm{cat}_X\,(\mathrm{cl}\,\mathcal{N}) = \mathrm{cat}_X\,K_\alpha$. By Corollary 5.4, there exist $\varepsilon > 0$ and an f-deformation $\eta : X \times [0,1] \to X$ of $f^{-1}(C_{\alpha+\varepsilon}) \setminus \mathcal{N}$ into $f^{-1}(A_{\alpha-\varepsilon})$, hence into $f^{-1}(C_{\alpha-\varepsilon})$.

It follows

$$\begin{aligned} j+m-1 &\leq \mathrm{cat}_{(X,f^{-1}(A))}\,f^{-1}(C_{\alpha+\varepsilon}) \leq \\ &\leq \mathrm{cat}_{(X,f^{-1}(A))}\left(f^{-1}(C_{\alpha+\varepsilon})\setminus\mathcal{N}\right) + \mathrm{cat}_X\,(\mathrm{cl}\,\mathcal{N}) \leq \\ &\leq \mathrm{cat}_{(X,f^{-1}(A))}\,f^{-1}(C_{\alpha-\varepsilon}) + \mathrm{cat}_X\,K_\alpha \leq \\ &\leq j-1+\mathrm{cat}_X\,K_\alpha\,, \end{aligned}$$

whence the assertion. □

The rest of this section is devoted to a discussion of condition (*).

Let C be a closed convex invariant subset of Y and let F be an invariant subset of Y, every point of which can be reached from int C. Let us mention that each $C^{+\alpha}$ is closed convex invariant in Y. For every $\alpha > 0$, we put $C_\alpha = C^{+\alpha} \cap F$ and $A_\alpha := \mathrm{int}\,(C^{+\alpha})$.

So, we have two increasing families of sets, for which we can try to see if (*) holds.

Proposition 5.7. *Suppose the set P_0 is of the form $p_0 + P$, for some $p_0 \in P \setminus \{0\}$. Then, with the above choice of the sets C_α and A_α, condition (*) holds at every level $\alpha > 0$.*

Proof. According to Remark 2.10, we have

$$C_\alpha = (C + \alpha p_0) \cap F\,, \quad A_\alpha = (\mathrm{int}\,C) + \alpha p_0\,.$$

Fix $0 < \alpha < \beta$. We claim at first that $C_\alpha \subset A_\beta$. Let $y \in C^{+\alpha} \cap F$. Then there exists $T \geq 0$ such that $y \in (\mathrm{int}\,C) + Tp_0$. Since int C is invariant, we may assume that $T \geq \beta - \alpha$. Thus, there exists a point $a \in \mathrm{int}\,C$ with $y = a + Tp_0$. Since $C \subset C^{+\alpha}$, we have that $a = y - Tp_0$ is contained in the interior of the closed convex set $C^{+\alpha}$ and $y \in C^{+\alpha}$. Then also $y - (\beta - \alpha)p_0$ is in the interior of $C^{+\alpha}$, namely in A_α. Thus $y \in A_\beta$ and the claim is proved.

We prove now the second condition of (*). By the previous step, we have

$$C_{\alpha+\frac{\vartheta}{3}} \subset A_{\alpha+\frac{2\vartheta}{3}} = \left(A_{\alpha-\frac{\vartheta}{3}}\right)^{+\vartheta}$$

and this completes the proof. □

An inspection to the proof above shows that, in the particular case when p_0 is also an interior point of the cone P, it is possible to take simply $C_\alpha = C^{+\alpha}$, without referring to any set F.

What happens if the set P_0 cannot be written in the form $P_0 = p_0 + P$? We can, of course, continue to make the choice above for sets C_α and A_α. But then condition (*) is not automatically satisfied for all $\alpha > 0$ and sets F, as the following example shows.

Example 5.8. Let $Y = \mathbb{R}^2$, suppose $P = \mathbb{R}^2_+$,

$$P_0 = \left\{(x,y) : x > 0 \wedge y \geq \frac{1}{x}\right\}\,, \qquad C = -P_0\,.$$

It can be shown that, for $0 < \alpha < 1$,

$$C^{+\alpha} = \left\{(x,y) : \ xy \geq (1-\alpha)^2, x < 0\right\} ,$$

and, for $\alpha \geq 1$,

$$C^{+\alpha} = -P \, .$$

Thus condition (*) holds for all α, for F a closed invariant subset of int $\mathbb{R}^2_-$, and for $\alpha < 1$ for $F = \mathbb{R}^2_-$. If we change and take $F = C = -P$, we see that F is reachable from int C, but condition (*) holds for no $\alpha > 0$.

We conclude the paper with the following observation. We have seen that in the case the set P_0 is of the form $p_0 + P$, for some $0 \neq p_0$, then the reachability condition simplifies, and condition (*) in the Ljusternik–Schnirelman theory holds automatically. Moreover, if p_0 is an interior point of the cone P, every (upper bounded) set is reachable from every set A. We made the choice to consider a general set P_0, not necessarily of the form $P_0 = p_0 + P$, because it allows more flexibility in finding critical points, and especially because, as Example 2.4 shows, a suitable choice of P_0 can make the Palais–Smale condition to be true.

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