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Functional Logarithm in the Sense of Convex Analysis

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Let $\Gamma_\circ(H)$ denote the usual cone of lower semicontinuous convex functionals from a Hilbert H into $I\!R \cup \{+\infty\}$ not identically equal to $+\infty$. In this paper we introduce a map $\mathcal{L} : \Gamma_\circ(H) \to \overline{I\!R}^H$ ($\overline{I\!R} = I\!R \cup \{-\infty, +\infty\}$) which extends the Logarithm from positive invertible operators to convex functionals. Afterwards, we present a functional limited development of $\mathcal{L}(f_\sigma + \lambda f)$ where $f_\sigma := \frac{1}{2}\| \, . \, \|^2$, $f \in \Gamma_\circ(H)$ and λ goes to 0^+. The paper will be illustrated with some examples which justify the chosen terminology and the importance of this work.

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1. Introduction

Recently, the extension of the means from positive real numbers to positive operators has extensive several developments and applications [4], [5], [9]. Since calculations involving operator-valued functions are feasible with large computers, many authors have used the operator means in solving some scientific problems. Let a and b be two positive real numbers and define a map ϕ by $\phi(a,b) = (\frac{a+b}{2}, \sqrt{ab})$. If ϕ^k denotes the k-th iterate of ϕ, it is not hard to prove that there is a positive number $M = M(a,b)$ such that $\lim_{k\uparrow+\infty} \phi^k(a,b) = (M,M)$.The number M is called the "arithmetico-geometric mean of a and b". An explicit form of $M(a,b)$ is given by the following elliptic integral [9]:

$$(M(a,b))^{-1} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \left(a^2cos^2\theta + b^2sin^2\theta\right)^{-\frac{1}{2}} d\theta. \tag{1}$$

The limit of the above iterative process $(\phi^k(a,b))_k$ can be used to compute the integral (1) which is of mathematical and physical interest.

The generalization of the preceding notions and results to positive linear operators is sufficiently studied. More precisely, let $\sigma = (\sigma_1, \sigma_2, ..., \sigma_m) \in I\!R^m$ be a probability vector,

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i.e. $\sigma_i > 0$, $1 \leq i \leq m$ and $\sum_{i=1}^m \sigma_i = 1$, R. D. Nussbaum and J. E. Cohen [9] have suggested that a reasonable analogue of $\prod_{i=1}^m a_i^{\sigma_i}$ (a_i positive numbers) is

$$exp\left(\sum_{i=1}^{m} \sigma_i Log(A_i)\right), \tag{2}$$

where A_i, $1 \leq i \leq m$, are positive invertible operators, exp is the exponential of operator, i.e. $expA = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ and Log is the logarithm of operator defined by ([9], page 253)

$$LogA = \int_0^1 \frac{I - ((1-t)I + tA)^{-1}}{t} dt, \tag{3}$$

where I denotes the identity operator.
It will be an interesting attempt to extend the previous notions and results from positive invertible operators to convex functionals. In this sense, M.Atteia and M.Raïssouli [1] have recently introduced the "Convex Geometric Functional Mean" from an iterative process that operates recursively on pairs $(f, g) \in \Gamma_\circ(H) \times \Gamma_\circ(H)$ where $\Gamma_\circ(H)$ denotes the cone of proper lower semicontinuous convex functionals defined on a real Hilbert space H, with values in $I\!R \cup \{+\infty\}$. In particular, the functional "square root" $R : \Gamma_\circ(I\!R^m) \to \Gamma_\circ(I\!R^m)$ was introduced in a way that for $f(x) = \frac{1}{2} < Ax, x >$, where A is a symmetric positive matrix, $[R(f)](x) = \frac{1}{2} < \sqrt{A}x, x >$ where $\sqrt{A}$ is the positive square root of A.
Very recently, M.Raïssouli and M.Chergui [11] have constructed the arithmetico-geometric functional mean $f\overset{+}{\tau}g$ of two functions $f, g \in \Gamma_\circ(H)$ as follows: $f\overset{+}{\tau}g = \lim_{k\uparrow+\infty} \Phi^k(f, g)$ where Φ^k denotes the k-th iterate of Φ defined by $\Phi(f, g) = (\frac{f+g}{2}, f\tau g)$, $f\tau g$ is the convex geometric functional mean of f and g constructed in [1]. An explicit form of $f\overset{+}{\tau}g$ analogue to (1) is, for the moment, unknown. For this, the solution of the reverse problem of the square root functional is very interesting, i.e. what should be the analogue of A^2 for convex functionals. The standard definition of A^2 comes from the product AB of two operators A and B. The extension of this product from operators to convex functionals is not obvious and appears to be interesting.
For positive invertible operators, another (equivalent) definition of A^2 can be given by

$$A^2 = \lim_{\lambda\downarrow 0} \frac{A - (\lambda I + A^{-1})^{-1}}{\lambda}. \tag{4}$$

In [10] the authors suggested that an analogue of the second member of (4) to $f \in \Gamma_\circ(H)$ is

$$\lim_{\lambda\downarrow 0} \frac{f - (\lambda f_\sigma + f^*)^*}{\lambda}, \tag{5}$$

where f^* denotes the Fenchel-conjugate of f, i.e. $f^*(x^*) = \sup_{x\in H}\{< x^*, x > -f(x)\}$ and $f_\sigma := \frac{1}{2}\|.\|^2$ is the only self- conjugate function. They have established that the limit (5) exists and

$$\lim_{\lambda\downarrow 0} \frac{f(x) - (\lambda f_\sigma + f^*)^*(x)}{\lambda} = f_\sigma(p_f(x)), \tag{6}$$

for all $x \in int(dom\ f)$, where $p_f(x) = P_{\partial f(x)}(0)$ is the unique point projection from 0 to the nonempty closed convex $\partial f(x)$ subdifferential of f at x.

Another interesting problem is to extend formula (2) from operators to functionals. This extension is important in order, for example, to explicit the geometric mean $f\tau g$ of f and g and secondly to define the geometric mean of three or more convex functionals. More precisely, what should be the regular analogue of Logarithm and Exponential from operators to functionals.

The fundamental goal of this work is to describe a reasonable analogue extension of the Logarithm from the case that the variable is positive invertible operator to the case that the variable is $\Gamma_\circ(H)$-functional.We suggest that a reasonable analogue of $LogA$ given by (3) to $f \in \Gamma_\circ(H)$ is:

$$\forall x \in H \quad [\mathcal{L}(f)](x) = \int_0^1 \frac{f_\sigma(x) - ((1-t)f_\sigma + tf)^*(x)}{t} dt. \tag{7}$$

Using (7), we present some functional-valued properties of the map $\mathcal{L}$ analogous to that of operator-valued Logarithm. Note that in particular the mapping $\mathcal{L} : \Gamma_\circ(H) \longrightarrow \overline{I\!R}^H$ is increasing, concave (with respect to the pointwise order on $\overline{I\!R}^H$) and satisfies that: $\forall f \in \Gamma_\circ(H) \quad \mathcal{L}(f^*) = -\mathcal{L}(f)$.

In case $f(x) = \frac{1}{2} < Ax, x >$ where A is a symmetric positive invertible linear operator of H, we obtain $[\mathcal{L}(f)](x) = \frac{1}{2} < (LogA)x, x >$ with $LogA$ is the Logarithm of A defined by (3).

To determine how to obtain the solution of the reverse problem of the functional Logarithm (i.e. the functional Exponential) is not obvious and appears to be interesting.

This paper will be divided into three parts: We begin by recalling some basic results from convex analysis that will be needed later. Secondly, we introduce the notion of functional Logarithm and we study its elementary properties. This section will be completed by illustrating the theoretical results with some examples. In the fourth section, we present a second order functional limited development of $\mathcal{L}(f_\sigma + \lambda f)$ where $f_\sigma := \frac{1}{2}\|\ .\ \|^2$, λ goes to 0 ($\lambda > 0$) and $f \in \Gamma_\circ(H)$. Some consequences and examples are given.

2. Background material and preliminary results

Let us recall some basic notions and results from convex analysis which are needed throughout this paper. H will denote a real Hilbert space with inner product $< .,. >$ and its associated hilbertian norm $\|.\|$.

Let $\overline{I\!R} = I\!R \cup \{-\infty, +\infty\}$, we prolong the structure of $I\!R$ on $\overline{I\!R}$ by setting:
$\forall x \in I\!R$, $-\infty < x < +\infty$, $+\infty + x = +\infty$, $-\infty + x = -\infty$, $+\infty + (-\infty) = +\infty$, $0.(+\infty) = +\infty$.

We can define a partial ordering on $\overline{I\!R}^H$ by

$$f \leq g \quad \Longleftrightarrow \quad \forall x \in H \qquad f(x) \leq g(x).$$

Consider a function $f : H \to \overline{I\!R}$, f^* denotes the Fenchel conjugate of f defined by the formula

$$\forall x^* \in H \qquad f^*(x^*) := \sup_{x\in H}\{< x^*, x > - f(x)\}.$$

If $f, g : H \to \overline{I\!R}$ are two given functionals, it is easy to see that if $f \le g$ then $g^* \le f^*$. The following inequality

$$\forall \alpha \in]0,1[\qquad (\alpha f + (1-\alpha)g)^* \le \alpha f^* + (1-\alpha)g^*, \tag{8}$$

holds for every $f, g \in \overline{I\!R}^H$.

Denote $dom\ f$ the effective domain of f defined by $dom\ f := \{x \in H\ ;\ f(x) \in I\!R\}$, and $\Gamma_\circ(H)$ the cone of lower semicontinuous convex functionals from H into $I\!R \cup \{+\infty\}$ not identically equal to $+\infty$. We recall that if $f \in \Gamma_\circ(H)$ then, $f^* \in \Gamma_\circ(H)$ and $f^{**} = f$.
Let S be a subset of H, we denote by Ψ_S the indicator functional of S defined as follows: $\Psi_S(x) = 0$ if $x \in S$ and $\Psi_S(x) = +\infty$ otherwise.
$\Psi_S \in \Gamma_\circ(H)$ if and only if S is a nonempty closed convex subset of H.
In particular, if $B = B(0,1)$ is the closed unit ball of center 0, then $\Psi_B^* = \|.\|$ and $\|.\|^* = \Psi_B$.

We set below $f_\sigma := \frac{1}{2}\|\ .\ \|^2$ the only self-conjugate functional defined on H. It is easy to see the following results:

$$\forall f \in \overline{I\!R}^H \quad f - f^* \le (2f - 2f_\sigma) \tag{9}$$

$$\forall a > 0 \quad (af_\sigma)^* = \frac{1}{a} f_\sigma. \tag{10}$$

$$(f_\sigma + \Psi_S)^* = f_\sigma - \frac{1}{2} d_S^2, \tag{11}$$

where S is a nonempty closed convex subset of H and d_S is defined by $d_S(x) := \inf_{y\in S}\|x - y\|$.

We recall that, ([2], page 66) for all real $\lambda > 0$ and $f \in \Gamma_\circ(H)$, the functional $(\lambda f_\sigma + f)^*$ is finite everywhere and for all $x \in H$ there holds

$$(\lambda f_\sigma + f)^*(x) = \inf_{y\in H}\{f^*(y) + \frac{1}{2\lambda}\|y - x\|^2\}, \tag{12}$$

moreover the application $x \to (\lambda f_\sigma + f)^*(x)$ from H into $I\!R$ is Frechet-differentiable.

Let $f : H \to \widetilde{I\!R} = I\!R \cup \{+\infty\}$, we say that p is a subgradient of f at the point x if $x \in dom\ f$ and:

$$f(y) \ge f(x) + < y - x, p >, \text{ for all } y \in H.$$

The subdifferential of f at x is defined by:

$$\partial f(x) := \{p \in H;\ \forall y \in H \ \ f(y) \ge f(x) + < y - x, p >\}.$$

Let us denote by $int(dom\ f)$ the topological interior of $dom\ f$, we recall that if $f \in \Gamma_\circ(H)$ and $int(dom\ f)$ is nonempty then, for all $x \in int(dom\ f)$, f is continuous at x and $\partial f(x) \ne \emptyset$.

3. Functional Logarithm in convex analysis

This section is devoted to introduce the functional Logarithm $\mathcal{L} : \Gamma_\circ(H) \to \overline{I\!R}^H$ together with its elementary properties. First, we state the following theorem which will be needed in the sequel.

Theorem 3.1. *For all $f \in \Gamma_\circ(H)$ and all $\lambda > 0$, the following formula holds*

$$\lambda(\lambda f_\sigma + f^*)^* + (f_\sigma + \lambda f)^* = f_\sigma$$

Remark 3.2. Theorem 3.1 is proved in ([8], page 284) for $\lambda = 1$ and ([7], page XIII-6), ([10], Theorem 3.1) for $\lambda > 0$. This result can be written under the following "convex" form (take $\lambda = \dfrac{1-t}{t}$):

$$\forall f \in \Gamma_\circ(H) \quad \forall t \in]0,1[\quad (1-t)\left((1-t)f_\sigma + tf^*\right)^* + t\left(tf_\sigma + (1-t)f\right)^* = f_\sigma \tag{13}$$

Let $f \in \Gamma_\circ(H)$ be a fixed functional, the mapping: $t \to ((1-t)f_\sigma + tf)^*$ is derivable on $]0,1[$, (cf. [10], Corollary 3.1) hence continuous on $]0,1[$ and consequently we introduce the following definition:

Definition 3.3. The mapping $\mathcal{L} : \Gamma_\circ(H) \to \overline{I\!R}^H$ defined by

$$\forall x \in H \qquad [\mathcal{L}(f)](x) = \int_0^1 \frac{f_\sigma(x) - \left((1-t)f_\sigma + tf\right)^*(x)}{t} dt, \tag{14}$$

is called the Functional Logarithm in the sense of Convex Analysis.

Remark 3.4. The denomination "Functional Logarithm in the sense of Convex Analysis" will be justified by Examples 3.13 and 4.4 below and the properties of $\mathcal{L}$ discussed throughout the paper.

Proposition 3.5. *Relation (14) is equivalent to the following one*

$$\forall x \in H \qquad [\mathcal{L}(f)](x) = \frac{1}{2}\int_0^1 \frac{1}{t}\left\{\left((1-t)f_\sigma + tf^*\right)^* - \left((1-t)f_\sigma + tf\right)^*\right\}(x)dt \tag{15}$$

Proof. According to relation (14), we have for all $x \in H$

$$[\mathcal{L}(f)](x) = \frac{1}{2}\int_0^1 \left\{\left(\frac{f_\sigma}{t} - \frac{((1-t)f_\sigma + tf)^*}{t}\right) + \left(\frac{f_\sigma}{t} - \frac{((1-t)f_\sigma + tf)^*}{t}\right)\right\}(x)dt$$

By a simple variable change ($s = 1-t$), it is easy to see that

$$\begin{aligned}
[\mathcal{L}(f)](x) &= \frac{1}{2}\int_0^1 \left\{\left(\frac{f_\sigma}{1-t} - \frac{(tf_\sigma + (1-t)f)^*}{1-t}\right) + \left(\frac{f_\sigma}{t} - \frac{((1-t)f_\sigma + tf)^*}{t}\right)\right\}(x)dt \\
&= \frac{1}{2}\int_0^1 \frac{1}{t}\left\{\frac{f_\sigma - t\left(tf_\sigma + (1-t)f\right)^*}{1-t} - \left((1-t)f_\sigma + tf\right)^*\right\}(x)dt,
\end{aligned}$$

and with (13), the desired result follows. □

Proposition 3.6. *The map* $\mathcal{L}:\Gamma_\circ(H)\to\overline{I\!R}^H$ *satisfies the following properties:*

(i) $\mathcal{L}(0)=-\Psi_{\{0\}}$ *and* $\forall f\in\Gamma_\circ(H)$ $\forall\alpha\in I\!R$ $\mathcal{L}(f+\alpha)=\mathcal{L}(f)+\alpha.$

(ii) $\mathcal{L}(f_\sigma)=0$ *and* $\forall a>0$ $\mathcal{L}(af_\sigma)=(Loga)f_\sigma.$

(iii) $\forall f\in\Gamma_\circ(H),$ $\mathcal{L}(f^*)=-\mathcal{L}(f).$

Proof. (i) It is immediate from the definition of $\mathcal{L}$.

(ii) Relation $\mathcal{L}(f_\sigma)=0$ is obvious. Let us show that $\mathcal{L}(af_\sigma)=(Loga)f_\sigma$, for all $a>0$. To simplify the writing below, we omit the x in relations (14) and (15), so by (14) and (10), we have successively

$$\begin{aligned}\mathcal{L}(af_\sigma)&=\int_0^1\frac{f_\sigma-((1-t)f_\sigma+taf_\sigma)^*}{t}dt=\int_0^1\frac{f_\sigma-((1-t+ta)f_\sigma)^*}{t}dt\\&=\int_0^1\left\{\frac{1}{t}\left(1-\frac{1}{1-t+ta}\right)\right\}f_\sigma dt=\int_0^1\left\{\frac{a-1}{1+t(a-1)}\right\}f_\sigma dt\\&=\left[Log(1+t(a-1))\right]_0^1 f_\sigma=\ (Loga)f_\sigma.\end{aligned}$$

(iii) Let $f\in\Gamma_\circ(H)$, then $f^{**}=f$ and by (15) one has

$$\mathcal{L}(f^*)=\frac{1}{2}\int_0^1\frac{((1-t)f_\sigma+tf)^*-((1-t)f_\sigma+tf^*)^*}{t}dt,$$

since, for all $f\in\Gamma_\circ(H)$, the mapping $t\to((1-t)f_\sigma+tf)^*$ is with finite values[1] then

$$\mathcal{L}(f^*)=-\frac{1}{2}\int_0^1\frac{((1-t)f_\sigma+tf^*)^*-((1-t)f_\sigma+tf)^*}{t}dt\quad=-\mathcal{L}(f).$$ □

Proposition 3.7. *With respect to the pointwise ordering on* $\overline{I\!R}^H$*, the map* $\mathcal{L}$ *is:*

(i) *Increasing:* $\forall f,g\in\Gamma_\circ(H)$*; if* $f\ge g$ *then* $\mathcal{L}(f)\ge\mathcal{L}(g)$.

(ii) *Concave:* $\forall f,g\in\Gamma_\circ(H)$, $\forall\alpha\in]0,1[$, $\mathcal{L}(\alpha f+(1-\alpha)g)\ge\alpha\mathcal{L}(f)+(1-\alpha)\mathcal{L}(g).$

Proof. (i) Let $f,g\in\Gamma_\circ(H)$ such that $f\ge g$, we derive for all $t\in]0,1[$

$$-((1-t)f_\sigma+tf)^*\ge-((1-t)f_\sigma+tg)^*.$$

Using (14) we deduce $\mathcal{L}(f)\ge\mathcal{L}(g)$, so $\mathcal{L}$ is increasing.

(ii) Show that $\mathcal{L}$ is concave: Let $f,g\in\Gamma_\circ(H)$, and $\alpha\in]0,1[$, we can write

$$(1-t)f_\sigma+t(\alpha f+(1-\alpha)g)=\alpha[(1-t)f_\sigma+tf]+(1-\alpha)[(1-t)f_\sigma+tg].$$

Then according to inequality (8), we deduce that

$$((1-t)f_\sigma+t(\alpha f+(1-\alpha)g))^*\le\alpha((1-t)f_\sigma+tf)^*+(1-\alpha)((1-t)f_\sigma+tg)^*,$$

from which we observe that

$$\begin{aligned}&f_\sigma-((1-t)f_\sigma+t(\alpha f+(1-\alpha)g))^*\ge\\&\quad\ge\alpha\left[f_\sigma-((1-t)f_\sigma+tf)^*\right]+(1-\alpha)\left[f_\sigma-((1-t)f_\sigma+tg)^*\right],\end{aligned}$$

[1]Note that if $f,g:H\to\overline{I\!R}$ the equality $f-g=-(g-f)$ is not always true.

and by relation (14), finally

$$\mathcal{L}(\alpha f + (1-\alpha)g) \geq \alpha\mathcal{L}(f) + (1-\alpha)\mathcal{L}(g),$$

which completes the proof. □

Proposition 3.8. *For all f in $\Gamma_\circ(H)$, the following formula holds*

$$f_\sigma - f^* \leq \mathcal{L}(f) \leq f - f_\sigma \tag{16}$$

Proof. Using (14), combined with (8), we get that

$$\mathcal{L}(f) = \int_0^1 \frac{f_\sigma - ((1-t)f_\sigma + tf)^*}{t}dt \geq \int_0^1 \frac{f_\sigma - ((1-t)f_\sigma + tf^*)}{t}dt,$$

and simplifying we obtain $\mathcal{L}(f) \geq f_\sigma - f^*$, for all $f \in \Gamma_\circ(H)$.
Replace f by f^* in the latter inequality to obtain, with Proposition 2.2, (iii), $\mathcal{L}(f) \leq f - f_\sigma$. The proof is complete. □

From the above proposition, it is easy to see the following remarks.

Remark 3.9. (i) If $dom\ f = dom\ f^* = H$ then $dom\mathcal{L}(f) = H$.
(ii) For all $f \in \Gamma_\circ(H)$, $\mathcal{L}(f)$ is not identically equal to $+\infty$ (resp. $-\infty$).

Remark 3.10. The integral $\displaystyle\int_0^1 \frac{f_\sigma(x) - ((1-t)f_\sigma + tf)^*(x)}{t}\,dt$ converges to a finite real number for every $x \in dom\ f \cap dom\ f^*$. In particular, if $dom\ f = dom\ f^* = H$ then the above integral converges for all $x \in H$.

Corollary 3.11. *Let $f \in \Gamma_\circ(H)$, then one has*

(i) $\mathcal{L}(f) = 0 \iff f = f_\sigma$.
(ii) $\mathcal{L}(f) \geq 0 \iff f \geq f_\sigma$ (*resp.* $\mathcal{L}(f) \leq 0 \iff f \leq f_\sigma$).

Proof. It follows immediately from (16). □

Corollary 3.12. *Let $(f_n)_{n\in\mathbb{N}}$ be defined recursively by*

$$\forall n \geq 0 \quad f_{n+1} = \frac{1}{2}f_n + \frac{1}{2}f_n^*, \tag{17}$$

with $f_\circ \in \Gamma_\circ(H)$. Then $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f_σ on $dom\ f_\circ \cap dom\ f_\circ^*$.

Proof. Since $f_\circ \in \Gamma_\circ(H)$ then, by induction, we deduce that $f_n \in \Gamma_\circ(H)$ for all $n \in \mathbb{N}$. On the other hand, $\mathcal{L}$ is concave so

$$\forall n \in \mathbb{N} \quad \mathcal{L}(f_{n+1}) \geq \frac{1}{2}\mathcal{L}(f_n) + \frac{1}{2}\mathcal{L}(f_n^*),$$

and according to Proposition 3.6, (iii) and the structure of $\overline{\mathbb{R}}$, we get

$$\forall n \in \mathbb{N} \quad \mathcal{L}(f_{n+1}) \geq \frac{1}{2}\mathcal{L}(f_n) - \frac{1}{2}\mathcal{L}(f_n) \geq 0,$$

and by Corollary 3.11, we deduce that

$$\forall n \in I\!N \quad f_{n+1} \geq f_\sigma \quad \text{and} \quad \forall n \in I\!N \quad f^*_{n+1} \leq f_\sigma.$$

Using these previous inequalities, relation (17) yields

$$\forall n \in I\!N^* \quad 0 \leq f_{n+1} - f_\sigma = \frac{1}{2}(f_n - f_\sigma) + \frac{1}{2}(f^*_n - f_\sigma) \leq \frac{1}{2}(f_n - f_\sigma),$$

which implies that

$$\forall n \in I\!N^* \quad 0 \leq f_n - f_\sigma \leq \frac{1}{2^{n-1}}(f_1 - f_\sigma).$$

If $x \in dom\ f_1 = dom\ f_\circ \cap dom\ f^*_\circ$ then $\lim_n \frac{1}{2^{n-1}}(f_1 - f_\sigma)(x) = 0,$

from which the desired result follows. □

We now end this section by illustrating our theoretical results with three examples. We begin by an example which explains that the functional Logarithm introduced above contains that of positive invertible operators.

Example 3.13. 1. Let $H = I\!R$ and $\forall x \in I\!R \quad f(x) = \frac{1}{2}ax^2 \quad (a > 0),$
due to Proposition 3.6, (ii) we have

$$\forall x \in I\!R \qquad [\mathcal{L}(f)](x) = \frac{1}{2}(Loga)x^2.$$

2. More generally, let $H = I\!R^n$ be the usual n-dimensional Euclidean space and f given by:

$\forall x \in I\!R^n,\ f(x) = \frac{1}{2} < Ax, x >$ where A is a symmetric positive definite matrix of order n.

In this case, we recall that ([2], page 38) f^* is given by

$$\forall x \in I\!R^n \qquad f^*(x) = \frac{1}{2} < A^{-1}x, x >.$$

Using the fact that, for all $t \in]0,1[$, the matrix $(1-t)I + tA$ is also symmetric positive definite, then

we obtain for all $x \in I\!R^n$,

$$((1-t)f_\sigma + tf)^*(x) = \frac{1}{2} < ((1-t)I + tA)^{-1} x, x >,$$

and relation (14) implies that

$$\forall x \in I\!R^n \qquad [\mathcal{L}(f)](x) = \int_0^1 \frac{1}{2} < \frac{I - ((1-t)I + tA)^{-1}}{t} x, x > dt.$$

By (3) and the bilinearity of $< .,. >$, we deduce that

$$\forall x \in I\!R^n \quad [\mathcal{L}(f)](x) = \frac{1}{2} < (LogA)x, x > .$$

If P is an invertible $n \times n$ matrix such that

$$A = P^{-1} \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p \end{pmatrix} P, \quad \text{with } \lambda_i > 0 \ (i = 1, 2, ...p) \text{ the eigenvalues of } A,$$

then $\quad \forall x \in I\!R^n \quad [\mathcal{L}(f)](x) = \frac{1}{2} < (LogA)x, x >,$

$$\text{where} \quad LogA = P^{-1} \begin{pmatrix} Log\lambda_1 & 0 & \cdots & 0 \\ 0 & Log\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Log\lambda_p \end{pmatrix} P.$$

3. The above example for finite-dimensional Hilbert spaces works, by the same argument, for general Hilbert spaces: Let $f_A(x) = \frac{1}{2} < Ax, x >$, for all $x \in H$, be the quadratic form associated to the symmetric positive invertible operator A then $\mathcal{L}(f_A) = f_{LogA}$ where $LogA$ is the Logarithm of A defined by (3).

Relation (iii) of Proposition 3.6, $\mathcal{L}(f^*) = -\mathcal{L}(f)$, can be interesting in order, for example, to compute $\mathcal{L}(f)$ when f^* has a simple expression (as f). The next example explains this situation.

Example 3.14. Let us consider the space $H = \mathcal{M}_n$ of symmetric square matrices of order n equipped with its usual inner product $< A, B >= TraceAB$.
The variance of a given matrix $A \in \mathcal{M}_n$ is defined by:

$$(var)(A) = \frac{1}{n} < A, A > - \left(\frac{TraceA}{n}\right)^2.$$

It is easy to see that the functional $var : \mathcal{M}_n \longrightarrow I\!R$ is convex and hence $var \in \Gamma_\circ(\mathcal{M}_n)$.
The conjugate $(var)^* : \mathcal{M}_n \longrightarrow I\!R \cup \{+\infty\}$ of var is given by ([13], page 161):

$$(var)^*(B) = \frac{n}{2} f_\sigma(B) + \Psi_{\mathcal{T}}(B) = \begin{cases} \frac{n}{2} f_\sigma(B) & \text{if } TraceB = 0 \\ +\infty & \text{otherwise} \end{cases}$$

where $\mathcal{T} := \{A \in \mathcal{M}_n;\ TraceA = 0\}$ is a closed subspace of $\mathcal{M}_n$.

So,we begin by computing $\mathcal{L}((var)^*)$.

By virtue of (11) and an elementary manipulation, we get that

$$((1-t)f_\sigma + t(var)^*)^* = \left((1 - t + \frac{n}{2}t)f_\sigma + \Psi_{\mathcal{T}}\right)^* = \frac{1}{1 - t + \frac{n}{2}t}\left(f_\sigma - \frac{1}{2}d^2_{\mathcal{T}}\right).$$

Using formula (14) combined with a simple computation, we obtain

$$\mathcal{L}(var)(A) = -\mathcal{L}((var)^*)(A) = -(Log\frac{n}{2}).f_\sigma(A) - \Psi_{\mathcal{T}}(A) = \begin{cases} -(Log\frac{n}{2}).f_\sigma(A) & \text{if } A \in \mathcal{T} \\ -\infty & \text{otherwise} \end{cases}$$

Example 3.15. 1. Let S be a nonempty closed convex subset of H and $f = \Psi_S$ the indicator function of S.
By (11) and a routine transformation, we have

$$((1-t)f_\sigma + t\Psi_S)^* = ((1-t)f_\sigma + \Psi_S)^* = \frac{1}{1-t}\left(f_\sigma - \frac{1}{2}(d_{(1-t)S})^2\right),$$

which, with relation (14), yields

$$\mathcal{L}(\Psi_S) = \int_0^1 \left\{-\frac{1}{1-t}f_\sigma + \frac{1}{2}\frac{1}{t(1-t)}(d_{(1-t)S})^2\right\} dt,$$

or equivalently (by a variable change $s = 1-t$)

$$\mathcal{L}(\Psi_S) = \frac{1}{2}\int_0^1 \frac{1}{t(1-t)}\left((d_{tS})^2 - f_\sigma\right) dt. \tag{18}$$

In particular, if $S = B = B(0,1)$, the united ball, Proposition 3.6, (iii) and relation (18) give (since $\Psi_B^* = \|.\|$):

$$\mathcal{L}(\|.\|) = \frac{1}{2}\int_0^1 \frac{1}{t(1-t)}\left(f_\sigma - (d_{tB})^2\right) dt.$$

2. With the above notations, assume that S is a closed subspace of H and $f = \Psi_S$. It is known ([12], pages 40, 50) that $f^* = \Psi_{S^\perp}$ where $S^\perp = \{x^* \in H;\ < x^*, x >= 0\}$ is the orthogonal of S.
Taking $\lambda = 1$ and $f = \Psi_S$, Theorem 3.1 combined with (11) yields the celebrate identity $d_S^2 + d_{S^\perp}^2 = \|.\|^2$, and relation (18) gives (since $tS = S$)

$$\mathcal{L}(\Psi_S) = \frac{1}{4}\int_0^1 \frac{1}{t(1-t)}(d_S^2 - d_{S^\perp}^2) dt,$$

which becomes in a different way

$$[\mathcal{L}(\Psi_S)](x) = \begin{cases} -\infty & \text{if } d_S(x) > d_{S^\perp}(x) \\ 0 & \text{if } d_S(x) = d_{S^\perp}(x) \\ +\infty & \text{if } d_S(x) < d_{S^\perp}(x) \end{cases}$$

or equivalently

$$\mathcal{L}(\Psi_S) = \Psi_{S^+} - \Psi_{S^-},$$

where $S^+ = \{x \in H;\ d_S(x) \geq d_{S^\perp}(x)\}$ and $S^- = \{x \in H;\ d_S(x) \leq d_{S^\perp}(x)\}$.

4. Functional limited developments (F.L.D) of $\mathcal{L}(f_\sigma + \lambda f)$

We preserve the same notations as previous. In this section, we shall give a second order development of $\mathcal{L}(f_\sigma + \lambda f)$, $\lambda \to 0^+$, which extends that of operators: $Log(I + \lambda A) = \lambda A - \frac{1}{2}\lambda^2 A^2 + \lambda^2 \theta_\lambda(A)$, with $\theta_\lambda(A)$ tends to 0 as λ. We begin by recalling the following lemma.

Lemma 4.1. ([10]) *Let $f \in \Gamma_\circ(H)$, then for all $x \in dom\ f$ one has*

$$(f_\sigma + \lambda f)^*(x) = f_\sigma(x) - \lambda f(x) + \lambda(\theta_\lambda(f))(x), \tag{19}$$

where $\theta_\lambda(f)$ tends pointwise to 0 when λ goes to 0^+.
If moreover int(dom f) is nonempty, the functional limited development (FLD2):

$$(f_\sigma + \lambda f)^*(x) = f_\sigma(x) - \lambda f(x) + \lambda^2 f_\sigma(p_f(x)) + \lambda^2(\theta_\lambda(f))(x), \tag{20}$$

holds for all $x \in int(dom\ f)$, where $\lim_{\lambda \downarrow 0} \theta_\lambda(f) = 0$ for the pointwise convergence and $p_f(x) = P_{\partial f(x)}(0)$ is the unique point projection from 0 to the nonempty closed convex $\partial f(x)$.

By applying the above lemma, we will prove the following theorem:

Theorem 4.2. *Let $f \in \Gamma_\circ(H)$, for each $x \in dom\ f$ we have*

$$[\mathcal{L}(f_\sigma + \lambda f)](x) = \lambda f(x) + \lambda(\theta_\lambda(f))(x), \tag{21}$$

where $\theta_\lambda(f)$ tends pointwise to 0 when $\lambda \to 0^+$.
If moreover int(dom f) is nonempty, the following development

$$[\mathcal{L}(f_\sigma + \lambda f)](x) = \lambda f(x) - \frac{1}{2}\lambda^2 f_\sigma(p_f(x)) + \lambda^2(\theta_\lambda(f))(x), \tag{22}$$

holds for every $x \in int(dom\ f)$, where $\lim_{\lambda \downarrow 0} \theta_\lambda(f) = 0$ in the pointwise convergence and $p_f(x) = P_{\partial f(x)}(0)$ is the unique point projection from 0 to the nonempty closed convex $\partial f(x)$.

Proof. Note that, first, the $\theta_\lambda(f)$ in (19) and (20) (resp. (21) and (22)) is not the same but only to simplify the writing below. For the same reason we omit the x in all relation and we write θ instead of $\theta_\lambda(f)$. Let us prove (21). According to Proposition 3.8, one has

$$f_\sigma - (f_\sigma + \lambda f)^* \le \mathcal{L}(f_\sigma + \lambda f) \le \lambda f,$$

and using (19), we get

$$\lambda f + \lambda \theta_\lambda(f) \le \mathcal{L}(f_\sigma + \lambda f) \le \lambda f,$$

which implies (21).

We now prove (22). From relation (14), it follows

$$\mathcal{L}(f_\sigma + \lambda f) = \int_0^1 \left\{ \frac{f_\sigma}{t} - \frac{(f_\sigma + \lambda t f)^*}{t} \right\} dt.$$

By virtue of (20), we obtain

$$\begin{aligned}
\mathcal{L}(f_\sigma + \lambda f) &= \int_0^1 \left\{ \frac{f_\sigma}{t} - \frac{f_\sigma - \lambda t f + \lambda^2 t^2 f_\sigma(p_f) + \lambda^2 t^2 \theta(t)}{t} \right\} dt \\
&= \int_0^1 \left\{ \lambda f - \lambda^2 t f_\sigma(p_f) - \lambda^2 t \theta(t) \right\} dt \\
&= \lambda f - \frac{\lambda^2}{2} f_\sigma(p_f) - \lambda^2 \int_0^1 t\theta(t) dt.
\end{aligned} \tag{23}$$

On the other hand, by combining Theorem 3.1 and relation (20), we infer that

$$\lambda t(\lambda t f_\sigma + f^*)^* = \lambda t f - \lambda^2 t^2 f_\sigma(p_f) - \lambda^2 t^2 \theta(t),$$

from which we deduce

$$\forall x \in int(dom\ f) \quad [\theta_\lambda(f)(t)](x) := [\theta(t)](x) = \frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t} - f_\sigma(p_f(x)). \quad (24)$$

To complete the proof we need the next lemma.

Lemma 4.3. *$(\theta_\lambda(f))_\lambda$ converges uniformly (with respect to t) to 0 when λ tends to 0^+, i.e.*

$$\forall x \in int(dom\ f) \quad \lim_{\lambda \downarrow 0} \sup_{t \in]0,1[} |[\theta_\lambda(f)(t)](x)| = 0.$$

Proof of Lemma 4.3. For each $x \in int(dom\ f)$, recall that we have $\partial f(x) \neq \emptyset$. Let $x^* \in \partial f(x)$ then, by definition, $f(x) - f(y) \leq \langle x^*, x - y \rangle$ for all $y \in H$.
Cauchy-Schwartz's inequality implies that, for all $y \in H$ and $0 < t < 1$,

$$f(x) - f(y) \leq \frac{1}{\lambda t} \|\lambda t x^*\| \|x - y\|,$$

and by Young's inequality

$$f(x) - f(y) \leq \frac{\lambda t}{2} \|x^*\|^2 + \frac{1}{2\lambda t} \|y - x\|^2,$$

which obviously yields

$$f(x) - f(y) - \frac{1}{2\lambda t} \|y - x\|^2 \leq \frac{\lambda t}{2} \|x^*\|^2.$$

If we take the supremum for all $y \in H$, this previous inequality gives

$$\sup_{y \in H} \{ f(x) - f(y) - \frac{1}{2\lambda t} \|y - x\|^2 \} \leq \frac{\lambda t}{2} \|x^*\|^2,$$

or equivalently

$$f(x) - \inf_{y \in H} \{ f(y) + \frac{1}{2\lambda t} \|y - x\|^2 \} \leq \frac{\lambda t}{2} \|x^*\|^2.$$

Thanks to relation (12), we conclude that

$$\forall x^* \in \partial f(x) \qquad f(x) - (\lambda t f_\sigma + f^*)^*(x) \leq \frac{\lambda t}{2} \|x^*\|^2. \quad (25)$$

Further, it is well known that, ([10], Lemma 3.2)

$$\lim_{\lambda \downarrow 0} \nabla f_\lambda(x) = \lim_{\lambda \downarrow 0} \nabla(\lambda f_\sigma + f^*)^*(x) = p_f(x) \in \partial f(x). \quad (26)$$

If we take $x^* = p_f(x)$ in (25) we obtain

$$\frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t} \leq f_\sigma(p_f(x)). \quad (27)$$

Moreover, we know that ([10], Theorem 3.1)

$$f_\sigma((\nabla f_{\lambda t}(x))) \leq \frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t},$$

which, with (27), gives

$$f_\sigma((\nabla f_{\lambda t}(x))) - f_\sigma(p_f(x)) \leq \frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t} - f_\sigma(p_f(x)) \leq 0,$$

and therefore

$$\left| \frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t} - f_\sigma(p_f(x)) \right| \leq f_\sigma(p_f(x)) - f_\sigma(\nabla f_{\lambda t}(x)). \tag{28}$$

Recalling that ([3], page II.10) $(\|\nabla f_\lambda\|)_\lambda$ is decreasing, it becomes for all $t \in]0,1[$, and $\lambda > 0$

$\|\nabla f_\lambda\| \leq \|\nabla f_{\lambda t}\|$ and thus $-f_\sigma(\nabla f_{\lambda t}) \leq -f_\sigma(\nabla f_\lambda)$.

It follows that

$$\left| \frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t} - f_\sigma(p_f(x)) \right| \leq f_\sigma(p_f(x)) - f_\sigma(\nabla f_\lambda(x)),$$

and

$$\sup_{t \in]0,1[} \left| \frac{f(x) - (\lambda t f_\sigma + f^*)^*(x)}{\lambda t} - f_\sigma(p_f(x)) \right| \leq f_\sigma(p_f(x)) - f_\sigma(\nabla f_\lambda(x)). \tag{29}$$

In (29) use (24) to write

$$\forall x \in int(dom\ f) \quad \sup_{t \in]0,1[} |[\theta_\lambda(f)(t)](x)| \leq f_\sigma(p_f(x)) - f_\sigma(\nabla f_\lambda(x)). \tag{30}$$

By virtue of relation (26) and the continuity of f_σ, the second member of (30) tends to 0 as λ. The proof of Lemma 4.3 is complete.

To end the proof of Theorem 4.2, we can write by virtue of Lemma 4.3:

$$\lim_{\lambda \downarrow 0} \int_0^1 t(\theta_\lambda(f))(t)dt = \int_0^1 \lim_{\lambda \downarrow 0} t(\theta_\lambda(f))(t)dt = \int_0^1 t \lim_{\lambda \downarrow 0} (\theta_\lambda(f))(t)dt = 0,$$

which, with (23), gives the desired result. □

We now illustrate Theorem 4.2 with the next example.

Example 4.4. 1. Let $H = I\!R$ and $\forall x \in I\!R \quad f(x) = \frac{1}{2}ax^2 \quad (a > 0)$.

Theorem 4.2 and Example 3.13, 1- give the following classical result:

$Log(1 + \lambda a) = \lambda a - \frac{1}{2}\lambda^2 a^2 + \lambda^2 \theta_\lambda(a)$, where $\theta_\lambda(a)$ tends to 0 when $\lambda \to 0$.

2. More generally, let A be a symmetric positive matrix of order n, and consider the functional

$$\forall x \in I\!R^n \quad f(x) := f_A(x) = \frac{1}{2} < Ax, x > .$$

Since f is convex and (Gâteaux) differentiable then $\partial f(x)=\{\nabla f(x)\}$,
A is symmetric gives $\partial f(x)=\{Ax\}$ and $f_\sigma(p_f(x))=\frac{1}{2}\|Ax\|^2=\frac{1}{2}<A^2x,x>$.

According to Theorem 4.2 and Example 3.13, 2., we can write

$$f_{Log(I+\lambda A)}=f_{\lambda A-\frac{1}{2}\lambda^2A^2}+\lambda^2\theta_\lambda(f), \tag{31}$$

from which we deduce that $\theta_\lambda(f)$ is a quadratic form, i.e. $\theta_\lambda(f)=f_{\theta_\lambda(A)}$ where $\theta_\lambda(A)$ is a symmetric matrix $\left(\text{because } \theta_\lambda(A)=\dfrac{Log(I+\lambda A)-\lambda A+\frac{1}{2}\lambda^2A^2}{\lambda^2}\right)$.

Then, relation (31) is equivalent to the following one

$$f_{Log(I+\lambda A)}=f_{\lambda A-\frac{1}{2}\lambda^2A^2+\lambda^2\theta_\lambda(A)} \tag{32}$$

From the relation $\theta_\lambda(f)=f_{\theta_\lambda(A)}$ and the fact that $[\theta_\lambda(f)](x)\longrightarrow_{\lambda\downarrow 0}0$ for all $x\in int(dom\ f)$ $=\mathbb{R}^n$, we deduce $<(\theta_\lambda(A))x,x>\longrightarrow_{\lambda\downarrow 0}0$ for all $x\in\mathbb{R}^n$, and by symmetry of $\theta_\lambda(A)$ we have $<(\theta_\lambda(A))x,y>\longrightarrow_{\lambda\downarrow 0}0$ for all $x,y\in\mathbb{R}^n$. It follows that $\forall x\in\mathbb{R}^n\ \ (\theta_\lambda(A))x\longrightarrow_{\lambda\downarrow 0}0$ in $\mathbb{R}^n$.
Using the fact that the matrices associated to the quadratic forms of (32) are symmetric we find the known development:

$$Log(I+\lambda A)=\lambda A-\frac{1}{2}\lambda^2A^2+\lambda^2\theta_\lambda(A).$$

Corollary 4.5. *Let $f\in\Gamma_\circ(H)$ such that $int(dom\ f)$ is nonempty. Then, for each s in a neighbourhood of 0, there holds*

$$\forall x\in int(dom\ f)\quad \frac{d}{ds}\mathcal{L}(f_\sigma+sf)(x)=(sf_\sigma+f^*)^*(x)+s(\theta_s(f))(x),$$

where $\theta_s(f)$ tends (pointwise) to 0 when $s\to 0^+$.

Proof. It follows by combining Theorem 3.1, Lemma 4.1 and Theorem 4.2. □

Proposition 4.6. *Let $f\in\Gamma_\circ(H)$ be a fixed functional such that $int(dom\ f)$ is nonempty. Then the mapping $s\longrightarrow\mathcal{L}(sf_\sigma+f)$ defined from $\mathbb{R}_+^*:=]0,+\infty[$ into $\overline{\mathbb{R}}^H$ is differentiable and*

$$\forall x\in int(dom\ f)\quad \frac{d}{ds}\mathcal{L}(sf_\sigma+f)(x)=\int_0^1 f_\sigma\left(\nabla((1-t+ts)f_\sigma+tf)^*\right)(x)dt \tag{33}$$

Proof. Here also, we omit the x and we write θ instead of $\theta_\lambda(f)$. Let $s\in\mathbb{R}_+^*$ and $\lambda\to 0^+$, we have

$$\begin{aligned}\mathcal{L}((\lambda+s)f_\sigma+f)&=\int_0^1\left\{\frac{f_\sigma}{t}-\frac{((1-t)f_\sigma+t(\lambda+s)f_\sigma+tf)^*}{t}\right\}dt\\&=\int_0^1\left\{\frac{f_\sigma}{t}-\frac{(\lambda tf_\sigma+(1-t+ts)f_\sigma+tf)^*}{t}\right\}dt.\end{aligned} \tag{34}$$

Setting $\overline{f} = (1-t+ts)f_\sigma + tf$, it is clear that $\overline{f} \in \Gamma_\circ(H)$ and $dom\ \overline{f} = dom\ f$. Using relation (20) with Theorem 3.1, we obtain

$$(\lambda t f_\sigma + \overline{f})^* = \overline{f}^* - \lambda t f_\sigma(p_{\overline{f}^*}) - \lambda t \tilde{\theta}(t),$$

which, with (34), yields

$$\begin{aligned}\mathcal{L}((\lambda+s)f_\sigma + f) &= \int_0^1 \left\{ \frac{f_\sigma}{t} - \frac{((1-t)f_\sigma + stf_\sigma + tf)^*}{t} + \lambda f_\sigma(p_{\overline{f}^*}) + \lambda\tilde{\theta}(t) \right\} dt \\ &= \mathcal{L}(sf_\sigma + f) + \lambda \int_0^1 f_\sigma\left(\nabla((1-t+ts)f_\sigma + tf)^*\right) dt + \lambda \int_0^1 \tilde{\theta}(t)dt,\end{aligned}$$

where

$$[\tilde{\theta}_\lambda(f)](t) := \tilde{\theta}(t) = \frac{\overline{f}^* - (\lambda t f_\sigma + \overline{f})^*}{\lambda t} - f_\sigma(p_{\overline{f}^*}).$$

Lemma 4.3 implies that

$$\lim_{\lambda \downarrow 0} \int_0^1 [\tilde{\theta}_\lambda(f)](t)dt = \int_0^1 \lim_{\lambda \downarrow 0} [\tilde{\theta}_\lambda(f)](t)dt = 0.$$

Let us observe that setting $\displaystyle\int_0^1 [\tilde{\theta}_\lambda(f)](t)dt = \theta_\lambda(f)$, it follows

$$\mathcal{L}((\lambda+s)f_\sigma + f) = \mathcal{L}(sf_\sigma + f) + \lambda \int_0^1 f_\sigma\left(\nabla((1-t+ts)f_\sigma + tf)^*\right) dt + \lambda\theta_\lambda(f),$$

and consequently

$$\frac{d}{ds}\mathcal{L}(sf_\sigma + f) = \int_0^1 f_\sigma\left(\nabla((1-t+ts)f_\sigma + tf)^*\right) dt. \qquad \square$$

Remark 4.7. Let A be a symmetric positive operator from H into H and take $f(x) = \frac{1}{2} < Ax, x >$ for all $x \in H$. In this case, relation (33) yields, after a simple computation, the known classical result:

$$\forall s \in I\!R_+^* \qquad \frac{d}{ds} Log(sI + A) = (sI + A)^{-1},$$

and the "Logarithm" terminology is again justified.

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