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Directional Derivative of a Class of Set-Valued Mappings and its Application*

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Calculating the directional derivative of a class of the set-valued mappings $G(x) = \{z \mid Az \leq h(x)\}$, in the sense of Tyurin (1965) and Banks & Jacobs (1970) is presented that can be viewed as an extension to the one given by Pecherskaya. Results obtained in this paper are used to get a bound of the Lipschitz constant for the solution sets of the perturbed Linear Programming. This new bound is smaller than the one, due to Li (1994).

Keywords: Set-valued mapping, directional derivative, perturbed linear programming, optimal solution set

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1. Introduction

In the last thirty years there are many publications dealing with differentiability of set-valued mappings (multifunctions), see for instance, Tyurin (1965), Banks and Jacobs (1970), Hogan (1973), Huard (1979), Aubin and Frankowska (1990), Percherskaya (1982, 1986), Minchenko and Volosevich (2000) .

Let $G : \Omega \to \mathcal{L}(\Re^m)$ be a set-valued mapping, where $\Omega \subset \Re^n$ is an open set and $\mathcal{L}(\Re^m)$ is the set of nonempty convex compact subsets in $\Re^m$. We denote by $\rho_H(S_1, S_2)$ the Hausdorff distance of sets S_1 and S_2, and by $\delta^*(\,\cdot \mid S\,)$ the support function of a convex set S.

The results presented in this paper is grounded on the space of pairs of convex sets

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with the operations of addition and scalar multiplication and on a corresponding quotient space defined by an equivalence relationship of pairs of convex sets, and the basic concepts can be referred to references, for instance, Hörmander (1954), Banks and Jacobs (1970), Demyanov and Rubinov (1983), and Pallaschke and Urbański (2000). Some main points that will be used later on are listed below.

In $\mathcal{L}(\Re^m)$, the algebraic operations of addition and multiplication by a nonnegative real number are defined in a natural way,

$$\begin{aligned} A+B &= \{ z \mid z = a+b,\, a \in A,\, b \in B \} \\ cA &= \{ ca \mid a \in A \}, \quad (c > 0). \end{aligned}$$

These operations possess usual properties, i.e., associativity, commutativity and distributivity. Consider the set K^2 defined as follows

$$K^2 = \mathcal{L}(\Re^m) \times \mathcal{L}(\Re^m) = \{ (A,\, B) \mid A \in \mathcal{L}(\Re^m), B \in \mathcal{L}(\Re^m) \}$$

The operations of addition and scalar multiplication in K^2 are defined by

$$\begin{aligned} (A_1,\, B_1) + (A_2,\, B_2) &= (A_1 + A_2,\, B_1 + B_2) \\ \alpha(A,\, B) &= \begin{cases} (\alpha A,\, \alpha B) & \text{if } \alpha > 0 \\ (|\alpha| B,\, |\alpha| A) & \text{if } \alpha < 0, \end{cases} \end{aligned} \tag{1}$$

where $(A_i,\, B_i) \in K^2$, $i = 1, 2$, and $(A,\, B) \in K^2$. A partial ordering '$\succ$' in K^2 is defined by

$$(A_1,\, B_1) \succ (A_2,\, B_2) \text{ if and only if } \mathrm{A}_1 + \mathrm{B}_2 \supseteq \mathrm{A}_2 + \mathrm{B}_1$$

for $(A_i,\, B_i) \in K^2$, $i = 1, 2$. The equivalence relation induced by '$\succ$', denoted by '$\sim$', (see for instance, Hörmander (1954)), is defined by

$$(A_1,\, B_1) \sim (A_2,\, B_2) \text{ if and only if } \mathrm{A}_1 + \mathrm{B}_2 = \mathrm{A}_2 + \mathrm{B}_1.$$

We define $K_1 = K^2/\sim$ and $[\,\cdot,\,\cdot\,]$ denotes its representative (element). The operations of addition and scalar multiplication in K_1 are the same substantially as (1), i.e., by

$$[A_1,\, B_1] + [A_2,\, B_2] = [A_1 + A_2,\, B_1 + B_2]$$

$$\alpha[A,\, B] = \begin{cases} [\alpha A,\, \alpha B] & \text{if } \alpha > 0 \\ [|\alpha| B,\, |\alpha| A] & \text{if } \alpha < 0, \end{cases}$$

where $[A_i,\, B_i] \in K_1$, $i = 1, 2$, $[A,\, B] \in K_1$.

A kind of directional differentiability for set-valued mappings based on the Hausdorff distance (see for instance, Tyurin (1965), Banks & Jacobs (1970) and Pecherskaya (1982, 1986)), which will be used in this paper, is given by the following definition.

Definition 1.1. A set-valued mapping G is said to be directionally differentiable at $x \in \Omega$ in a direction $v \in \Re^n$, if there exists a pair of nonempty compact convex sets, $[G_x^+(v),\, G_x^-(v)]$, such that for a sufficiently small positive number α,

$$\rho_H(G(x+\alpha v) + \alpha G_x^-(v), G(x) + \alpha G_x^+(v)) = o_{x,v}(\alpha) \tag{2}$$

is valid, where $o_{x,v}(\alpha)/\alpha \xrightarrow{\alpha \downarrow 0} 0$, $G_x^+(v), G_x^-(v) \in \mathcal{L}(\Re^m)$. The pair $[G_x^+(v),\, G_x^-(v)]$ is called a directional derivative of G at x in v.

The next definition is alternative one equivalent to the definition above.

Definition 1.2. A set-valued mapping G is said to be differentiable at x in a direction v, if for $x \in \Omega$, $v \in \Re^n$ and a sufficiently small positive number α, there exists a pair of nonempty convex compact sets, $[\, G_x^+(v),\ G_x^-(v)\,]$, such that the limit

$$\lim_{\alpha \downarrow 0} \frac{\delta^*(\cdot \mid G(x+\alpha v)) - \delta^*(\cdot \mid G(x))}{\alpha} = \delta^*(\cdot \mid G_x^+(v)\,) - \delta^*(\cdot \mid G_x^-(v)\,) \tag{3}$$

exists. The pair $[\, G_x^+(v), G_x^-(v)\,]$ is called the directional derivative of a set-valued mapping G at x in the direction v.

The differentiability of the set-valued mapping $F(x)$ defined by

$$F(x) := \{z \mid Az \le x\} \tag{4}$$

was investigated by Percherskaya (1982, 1986). The linear programming (LP)

$$\begin{cases} \min & c^T z \\ \text{s.t.} & Az \le h(x) \end{cases} \tag{5}$$

where A is an $n \times m$ matrix, $n > m$ and $h : \Omega \to \Re^n$ is a vector function. Let $\varphi(x) = \min\{\, c^T z \,|\, Az \le h(x)\,\}$, $\quad S(x) = \{\, z \,|\, Az \le h(x), \quad c^T z \le \varphi(x)\,\}$. It follows from Klatte (1987) that S is Lipschitz continuous, i.e., there exists $\gamma > 0$, such that for a sufficiently small positive number $\alpha > 0$, $x \in \Omega$, and $v \in \Omega$, we have that

$$\rho_H(S(x), S(x+\alpha v)) \le \gamma \|\, h(x+\alpha v) - h(x)\,\|. \tag{6}$$

There are some papers dealing with the topic about estimation to γ, see for instance Cook (1986), Li (1993, 1994), Mangasarian (1987), Robinson (1973). A bound to γ due to Li (1994) is smaller than or equal to the one given by Mangasarian and Shiau (1987). A new bound of γ, obtained by using differential results of the set-valued mapping $S(x)$ given above, will be given in this paper. It will be shown that the new result is smaller than others under some assumptions.

This paper is organized as follows. In Section 2, the differentiability of the set-valued mapping $G(x) := \{z \mid Az \le h(x)\}$ will be given, where $h(\cdot)$ is directional differentiable at x in the common sense. Based on a perturbation to linear programming (5) by $h(x+\alpha v)$, a new bound to γ, is presented in Section 3.

2. Directional derivative for a class of set-valued mappings

Consider a class of set-valued mappings being of the form

$$G(x) := \{\, z \mid Az \le h(x)\}, \tag{7}$$

where A is an $n \times m$ matrix, $n > m$, $h : \Omega \to \Re^n$ is a vector function in the open set $\Omega \subset \Re^n$.

Let a_i be a column vector composed of elements of the i-th row vector in A, where $i = 1, 2, \cdots, n$. Without loss of generality, we may assume that $\|a_i\| = 1$, where $\|\cdot\|$ denotes the Euclidean norm.

The following assumptions will be used later on.

AM1 $h : \Omega \to \Re^n$ is directionally differentiable at x in a direction v.

AM2 $G(x)$ is bounded and nondegenerate in the sense of all extremal points. i.e., At any extremal point $z \in G(x)$, the set of vectors $\{\, a_j \,|\, j \in J(z)\}$ are linearly independent where $J(z) = \{\, j \,|\, a_j^T z = h_j(x)\}$.

AM3 $G(x)$ does not contain unnecessary polyhedral constraints for every $x \in \mathrm{dom}G(x) = \{\, x \in \Omega \,|\, G(x) \neq \emptyset \,\}$, i.e., every a_i is a normal vector to some (m-1)-dimensional face of $G(x)$.

It is clear that the set-valued mapping G defined by formula (7) is a closed convex set at every point. If $AM2$ is satisfied , then $G(x)$ is bounded. Consequently, $G(x)$ is a compact convex set.

Let $B(x,u) = \{\, z \in G(x) \mid \delta^*(u \mid G(x)) = \langle u, z\rangle \,\}$. If $z \in B(x,u)$, let $J_u(z) = \{\, i \mid \langle a_i, z\rangle = h_i(x), \quad i = 1,2,\cdots,n\}$. Without loss of generality, we may assume that $J_u(z) = \{1,2,3,\cdots,s\}$.

The following Lemma is a direct consequence of [Lemma 2.5, 4].

Lemma 2.1. *Let $G(x)$ be defined by formula (7). If $AM1$ and $AM2$ are satisfied, then the set-valued mapping $G \in \mathcal{L}(\Re^m)$ is locally bounded at x.* □

Define

$$G_x^+(v) \;=\; \{g \mid Ag \le h'_+(x;v) + N_{x,v}\mathbf{e}\} \tag{8}$$
$$G_x^-(v) \;=\; \{g \mid Ag \le h'_-(x;v) + N_{x,v}\mathbf{e}\}, \tag{9}$$

where $N_{x,v}$ is sufficiently large positive number depending on x, v,

$$\mathbf{e} = (1,1,\cdots,1)^T \in \Re^n$$

$$[h'_+(x;v)]^i = \begin{cases} [h'(x;v)]^i & \text{if } [h'(x;v)]^i > 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad i \in \{1,2,\cdots,n\}$$

$$[h'_-(x;v)]^i = \begin{cases} -[h'(x;v)]^i & \text{if } [h'(x;v)]^i < 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad i \in \{1,2,\cdots,n\}.$$

Lemma 2.2. *For any fixed $x \in \Omega$, if $AM1 - AM3$ hold, then there exists a positive number $N_{x,v}^0$ such that $G_x^+(v)$ and $G_x^-(v)$ defined by (8) and (9), respectively, are bounded and nondegenerate polyhedrals, and they do not contain unnecessary constraints for $N_{x,v} \ge N_{x,v}^0$.*

Proof. Let $x \in \Omega$, $v \in \Re^n$, and $P_N = F(N\mathbf{e})$ if $N > 0$ where $e = (1,1,\cdots,1)^T \in \Re^m$. One has from (4) that

$$\begin{aligned} P_N &= NF(e) \\ &= \{Nz | Az \le e\}. \end{aligned}$$

It follows from AM1-AM2 that $F(x)$ is bounded and nondegenerate, on $\operatorname{dom} F = \{x \in \Omega |\, F(x) \neq \emptyset\}$, where F is defined by (4). The fact that $F(\cdot)$ does not contain unnecessary polyhedral constraints on domF comes directly from AM3 and $G(x) = F(x)$ when $h(x)$ is taken as x. As a result, one has that P_N is bounded and doesn't contain unnecessary polyhedral constraints. Since it is assumed, at the beginning of this section, that $\|a_i\| = 1$, $i = 1, 2, \cdots, n$, it follows that $B_m(N)$ is the largest ball inscribed in P_N and Na_i is a point, on $H(P_N, a_i) = \{Nz | a_i^T z = 1\}$ which is a facet of P_N, tangent to $B_m(N)$, where $B_m(N)$ is the $m-$dimensional ball with the origin as the center and N as the radius.

According to (8) and (9) and (4), one has that

$$\begin{aligned} G_x^+(v) &= F(h'_+(x; v) + N_{x,v}\mathbf{e}) \\ G_x^-(v) &= F(h'_-(x; v) + N_{x,v}\mathbf{e}). \end{aligned}$$

We denote by $B^i_{m-1}(1)$ the unit ball of the complementary subspace spanned by $a_i \in \Re^m$. Since $H(G_x^+(v), a_i)$ and $[h'_+(x; v)]^i$ are known, $N_{x,v}$ can be obtained by the following equation

$$([h'_+(x; v)]^i + N_{x,v})a_i + B^i_{m-1}(1) \subset H(G_x^+(v), a_i), \qquad i = 1, 2, \cdots, n.$$

Then it follows that

$$[([h'_+(x, v)]^i + N_{x,v})a_i + b_i]^T a_j \leq [h'_+(x; v)]^j + N_{x,v} \quad i = 1,\, 2,\, \cdots,\, n, \tag{10}$$

where $b_j \in B^i_{m-1}(1)$. From the system of inequalities, (10), we have that

$$N^i_{x,v} \geq \max_{1 \leq j \leq n} \max_{b \in B^i_{m-1}} \frac{a_i^T a_j [h'_+(x; v)]^i - [h'_+(x; v)]^j - b^T a_j}{1 - a_i^T a_j}, \quad j \neq i.$$

Similarly,

$$\bar{N}^i_{x,v} \geq \max_{1 \leq j \leq n} \max_{b \in B^i_{m-1}} \frac{a_i^T a_j [h'_-(x; v)]^i - [h'_-(x; v)]^j - b^T a_j}{1 - a_i^T a_j}, \quad j \neq i.$$

Take $N^0_{x,v} = \max\{\max_{1 \leq i \leq n} N^i_{x,v}\ ,\ \max_{1 \leq i \leq n} \bar{N}^i_{x,v}, 0\}$. Therefore, formula (10) is always valid for $N_{x,v} \geq N^0_{x,v}$. □

Lemma 2.3. *Assume that* $\{\, a_i \,|\, i = 1, 2, \cdots, s\} \subset \Re^m$, $s \leq m$, *are linearly independent, then the general solution of the system of equations*

$$\langle a_i, y \rangle = b_i, \quad i = 1, 2, \cdots, s$$

can be represented as

$$y = (A\bar{A}^{-1})b + y',$$

where $A = (a_1, a_2, \cdots, a_s)$, $\bar{A} = A^T A$, $b = (b_1, b_2, \cdots, b_s)^T$, $y' \in [\operatorname{span}\{a_i |\, i = 1, 2, \cdots, s\}]^\perp$. □

Lemma 2.4. *Let* $T_{G(x+\alpha v)}(u) = \{y \in G(x + \alpha v) \,|\, \delta^*(u|G(x + \alpha v)) = \langle u, y\rangle\}$. *Then for* $\alpha > 0$ *small enough, there exists* $y' \in T_{G(x+\alpha v)}(u)$ *satisfying the following system:*

$$(\textit{ System I }) \qquad \begin{aligned} &\langle a_i, \bar{z} \rangle = h_i(x + \alpha v), && i \in J_u(z) \\ &\langle a_j, \bar{z} \rangle \leq h_j(x + \alpha v), && j \in \{1, 2, \cdots, m\} \setminus J_u(z). \end{aligned}$$

Proof. Proceeding by contradiction. Suppose that no element of $T_{G(x+\alpha v)}(u)$ satisfies the system I.

For any solution $\bar{z}$ of the system I, we have that $\delta^*(u|G(x+\alpha v)) > \langle u, \bar{z}\rangle$. By Lemma 2.3, $\bar{z}$ can be formulated in $\bar{z} = (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})\bar{h}(x+\alpha v) + z'$, where $z' \in [\text{span}\{a_i | i \in J_u(z)\}]^{\perp}$ and $\bar{h}(x+\alpha v) = (h_1(x+\alpha v), h_2(x+\alpha v), \cdots, h_s(x+\alpha v))$. Then one has that $\delta^*(u|G(x+\alpha v)) > \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})\bar{h}(x+\alpha v)\rangle$. By the Mean Value Theorem, we have that

$$\bar{h}(x+\alpha v) = \bar{h}(x) + \alpha \int_0^1 \bar{h}'(x+\alpha t v; v)dt, \tag{11}$$

where $0 \leq t \leq 1$. It is clear that $\delta^*(u|G(x)) = \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})\bar{h}(x)\rangle$ in terms of definitions of $B(x,u)$ and $J_u(z)$ and Lemma 2.3. Thus one obtains that

$$\begin{aligned}\delta^*(u \mid G(x+\alpha v)) &> \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})(\bar{h}(x) + \alpha \int_0^1 \bar{h}'(x+\alpha t v; v)dt)\rangle \\ &= \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})\bar{h}(x)\rangle + \alpha\langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)}) \int_0^1 \bar{h}'(x+\alpha t v; v)dt\rangle.\end{aligned}$$

in other words,

$$\delta^*(u|G(x+\alpha v)) - \delta^*(u|G(x)) > \alpha\langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)}) \int_0^1 \bar{h}'(x+\alpha t v; v)dt\rangle. \tag{12}$$

On the other hand, for any $y \in T_{G(x+\alpha v)}(u)$, there exists $k \in J_u(z)$ such that $\langle a_k, y\rangle < h_k(x+\alpha v)$. Let $\int_0^1 \bar{h}'(x+\alpha t v; v)dt = (f_1(\alpha,x,v), f_2(\alpha,x,v), \cdots, f_s(\alpha,x,v))^T$. It follows from formula (11) that $\langle a_k, y\rangle < h_k(x) + \alpha f_k(\alpha,x,v)$. Since $\{a_j | j \in J_u(z)\}$ are linearly independent, there exists $p \in \Re^m$ such that $\langle a_i, p\rangle = f_i(\alpha,x,v), \quad i \in J_u(z)$. One has that $\langle a_k, y - \alpha p\rangle < h_k(x)$ and hence $\langle a_k, y - \alpha p - z\rangle < 0$. Since $\langle a_j, y\rangle \leq h_j(x+\alpha v), \quad j \in J_u(z)$, it follows that $\langle a_j, y - \alpha p - z\rangle \leq 0, \quad j \in J_u(z)$, i.e., $y - \alpha p - z \in K_{G(x)}(z)$, where $K_{G(x)}(z)$ denotes the tangent cone of $G(x)$ at z. Thus, one has $\langle u, y - \alpha p - z\rangle \leq 0$. Furthermore, $\langle u, y\rangle - \langle u, z\rangle \leq \alpha\langle u, p\rangle$. In consequence, one has that

$$\delta^*(u|G(x+\alpha v)) - \delta^*(u|G(x)) \leq \alpha\langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)}) \int_0^1 \bar{h}'(x+\alpha t v; v)dt\rangle. \tag{13}$$

It contradicts with the fact that formula (12) and formula (13) hold simultaneously. The proof is completed. □

Lemma 2.5. *Let* $T_{G_x^+(v)}(u) = \{g_+ \in G_x^+(v) \,|\, \delta^*(u|G_x^+(v)) = \langle u, g_+\rangle\}$, *then there exists* $g'_+ \in T_{G_x^+(v)}(u)$ *which satisfies the following system:*

$$(\textit{ System II }) \qquad \begin{array}{ll} \langle a_i, g\rangle = [\, h'_+(x;v)\,]^i + N_{x,v}, & i \in J_u(z) \\ \langle a_j, g\rangle \leq [\, h'_+(x;v)\,]^j + N_{x,v} & j \in \{1,2,\cdots,m\} \setminus J_u(z). \end{array}$$

Proof. Proceeding by the contradiction. Assume that no element of $T_{G_x^+(v)}(u)$ satisfies the system II.

For any solution g of the system II, it follows that $\delta^*(u|G_x^+(v)) > \langle u, g\rangle$. we know from Lemma 2.3 that $g = (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_+(x;v) + N_{x,v}\bar{\mathbf{e}}] + g'$, where

$$g' \in [\text{span}\{a_i | i \in J_u(z)\}]^\perp$$
$$[\bar{h}'_+(x;v)]^i = \begin{cases} [h'(x;v)]^i & \text{if } [h'(x;v)]^i > 0 \\ 0 & \text{otherwise} \end{cases} \quad i \in J_u(z)$$
$$\bar{\mathbf{e}} = (1, 1, \cdots, 1)^T \in \Re^s.$$

Consequently, one has

$$\delta^*(u|G_x^+(v)) > \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_+(x;v) + N_{x,v}\bar{\mathbf{e}}]\rangle. \tag{14}$$

On the other hand, For any $g \in T_{G_x^+(v)}(u)$, there exists $k \in J_u(z)$ such that the inequality $\langle a_k, g_+\rangle < [h'_+(x;v)]^k + N_{x,v}$ holds. Since $\{a_i | i \in J_u(z)\}$ are linearly independent, there exists $p \in \Re^m$ such that $\langle a_i, p\rangle = h_i(x) - [h'_+(x;v)]^i - N_{x,v}, \quad i \in J_u(z)$. Then one has that $\langle a_k, g'_+ + p\rangle < h_k(x)$. Thus, one has that $\langle a_i, g'_+ + p - z\rangle \le 0, \quad i \in J_u(z)$, i.e., $g'_+ + p - z \in K_{G(x)}(z)$, where $K_{G(x)}(z)$ denotes the tangent cone of $G(x)$ at z. Thus, $\langle u, g'_+ + p - z\rangle \le 0$, i.e., $\langle u, g'_+\rangle \le \langle u, z\rangle - \langle u, p\rangle$. It follows that

$$\delta^*(u|G_x^+(v)) \le \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_+(x;v) + N_{x,v}\bar{\mathbf{e}}]\rangle. \tag{15}$$

It contradicts with the fact that formula (14) and formula (15) hold simultaneously. The proof is completed. □

The following lemma can be proved in a way similar to the proof of Lemma 2.5.

Lemma 2.6. *Let* $T_{G_x^-(v)}(u) = \{g_- \in G_x^-(v) \,|\, \delta^*(u|G_x^-(v)) = \langle u, g_-\rangle\}$, *then there exists* $g'_- \in T_{G_x^-(v)}(u)$ *which satisfies the following system:*

$$(\textit{ System III }) \qquad \begin{array}{ll} \langle a_i, g\rangle = [h'_-(x;v)]^i + N_{x,v}, & i \in J_u(z) \\ \langle a_j, g\rangle \le [h'_-(x;v)]^j + N_{x,v}, & j \in \{1, 2, \cdots, m\} \setminus J_u(z). \end{array}$$

□

Lemma 2.7. *Let* $N^0_{x,v}$ *be the same as in Lemma 2.2,* $\forall N^1_{x,v} \ge N^0_{x,v}$, $\forall N^2_{x,v} \ge N^0_{x,v}$. *Then one has that*

$$[G(h'_+(x,v) + N^1_{x,v}\boldsymbol{e}),\, G(h'_-(x;v) + N^1_{x,v}\boldsymbol{e})] \sim [G(h'_+(x;v) + N^2_{x,v}\boldsymbol{e}),\, G(h'_-(x;v) + N^2_{x,v}\boldsymbol{e})].$$

Proof. Take $u \in S$. We can obtain from Lemma 2.3, Lemma 2.5 and Lemma 2.6 that

$$\begin{aligned}
\delta^*(u|G(h'_+(x,v) + N^1_{x,v}\mathbf{e})) &= \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_+(x;v) + N^1_{x,v}\bar{\mathbf{e}}]\rangle, \\
\delta^*(u|G(h'_-(x,v) + N^1_{x,v}\mathbf{e})) &= \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_-(x;v) + N^1_{x,v}\bar{\mathbf{e}}]\rangle, \\
\delta^*(u|G(h'_+(x,v) + N^2_{x,v}\mathbf{e})) &= \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_+(x;v) + N^2_{x,v}\bar{\mathbf{e}}]\rangle, \\
\delta^*(u|G(h'_-(x,v) + N^2_{x,v}\mathbf{e})) &= \langle u, (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_-(x;v) + N^2_{x,v}\bar{\mathbf{e}}]\rangle.
\end{aligned}$$

Then

$$\begin{aligned}\delta^*(u|G(h'_+(x,v)+N^1_{x,v}\mathbf{e}))-\delta^*(u|G(h'_-(x,v)+N^1_{x,v}\mathbf{e})) &= \langle u,(A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})\bar{h}'(x;v)\rangle,\\ \delta^*(u|G(h'_+(x,v)+N^2_{x,v}\mathbf{e}))-\delta^*(u|G(h'_-(x,v)+N^2_{x,v}\mathbf{e})) &= \langle u,(A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})\bar{h}'(x;v)\rangle.\end{aligned}$$

Let

$$\begin{aligned} A=G(h'_+(x,v)+N^1_{x,v}\mathbf{e}),\quad B=G(h'_-(x,v)+N^1_{x,v}\mathbf{e}),\\ C=G(h'_+(x,v)+N^2_{x,v}\mathbf{e}),\quad D=G(h'_-(x,v)+N^2_{x,v}\mathbf{e}).\end{aligned}$$

Then, one has that

$$\begin{aligned}\rho_H(A+D,B+C) &= \max_{u\in S}|\delta^*(u|A)+\delta^*(u|D)-[\delta^*(u|B)+\delta^*(u|C)]|\\ &= \max_{u\in S}|[\delta^*(u|A)-\delta^*(u|B)]-[\delta^*(u|C)-\delta^*(u|D)]|\\ &= 0,\end{aligned}$$

i.e., $[A,\,B]\sim[C,\,D]$. The proof is completed. □

Theorem 2.8. [15] *Let $A,D\in\mathcal{L}(\Re^m)$. Then*

$$\rho_H(A,D)=\mid\delta^*(\cdot\mid A)-\delta^*(\cdot\mid D)\mid_{C(S)}, \tag{16}$$

where $C(S)$ is a space of all functions with a uniform norm which are continuous. □

According to (2) and (16), we have

$$\max_{u\in S}\mid\delta^*(u\mid G(x+\alpha v)+\alpha G^-_x(v))-\delta^*(u\mid G(x)+\alpha G^+_x(v))\mid=o_{x,v}(\alpha), \tag{17}$$

where $o_{x,v}(\alpha)/\alpha\xrightarrow{\alpha\downarrow 0}0$. By Minkowski duality, see [11], (17) can be formulated in

$$\mid\delta^*(u\mid G(x+\alpha v))-\delta^*(u\mid G(x))-\alpha(\delta^*(u\mid G^+_x(v))-\delta^*(u\mid G^-_x(v)))\mid=o_{x,u}(\alpha), \tag{18}$$

where $o_{x,u}(\alpha)/\alpha\xrightarrow{\alpha\downarrow 0}0$, in which the limit is uniform with respect to u.

Theorem 2.9. *Suppose the set-valued mapping $G(x)$ is defined by formula (7). If $AM1-AM3$ are satisfied, then the set-valued mapping G is directionally differentiable at x in a direction v. The pair $[G^+_x(v),G^-_x(v)]$ defined by formula (8) and formula (9) is the directional derivative of G at x in v.*

Proof. It is enough to prove that formula (18). For $u\in S$ and $\alpha>0$ small enough, consider equations

$$\delta^*(u\mid G(x))=\langle u,z\rangle, \tag{19}$$
$$\delta^*(u\mid G(x+\alpha v))=\langle u,y'\rangle, \tag{20}$$
$$\delta^*(u\mid G^+_x(v))=\langle u,g'_+\rangle, \tag{21}$$
$$\delta^*(u\mid G^-_x(v))=\langle u,g'_-\rangle. \tag{22}$$

From Lemmas (2.4)-(2.6), one has that z, y', g'_+, and g'_- in formulae (19)—(22) are solutions of the following four systems

$$\langle a_i, z\rangle - h_i(x) = 0, \quad i \in J_u(z), \tag{23}$$

$$\langle a_i, \bar{z}\rangle - h_i(x+\alpha v) = 0, \quad i \in J_u(z), \tag{24}$$

$$\langle a_i, g\rangle - [h'_+(x;v)]^i - N = 0, \quad i \in J_u(z), \tag{25}$$

$$\langle a_i, \bar{g}\rangle - [h'_-(x;v)]^i - N = 0, \quad i \in J_u(z) \tag{26}$$

respectively.

In what follows, we only discuss equation (21) (the others can be similar to be discussed). Since $\{a_i \mid i \in J(z)\}$ are linearly independent, every solution of equation (25) can be represented as

$$g = (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})[\bar{h}'_+(x;v) + N_{x,v}\bar{\mathbf{e}}] + g',$$

where $g' \in [\mathrm{span}\{a_i | i \in J_u(z)\}]^\perp$, $\bar{A}_{J_u(z)} = A^T_{J_u(z)}A_{J_u(z)}$. Consequently, one has that

$$\begin{aligned} \delta^*(u|G^+_x(v)) &= \langle u, g'_+\rangle \\ &= \langle u, A_{J_u(z)}\bar{A}^{-1}_{J_u(z)}(\bar{h}'_+(x;v) + N_{x,v}\bar{\mathbf{e}})\rangle \\ &= \langle (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u, \bar{h}'_+(x;v) + N_{x,v}\bar{\mathbf{e}}\rangle. \end{aligned}$$

Similarly, one has that

$$\begin{aligned} \delta^*(u|G^-_x(v)) &= \langle (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u, \bar{h}'_-(x;v) + N_{x,v}\bar{\mathbf{e}}\rangle, \\ \delta^*(u|G(x)) &= \langle (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u, \bar{h}(x)\rangle, \\ \delta^*(u|G(x+\alpha v)) &= \langle (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u, \bar{h}(x+\alpha v)\rangle. \end{aligned}$$

Let $|J_u(z)|$ denote the number of elements $J_u(z)$. Then $|J_u(z)| \le \mathrm{rank}(A)$. Suppose $\mathrm{rank}(A) = |I|$, then one has that

$$\begin{aligned} & |\delta^*(u|G(x+\alpha v)) - \delta^*(u|G(x)) - \alpha[\delta^*(u|G^+_x(v)) - \delta^*(u|G^-_x(v))]| \\ = \ & |\langle (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u, \bar{h}(x+\alpha v) - \bar{h}(x) - \alpha[\bar{h}'_+(x;v) - \bar{h}'_-(x;v)]\rangle| \\ = \ & |\langle (A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u, \tilde{o}_{x,v}(\alpha)\rangle| \\ \le \ & \|(A_{J_u(z)}\bar{A}^{-1}_{J_u(z)})^T u\| \cdot \|\tilde{o}_{x,v}(\alpha)\| \\ \le \ & \max_{I\in\mathcal{F}} \|A^+_I\| \cdot \|u\| \cdot \|\tilde{o}_{x,v}(\alpha)\|, \end{aligned}$$

where $\mathcal{F} = \{I \,|\, |I| = \mathrm{rank}(A) = \mathrm{rank}(A_I)\}$. It follows that

$$\max_{u\in S} |\delta^*(u|G(x+\alpha v)) - \delta^*(u|G(x)) - \alpha[\delta^*(u|G^+_x(v)) - \delta^*(u|G^-_x(v))]| = o_{x,v}(\alpha),$$

where $\frac{o_{x,v}(\alpha)}{\alpha} \xrightarrow{\alpha\downarrow 0} 0$ (uniformly with respect to u). □

Corollary 2.10. *Let G be defined by formula (7). If $AM1 - AM3$ are satisfied, and $h(x)$ is differentiable, then G is directional differentiable at x in a direction v. It's directional derivative $[G^+_x(v), G^-_x(v)]$ is defined by the following form*

$$\begin{aligned} G^+_x(v) &= \{g \mid Ag \le [\nabla h(x)v]_+ + N\boldsymbol{e}\}, \\ G^-_x(v) &= \{g \mid Ag \le [\nabla h(x)v]_- + N\boldsymbol{e}\}. \end{aligned}$$

where $\mathbf{e}$ *and* N *are defined by the way that is the same as defined in Theorem 2.9.*

$$[\nabla h(x)v]_+^i = \begin{cases} [\nabla h(x)v]^i & \text{if } [\nabla h(x)v]^i > 0 \\ 0 & \text{otherwise} \end{cases} \quad i \in \{1, 2, \cdots, n\},$$

$$[\nabla h(x)v]_-^i = \begin{cases} -[\nabla h(x)v]^i & \text{if } [\nabla h(x)v]^i < 0 \\ 0 & \text{otherwise} \end{cases} \quad i \in \{1, 2, \cdots, n\}.$$

Proof. It can be directly obtained from Theorem 2.9. □

3. Application to linear programming

In this section, we analyse the stability of the set of optimal solutions for a perturbed linear programming, using results obtained in the last section. Consider the following linear programming problem

$$\begin{cases} \min & c^T z \\ \text{s.t.} & Az \le h(x), \end{cases} \tag{27}$$

where A is an $n \times m$ matrix, $n > m$, c is a m dimensional vector, and $h : \Omega \to \Re^n$ is a vector function. Define

$$\begin{aligned} G(x) &= \{z \mid Az \le h(x)\}, \\ \varphi(x) &= \min\{c^T z \mid z \in G(x)\}, \\ S(x) &= \{z \in G(x) \mid \varphi(x) = c^T z\}. \end{aligned}$$

Consider the perturbed linear programming

$$\begin{cases} \min & c^T z \\ \text{s.t.} & Az \le h(x + \alpha v). \end{cases} \tag{28}$$

Define

$$\begin{aligned} G(x + \alpha v) &= \{z \mid Az \le h(x + \alpha v)\}, \\ \varphi(x + \alpha v) &= \min\{c^T z \mid z \in G(x + \alpha v)\}, \\ S(x + \alpha v) &= \{z \in G(x + \alpha v) \mid \varphi(x + \alpha v) = c^T z\}. \end{aligned}$$

The following assumption will be required in the rest of this section.

AM4 The set $S(x)$ is not a singleton.

Under the assumption **AM4**, we can immediately obtain the following lemmas.

Lemma 3.1. *The following results hold:*

(i) *The solution set* $S(x)$ *is a polyhedra, and it is a face of the polyhedra* $G(x)$.

(ii) *There exist vectors* $a_i, i \in I$, *satisfying* $a_i^T z = h_i(x)$ *for any* $i \in I$ *and* $z \in S(x)$ *such that* $S(x) = \{ z \mid A_I^T z = h_I(x), A_{\bar{I}}^T z \le h_{\bar{I}}(x)\}$ *where* $I \subset \{1, 2, \cdots, n\}$, $|I| < m$, *and* $\bar{I} = \{1, 2, \cdots, n\} \setminus I$.

For example, consider the problem:

$$\begin{cases} \min & z_1 + 2z_2 \\ \text{s.t.} & -3z_1 - z_2 \le -6 + x \\ & -z_1 - 2z_2 \le -4 + x \\ & \qquad z_1 \le 5 + x \\ & \qquad z_2 \le 7 + x. \end{cases}$$

The solution set $S(x) = \{(z_1, z_2) \mid -z_1 - 2z_2 = -4 + x,\ -3z_1 - z_2 \le -6 + x,\ z_1 \le 5 + x,\ z_2 \le 7 + x\}$ is a polyhedra , and it is a face of $G(x)$ for a fixed x.

Without loss of generality, we assume that $I = \{1, 2, \cdots, p\}$. Let $z = (\bar{z}^T, w^T)^T$ where $\bar{z} \in \Re^p$ and $w \in \Re^{m-p}$, $A_I^T = (A_{IB}^T A_{IN}^T)$ where $A_{IB} \in \Re^{p\times p}$ and $A_{IN} \in \Re^{(m-p)\times p}$, and $A_{\bar{I}}^T = A \backslash A_I$. Then the set $S(x)$ becomes

$$S(x) = \{(\bar{z}^T, w^T)^T | A_{\bar{I}}^T \begin{pmatrix} -A_{IB}^T A_{IN}^T \\ I_{m-p} \end{pmatrix} w \le h_{\bar{I}}(x) - A_{\bar{I}}^T \begin{pmatrix} A_{IB}^{-T} h_I(x) \\ 0 \end{pmatrix},$$
$$\bar{z} = A_{IB}^{-T} h_I(x) - A_{IB}^{-T} A_{IN}^T w\}.$$

Define

$$\begin{aligned} Q(x) &= \{z | Hw \le f(x)\}, \\ Q(x + \alpha v) &= \{z | Hw \le f(x + \alpha v)\}, \end{aligned}$$

where $f(x) = h_{\bar{I}}(x) - A_{\bar{I}}^T \begin{pmatrix} A_{IB}^{-T} h_I(x) \\ 0 \end{pmatrix}$, $f(x+\alpha v) = h_{\bar{I}}(x+\alpha v) - A_{\bar{I}}^T \begin{pmatrix} A_{IB}^{-T} h_I(x+\alpha v) \\ 0 \end{pmatrix}$, $H = A_{\bar{I}}^T \begin{pmatrix} -A_{IB}^T A_{IN}^T \\ I_{m-p} \end{pmatrix}$.

Lemma 3.2. *If z is an extremal point of the set $G(x)$, then the point w satisfying $z_1 = \begin{pmatrix} A_{IB}^{-T} h_I(x) \\ 0 \end{pmatrix} + \begin{pmatrix} -A_{IB}^{-T} A_{IN}^T \\ I_{m-p} \end{pmatrix} w$ must be an extremal point of $Q(x)$.*

According to AM2, there exist an index set $\tilde{J}$ satisfying $|\tilde{J}| = m$. Without loss of generality, we assume that $\tilde{J} = \{1, 2, \cdots, m\}$, then there exists $z \in G(x)$ such that $a_i^T z = h_i(x), i = 1, 2, \cdots, m$.

Lemma 3.3. *The collection of $\{(-A_{IN} A_{IB}^{-1} \quad I_{m-p}) a_i\}_{i=p+1}^m$ is linearly independent.*

Proof. Let

$$\lambda_{p+1}(-A_{IN} A_{IB}^{-1} \quad I_{m-p}) a_{p+1} + \cdots + \lambda_m (-A_{IN} A_{IB}^{-1} \quad I_{m-p}) a_m = 0. \tag{29}$$

All we done are to prove that $\lambda = (\lambda_{p+1}, \lambda_{p+2}, \cdots, \lambda_m)^T = 0$. In fact, (29) is translated into

$$(-A_{IN} A_{IB}^{-1} \quad I_{m-p})(a_{p+1}, a_{p+2}, \cdots, a_m) \begin{pmatrix} \lambda_{p+1} \\ \vdots \\ \lambda_m \end{pmatrix} = 0. \tag{30}$$

Let $(a_{p+1},\cdots,a_m)^T=(\tilde{A}_{\bar{I}B}^T,\tilde{A}_{\bar{I}N}^T)^T$, then (30) becomes

$$(\tilde{A}_{\bar{I}N}-A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B})\lambda=0.$$

According to AM2, one has from Linear Algebra that the matrix $\tilde{A}_{\bar{I}N}-A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B}$ is nonsigular. Hence, $\lambda=0$. □

Lemma 3.4. *If $AM1\sim AM4$ are satisfied, then*

- **(i)** *$Q(x)$ is bounded and nondegenerated, i.e., At any extremal point $w\in Q(x)$, the collection of $\{H_j|j\in J(w)\}$, where $J(w)=\{j|H_j^Tw=f_j(x)\}$, are linearly independent.*
- **(ii)** *$Q(x)$ does not contain unnecessary polyhedral constraints for any $x\in \mathrm{dom}Q(x)$.*
- **(iii)** *$f:\Omega\to\Re^{n+1}$ is differentiable at x in a direction v.*

Proof. It is clear from $AM1\sim AM4$ and Lemma 3.2–3.3. □

Let $B(x,r)=\{w|\langle r,w\rangle=\delta^*(r|Q((x))\}$ for $r\in\bar{S}\subset\Re^{n-p}$ where $\bar{S}$ is the unit ball in $\Re^{n-p}$. If $w\in B(x,r)$, then we define $J_r(z)=\{j|\langle H_j,w\rangle=f_j(x),\ j=1,2,\cdots,n-p\}$ where $H_j=(-A_{IN}A_{IB}^{-1}\quad I_{m-p})a_{p+j}$, $f_j(x)=h_{p+j}(x)-a_{p+j}^T(\begin{matrix}A_{IB}^{-T}h_I(x)\\0\end{matrix})$ for $j=1,2,\cdots,n-p$.

Without loss of generality, assume that $J_r(w)=\{1,2,\cdots,q\}$.

Define

$$\begin{aligned}B_{J_r(w)}&=(-A_{IN}A_{IB}^{-1}\quad I_{m-p})(a_{p+1},a_{p+2},\cdots,a_{p+q}),\\ \bar{B}_{J_r(w)}&=B_{J_r(w)}^TB_{J_r(w)}.\end{aligned}$$

Theorem 3.5. *If $AM1\sim AM4$ are satisfied, and a linear programming is given by formula (27), then*

$$\rho_H(\,S(x+\alpha v),\,S(x)\,)\le\gamma\,\|\,h(x+\alpha v)-h(x)\,\|,$$

where

$$\begin{aligned}\gamma&=\max_{u\in S}\max_{J_r(w)\in\mathcal{F}(x,r)}\|\,(A_I\quad A_{J_r(w)})^+u\,\|,\\ \mathcal{F}(x,r)&=\{\,J_r(w)\,|\,J_r(w)=\{\,i\,|\,\langle H_i,w\rangle=f_i(x),\quad w\in B(x,r)\,\}\,\},\\ r&=\|(-A_{IN}A_{IB}^{-1}\quad I_{m-p})u\|^{-1}(-A_{IN}A_{IB}^{-1}\quad I_{m-p})u.\end{aligned}$$

Proof. We have from Theorem 2.8 that

$$
\begin{aligned}
&\rho_H(S(x+\alpha v), S(x)) \\
= \ &\rho_H(\begin{pmatrix} A_{IB}^{-T} h_I(x+\alpha v) \\ 0 \end{pmatrix} + \begin{pmatrix} -A_{IB}^{-T} A_{IN}^T \\ I_{m-p} \end{pmatrix} Q(x+\alpha v), \begin{pmatrix} A_{IB}^{-T} h_I(x) \\ 0 \end{pmatrix} \\
&+\begin{pmatrix} -A_{IB}^{-T} A_{IN}^T \\ I_{m-p} \end{pmatrix} Q(x)) \\
= \ &\max_{u\in S} \mid \delta^*(u|\begin{pmatrix} A_{IB}^{-T} h_I(x+\alpha v) \\ 0 \end{pmatrix} + \begin{pmatrix} -A_{IB}^{-T} A_{IN}^T \\ I_{m-p} \end{pmatrix} Q(x+\alpha v)) \\
&-\delta^*(u|\begin{pmatrix} A_{IB}^{-T} h_I(x) \\ 0 \end{pmatrix} + \begin{pmatrix} -A_{IB}^{-T} A_{IN}^T \\ I_{m-p} \end{pmatrix} Q(x)) \mid \\
= \ &\max_{u\in S} \| \langle u, \begin{pmatrix} A_{IB}^{-T}(h_I(x+\alpha v) - h_I(x)) \\ 0 \end{pmatrix} \rangle + \delta^*((-A_{IN}A_{IB}^{-1} \quad I_{m-p})u \,|\, Q(x+\alpha v)) \\
&-\delta^*((-A_{IN}A_{IB}^{-1} \quad I_{m-p})u \,|\, Q(x)) \|.
\end{aligned}
$$

It follows from Lemma 3.4 and Theorem 2.9 that the set-valued mapping Q is differentiable at x in a direction v, and

$$
\begin{aligned}
Q_x^+(v) &= \{ g \,|\, Hg \le f'_+(x;v) + N_{x,v}\mathbf{e} \}, \\
Q_x^-(v) &= \{ g \,|\, Hg \le f'_-(x;v) + N_{x,v}\mathbf{e} \},
\end{aligned}
$$

where

$$
[f'_+(x;v)]^i = \begin{cases} [f'(x;v)]^i & \text{if } [f'(x;v)]^i > 0 \\ 0 & \text{otherwise} \end{cases} \qquad i \in \{1, 2, \cdots, n-p\},
$$

$$
[f'_-(x;v)]^i = \begin{cases} -[f'(x;v)]^i & \text{if } [f'(x;v)]^i < 0 \\ 0 & \text{otherwise} \end{cases} \qquad i \in \{1, 2, \cdots, n-p\}.
$$

For $r = \|(-A_{IN}A_{IB}^{-1} \quad I_{m-p})u\|^{-1}(-A_{IN}A_{IB}^{-1} \quad I_{m-p})u \in \bar{S}$ and $w \in B(x,r)$, following the proof line in Theorem 2.9, we obtain that

$$
\begin{aligned}
\delta^*(r|Q_x^+(v)) &= \langle (B_{J_r(w)}\bar{B}_{J_r(w)}^{-1})^T r, \bar{f}'_+(x;v) + N_{x,v}\bar{\mathbf{e}} \rangle, \\
\delta^*(r|Q_x^-(v)) &= \langle (B_{J_r(w)}\bar{B}_{J_r(w)}^{-1})^T r, \bar{f}'_-(x;v) + N_{x,v}\bar{\mathbf{e}} \rangle,
\end{aligned}
$$

where

$$[\bar{f}'_+(x;v)]^i = \begin{cases} [f'(x;v)]^i & \text{if } [f'(x;v)]^i > 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad i \in J_r(w),$$

$$[\bar{f}'_-(x;v)]^i = \begin{cases} -[f'(x;v)]^i & \text{if } [f'(x;v)]^i < 0 \\ \\ 0 & \text{otherwise} \end{cases} \qquad i \in J_r(w).$$

$\bar{\mathbf{e}} = (1, 1, \cdots, 1) \in \Re^{|J_r(w)|}$, $|J_r(w)|$ denotes the number of elements of the index set $J_r(w)$. Then $\delta^*(r|Q_x^+(v)) - \delta^*(r|Q_x^-(v)) = \langle P(r), f'(x;v)\rangle$, where $P(r) = [r(B_{J_r(w)}\bar{B}^{-1}_{J_r(w)}), 0, \cdots, 0]^T$. By Taylor expansion, we have that

$$\begin{aligned} \delta^*(r|Q(x+\alpha v)) &= \delta^*(r|Q(x)) + \alpha[\delta^*(r|Q_x^+(v)) - \delta^*(r|Q_x^-(v))] + o_{x,v,r}(\alpha), \\ \bar{f}(x+\alpha v) &= \bar{f}(x) + \alpha \bar{f}'(x;v) + \bar{o}_{x,v}(\alpha), \\ f(x+\alpha v) &= f(x) + \alpha f'(x;v) + o_{x,v}(\alpha). \end{aligned}$$

Following the line of proof in Theorem 2.9, one has that

$$\begin{aligned} &\delta^*(r|Q(x+\alpha v)) - \delta^*(r|Q(x)) - \alpha[\delta^*(r|Q_x^+(v)) - \delta^*(r|Q_x^-(v))] \\ =\ &\langle (B_{J_r(w)}\bar{B}^{-1}_{J_r(w)})^T r, \bar{o}_{x,v}(\alpha)\rangle \\ =\ &\langle P(r), o_{x,v}(\alpha)\rangle. \end{aligned}$$

Therefore, one obtains that

$$\begin{aligned} &\alpha[\delta^*(r|Q_x^+(v)) - \delta^*(r|Q_x^-(v))] + o_{x,v,r}(\alpha) \\ =\ &\langle P(r), \alpha f'(x;v) + o_{x,v}(\alpha)\rangle \\ =\ &\langle P(u), f(x+\alpha v) - f(x)\rangle \\ =\ &P(r)^T(-A^T_{\bar{I}N}\begin{pmatrix} A^{-T}_{IB} \\ 0 \end{pmatrix} \quad I_{m-p})(h(x+\alpha v) - h(x)). \end{aligned}$$

Let $A^T_{\bar{I}} = \begin{pmatrix} \tilde{A}^T_{\bar{I}B} & \tilde{A}^T_{\bar{I}N} \\ \hat{A}^T_{\bar{I}B} & \hat{A}^T_{\bar{I}N} \end{pmatrix}$. Computing

$$\begin{aligned} &P(r)^T(-A^T_{\bar{I}N}\begin{pmatrix} A^{-T}_{IB} \\ 0 \end{pmatrix} \quad I_{m-p}) \\ =\ &\|(-A_{IN}A^{-1}_{IB} \quad I_{m-p})u\|^{-1}u\begin{pmatrix} -A^{-T}_{IB}A^T_{IN} \\ I_{m-p} \end{pmatrix}(\tilde{A}_{\bar{I}N} - A_{IN}A^{-1}_{IB}\tilde{A}_{\bar{I}B}) \\ &\cdot[(\tilde{A}^T_{\bar{I}N} - \tilde{A}^T_{\bar{I}B}A^{-T}_{IB}A^T_{IN})(\tilde{A}_{\bar{I}N} - A_{IN}A^{-1}_{IB}\tilde{A}_{\bar{I}B})]^{-1}(-\tilde{A}^T_{\bar{I}B}A^{-T}_{IB} \quad I_q \quad 0_{q\times(n-p-q)}) \end{aligned}$$

leads to

$$\begin{aligned}
&\alpha[\delta^*(r|Q_x^+(v)) - \delta^*(r|Q_x^-(v))] + o_{x,v,r}(\alpha) \\
= \;& \|(-A_{IN}A_{IB}^{-1} \quad I_{m-p})u\|^{-1}u\begin{pmatrix} -A_{IB}^{-T}A_{IN}^T \\ I_{m-p}\end{pmatrix}(\tilde{A}_{\bar{I}N} - A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B}) \\
&\cdot[(\tilde{A}_{\bar{I}N}^T - \tilde{A}_{\bar{I}B}^T A_{IB}^{-T}A_{IN}^T)(\tilde{A}_{\bar{I}N} - A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B})]^{-1}(-\tilde{A}_{\bar{I}B}^T A_{IB}^{-T} \quad I_q \quad 0_{q\times(n-p-q)}) \\
&\cdot(h(x+\alpha v) - h(x)).
\end{aligned}$$

Consequently, we have the following inequality

$$\begin{aligned}
&\delta^*(r|Q(x+\alpha v)) - \delta^*(r|Q(x)) \\
= \;& \alpha[\delta^*(r|Q_x^+(v)) - \delta^*(r|Q_x^-(v))] + o_{x,v,r}(\alpha) \\
= \;& \|(-A_{IN}A_{IB}^{-1} \quad I_{m-p})u\|^{-1}u^T\begin{pmatrix} -A_{IB}^{-T}A_{IN}^T \\ I_{m-p}\end{pmatrix}(\tilde{A}_{\bar{I}N} - A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B}) \\
&\cdot[(\tilde{A}_{\bar{I}N}^T - \tilde{A}_{\bar{I}B}^T A_{IB}^{-T}A_{IN}^T)(\tilde{A}_{\bar{I}N} - A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B})]^{-1}(-\tilde{A}_{\bar{I}B}^T A_{IB}^{-T} \quad I_q \quad 0_{q\times(n-p-q)}) \\
&\cdot(h(x+\alpha v) - h(x)).
\end{aligned}$$

Then one has that

$$\begin{aligned}
&\rho_H(S(x+\alpha v), S(x)) \\
= \;& \max_{u\in S} \| \langle u, \begin{pmatrix} A_{IB}^{-T}(h_I(x+\alpha v) - h_I(x)) \\ 0 \end{pmatrix}\rangle + u^T\begin{pmatrix} -A_{IB}^{-T}A_{IN}^T \\ I_{m-p}\end{pmatrix}(\tilde{A}_{\bar{I}N} - A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B}) \\
&\cdot[(\tilde{A}_{\bar{I}N}^T - \tilde{A}_{\bar{I}B}^T A_{IB}^{-T}A_{IN}^T)(\tilde{A}_{\bar{I}N} - A_{IN}A_{IB}^{-1}\tilde{A}_{\bar{I}B})]^{-1}(-\tilde{A}_{\bar{I}B}^T A_{IB}^{-T} \quad I_q \quad 0_{q\times(n-p-q)}) \\
&\cdot(h(x+\alpha v) - h(x)) \| \\
= \;& \max_{u\in S} \|[(A_I \quad A_{J_r(w)})^+u]^T(h(x+\alpha v) - h(x))\| \\
\le \;& \max_{u\in S} \|(A_I \quad A_{J_r(w)})^+u\| \cdot \|(h(x+\alpha v) - h(x))\| \\
\le \;& \max_{u\in S} \max_{J_r(w)\in\mathcal{F}(x,r)} \|(A_I \quad A_{J_r(w)})^+u\| \cdot \|(h(x+\alpha v) - h(x))\|.
\end{aligned}$$

□

Lemma 3.6. [6] *Let $A = (a_1, a_2, \cdots, a_m) \in \Re^{n\times m}$, $n \ge m$, be a column partitioned matrix, and $A_r = (a_1, a_2, \cdots, a_r)$. Then for $r = 1, 2, \cdots, m-1$, it follows that*

$$\sigma_1(A_{r+1}) \ge \sigma_1(A_r) \ge \ldots \ge \sigma_r(A_{r+1}) \ge \sigma_r(A_r) \ge \sigma_{r+1}(A_{r+1}).$$

□

Remark 3.7. A result, due to Li (1994), is similar to Theorem 3.5 (but stated without equality constraints $Cx = d$) if the feasible set is nonempty bounded (this assumption is relaxed in Li (1994)). The assumptions of the feasible set in this section is stronger than ones of Li (1994). However, the Lipschitz constant given here is smaller than or equal to the one, $\gamma_{2,2}(A, \emptyset)$, due to Li (1994) under the assumptions given in this paper. The relationship between the two Lipschitz constants is formulated in the following. □

Theorem 3.8. $\gamma \leq \gamma_{2,2}(A, \emptyset)$.

Proof. Let $\|(A_I \quad A_{J_u(w)})^+ u\| = \max_{J_r(w)\in\mathcal{F}(x,r)} \|(A_I \quad A_{J_r(w)})^+ u\|$. One has

$$\begin{aligned}\gamma &= \max_{u\in S} \max_{J_r(w)\in\mathcal{F}(x,r)} \|(A_I \quad A_{J_r(w)})^+ u\| \\ &= \max_{u\in S} \|(A_I \quad A_{J_u(w)})^+ u\| \\ &\leq \max_{u\in S} \|(A_I \quad A_{J_u(w)})^+\|.\end{aligned}$$

Let $\|(A_I \quad A_{J_1(w)})^+\| = \max_{u\in S} \|(A_I \quad A_{J_u(w)})^+\|$. From the definition of $J(z)$ and assumptions, we know that $|I| + |J_1(w)| \leq \mathrm{rank}(A)$. Take $K \subset \{1, 2, \cdots, n\} \setminus \{J_1(w) + I\}$ such that $\bar{J} = K \cup J_1(w) \cup I$ and $\mathrm{rank}(A_J) = \mathrm{rank}(A)$. Let σ_p be the smallest singular value of $(A_I \ \ A_{J_1(z)})$, σ_q the smallest singular value of $A_{\bar{J}}$. From Lemma 3.6, we have that $\sigma_p \geq \sigma_q$. Then $\sigma_p^{-1} \leq \sigma_q^{-1}$. Furthermore, $\|(A_I \quad A_{J_1(w)})^+\| \leq \|A_{\bar{J}}^+\|$. Let $\mathcal{F} = \{\tilde{I} | \mathrm{rank} A = |\tilde{I}|\}$. In consequence, we have that

$$\begin{aligned}\gamma &= \max_{u\in S} \max_{J_r(w)\in\mathcal{F}(x,r)} \|(A_I \quad A_{J_r(w)})^+ u\| \\ &= \max_{u\in S} \|(A_I \quad A_{J_u(w)})^+ u\| \\ &\leq \|(A_I \quad A_{J_1(w)})^+\| \\ &\leq \|A_{\bar{J}}^+\| \\ &\leq \max_{\tilde{I}\in\mathcal{F}} \|A_{\tilde{I}}^+\| \\ &= \gamma_{2,2}(A, \emptyset).\end{aligned}$$

□

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References

[1] J.-P. Aubin, H. Frankowska: Set-Valued Analysis, Birkhäuser, Boston (1990).

[2] H. T. Banks, M. Q. Jacobs: A differential calculus for multifunctions, J. of Mathematical Analysis and Application 29 (1970) 246–272.

[3] W. Cook, A. M. H. Gerards, A. Schrijver, E. Tardos: Sensitivity theorems in integer linear programming, Math. Programming 34 (1986) 251–264.

[4] M. R. Davidson: Stability of the extreme point set of a polyhedron, J. Optim. Theory Appl. 90 (1996) 357–380.

[5] V. F. Demyanov, A. M. Rubinov: On quasidifferentiable mappings, Math. Operationsforsch. Statist. Ser. Optimization 14 (1983) 3–21.

[6] G. H. Golub, C. F. van Loan: Matrix Computations, 3rd Edition, Johns Hopkins Univ. Press (1995).

[7] W. W. Hogan: Directional derivatives for extremal value functions with application to the completely convex case, Oper. Res. 21 (1973) 188–209.

[8] L. Hörmander: Sur la fonction d'appui des ensembles convexes dans un espace localement convexe, Arkiv för Matematik 3 (1954) 181–186.

[9] P. Huard (ed.): Point-to-set maps and mathematical programming, Mathematical Programming Study 10 (1979) 1–190.

[10] D. Klatte: Lipschitz continuity of infima and optimal solutions in parametric optimization and related topics, Math. Res. 35, Akademic-Verlag, Berlin (1987) 229–248.

[11] S. S. Kutateladze, A. M. Rubinov: Minkowski Duality and its Applications, Nauka, Moscow (1976).

[12] W. Li: The sharp Lipschitz constants for feasible and optimal solutions of a perturbed linear program, Linear Algebra Appl. 187 (1993) 15–40.

[13] W. Li: Sharp Lipschitz constants for basic optimal solutions and basic feasible solutions of linear programs, SIAM J. Control. Optim. 32 (1994) 140–153.

[14] O. L. Mangasarian, T.-H. Shiau: Lipschitz continuity of solutions of linear inequalities, programs, and complementarity problems, SIAM J. Control Optim. 25 (1987) 583–595.

[15] V. L. Makarov, A. M. Rubinov: Mathematical Theory of Economic Dynamics and Equilibria, Nauka, Moscow (1973).

[16] L. Minchenko, A. Volosevich: Strongly differentiable multifunctions and directional differentiability of marginal functions, in: Quasidifferentiability and Related Topics, V. Demyanov, A. Rubinov (eds.), Kluwer Academic Publishers, Netherlands (2000).

[17] D. Pallaschke, R. Urbański: Minimal pairs of compact convex sets with application to quasidifferential caculus, in: Quasidifferentiability and Related Topics, V. Demyanov, A. Rubinov (eds.), Kluwer Academic Publishers, Netherlands (2000).

[18] N. A. Percherskaya: Differentiability of set-valued mappings, in: Nonsmooth Problems of Control and Optimization, V. F. Demyanov (ed.), Leningrad University Press (1982) 128–147.

[19] N. A. Pecherskaya: Quasidifferentiable mappings and the differentiability of maximum functions, Mathematical Programming Study 29 (1986) 145–159.

[20] S. M. Robinson: Bounds for error in the solution set of a perturbed linear program, Linear Algebra Appl. 6 (1973) 69–81.

[21] A. M. Rubinov: Superlinear Multimappings and their Application to Economic and Mathematical Problems, Nauka (1980).

[22] Y. N. Tyurin: A simplified model of production planning, Econ. and Math. Methods 1 (1965) 391–410.