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# Remarks on Differentiability of Metric Projections onto Cones of Nonnegative Functions

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The differentiability properties of the metric projections onto the cones of nonnegative functions in $L^p(0,1)$ and in $W^{1,p}(0,1)$ are considered. It is shown that the metric projection mapping is Bouligand differentiable in $L^p(0,1)$, but it is not Bouligand differentiable in $W^{1,p}(0,1)$.

*Keywords:* Metric projection, cones of nonnegative funcions, Bouligand differentiability

## 1. Preliminaries

Differentiability properties of the matric projection onto closed convex sets are of interest in variational inequalities and optimal control problems. The examples constructed by J. Kruskal [6] and A. Shapiro [19] show that, in general such a projection is not directionally differentiable, even in finite dimensional spaces. An important contribution was made by A. Haraux [4] and F. Mignot [12], who proved existence and characterized the conical differentials of the projection for a class of closed convex sets in Hilbert spaces. This direction of research has been further developed (see e.g., [14]) and found numerous applications in sensitivity analysis of variational inequalities and optimal control (see e.g., [5, 9, 10, 20]). Recently, a quite general results on differentiability, in a weak sense, of solutions to variational inequalities were obtained by A. B. Levy [7].

This paper is devoted to studying a stronger type of differentiability, the so called *Bouligand differentiability*, of the maps of metric projection onto the cones of nonnegative functions in $L^p(0,1)$ and $W^{1,p}(0,1)$. Just these cones are connected with control and state constraints in optimal control problems for ODEs. Let us recall the notions of *conical* and *Bouligand* differentiability (see [12] and [15, 17], respectively):

**Definition 1.1.** Let $X$ and $Y$ be Banach spaces. A function $f: X \to Y$ is called *conically* differentiable at $x$ if there exists a positively homogeneous mapping $df(x): X \to Y$, called conical-derivative, with the property that, for every compact set $C \subset X$ and every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} &\|f(y) - f(x) - (df(x); y - x)\|_Y \le \epsilon \|y - x\|_X \\ &\text{for every } y \in C \text{ such that } \|y - x\|_X \le \delta. \end{aligned} \tag{1}$$

If (1) holds uniformly for all $y \in X, \|y - x\|_X \le \delta$, then we say that $f$ is *Bouligand*, or B-differentiable at $x$ and the corresponding B-derivative is denoted by $Df(x)$.

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Clearly, the notions of conical and B-differentials coincide in finite dimensional spaces. Moreover, if $Df(x)$ is linear, then it becomes Fréchet derivative. That is the reason, that sometimes B-derivative is called *directional* Fréchet derivative [2]. It turns out that the concept of B-differentiability is useful in sensitivity analysis of parametric mathematical programs [15, 16, 18]. Recently this concept was also used in sensitivity analysis of optimal control problems [3, 11]. In this paper, we are going to study B-differentiability properties of the metric projection onto the cone of nonnegative functions in $L^p(0,1)$) and in $W^{1,p}(0,1)$.

Our starting point are the differential properties of the metric projection onto closed convex cones in abstract Hilbert spaces derived in [12] (see also [4]). To recall these results, we have to start with some definitions. Let $X$ be a real Hilbert space, with the inner product denoted by $(\cdot,\cdot)$. Let $K \subset X$ be a closed convex cone and $P_K$ denote the metric projection onto $K$, i.e., for any $x \in X$, $P_K x \in K$ is a unique element such that

$$\|x - P_K x\|_X = \min_{y \in K} \|x - y\|_X.$$

By $[x]$ we donote the one-dimensional space generated by $x$ and by $[x]^\perp = \{y \in X \mid (x,y) = 0\}$ the subspace orthogonal to $[x]$. We will need the following notion (see [4]):

**Definition 1.2.** A closed convex cone $K \subset X$ is called *polyhedric*, if for any $x \in X$

$$\overline{(K + [P_K x]) \cap [x - P_K x]^\perp} = \overline{(K + [P_K x])} \cap [x - P_K x]^\perp. \tag{2}$$

Note that *polihedricity* is a natural extention of the polyhedral property in finite dimension. We will denote

$$\Xi(x) = \overline{(K + [P_K x])} \cap [x - P_K x]^\perp. \tag{3}$$

The following theorem of Mignot (see Theorem 2.1 in [12]) shows that the notion of polyhedricity plays an important role in differentiability properties of the projection onto a cone.

**Theorem 1.3.** *Let $K$ be a polyhedric cone in a Hilbert space $X$. Then the metric projection mapping $P_K : X \to K$ is conically differentiable and, for any $x, y \in X$,*

$$(dP_K x; y) = P_{\Xi(x)} y, \tag{4}$$

*where $P_{\Xi(x)}$ denotes the metric projection onto $\Xi(x)$.*

Haraux studied polyhedric properties of positive cones of Hilbert lattices and he proved the following result (Corollary 2 in [4]):

**Theorem 1.4.** *Let $X$ be a Hilbert lattice, with $K = \{x \in X \mid x \geq 0\}$ being closed. Denote $x^+ = \sup\{x, 0\}$. If there exists constant $M > 0$ such that*

$$\|x^+\|_X \leq M \|x\|_X \qquad \text{for all } x \in X,$$

*then the cone $K$ is polyhedric.*

It follows from Theorem 1.4 that the cones of nonnegative functions in $L^2(0,1)$ and $W^{1,2}(0,1)$ are polyhedric, i.e., by Theorem 1.3, the metric projection mappings onto these cones are conically differentiable. We are going to study Bouligand-differentiability property of the projection mappings onto these cones. We show that the metric projection onto the cone of nonnegative functions is B-differentiable in $L^p(0,1))$ for $p < \infty$ (but not for $p = \infty$) and it is *not* B-differentiable in $W^{1,p}(0,1)$ for any $p \in [1,\infty]$.

## 2. Metric projection in $L^p(0,1)$

In this section we investigate B-differentiability of the projection onto the cone of non-negative functions in $L^p(0,1)$. The analysis is performed for one-dimensional problems. However, the resuls can be extended to multi-dimensional case. Consider the cone

$$K^p = \{x \in L^p(0,1) \mid x(t) \geq 0 \quad \text{for a.a. } t \in [0,1]\}. \tag{5}$$

Clearly, for any $p \in [1,\infty)$ we have

$$(P_{K^p}x)(t) = x^+(t) := \sup\{x(t), 0\} \quad \text{for a.a. } t \in [0,1]. \tag{6}$$

For $p = \infty$, the projection is not unique. In the sequel, by $P_{K^\infty}x$ we will understand the selection given by (6). Note that, a full characterization of a tangent set to the cone of non-negative functions is $L^\infty$ is presented in [1]. For a given $x \in L^p(0,1)$ introduce the sets

$$\begin{aligned} N_+(x) &= \{t \in [0,1] \mid x(t) > 0\},\\ N_0(x) &= \{t \in [0,1] \mid x(t) = 0\},\\ N_-(x) &= \{t \in [0,1] \mid x(t) < 0\}. \end{aligned} \tag{7}$$

Define the cones

$$C^p(x) = \left\{ y \in L^p(0,1) \mid y(t) \begin{cases} \text{free} & \text{for a.a. } t \in N_+(x),\\ \geq 0 & \text{for a.a. } t \in N_0(x),\\ = 0 & \text{for a.a. } t \in N_-(x). \end{cases} \right\}. \tag{8}$$

Clearly, $C^2(x)$ coincide with the set $\Xi(x)$ defined in (3).

**Theorem 2.1.** *The mapping $P_{K^\infty} : L^\infty(0,1) \to L^p(0,1)$ is Bouligand differentiable for any $p \in [1,\infty)$ and for any $x, y \in L^\infty(0,1)$ the B-differential is given by*

$$(DP_{K^\infty}x; y) = P_{C^\infty(x)}y. \tag{9}$$

**Proof.** Clearly, the mapping $P_{C^\infty(x)} : L^\infty(0,1) \to L^p(0,1)$ is positively homogeneous. Hence to prove the theorem, it is enough to show that (1) holds for any $y \in X := L^\infty(0,1)$. In view of (8) we have

$$(P_{C^\infty(x)}y)(t) = \begin{cases} y(t) & \text{for a.a. } t \in N_+(x),\\ y^+(t) & \text{for a.a. } t \in N_0(x),\\ 0 & \text{for a.a. } t \in N_-(x). \end{cases} \tag{10}$$

Let us fix any $x \in L^\infty(0,1)$ and choose $\delta > 0$. Introduce the following subsets of the interval $[0,1]$:

$$\begin{aligned} \mathcal{A}_\delta &= \{t \in [0,1] \mid x(t) \geq \delta\}, & \mathcal{B}_\delta &= \{t \in [0,1] \mid x(t) \leq -\delta\},\\ \mathcal{C}_\delta &= \{t \in [0,1] \mid x(t) = 0\}, & \mathcal{D}_\delta &= \{t \in [0,1] \mid x(t) \in (-\delta,\delta)\}. \end{aligned} \tag{11}$$

Let $||y - x\|_\infty < \delta$. Consider

$$s(t) := (P_{K^\infty} y)(t) - (P_{K^\infty} x)(t) - (P_{C^\infty(x)}(y - x))(t)$$

on each of the subintervals $\mathcal{A}_\delta - \mathcal{D}_\delta$ successively.

Subset $\mathcal{A}_\delta$

In view of (6), (10) and (11), we have

$$(P_{K^\infty} y)(t) = y(t),\ (P_{K^\infty} x)(t) = x(t),\ (P_{C^\infty(x)}(y - x))(t) = y(t) - x(t),$$

hence

$$s(t) = 0 \qquad \text{for a.a. } t \in \mathcal{A}_\delta. \tag{12}$$

Subset $\mathcal{B}_\delta$

We have

$$(P_{K^\infty} y)(t) = 0,\ (P_{K^\infty} x)(t) = 0,\ (P_{C^\infty(x)}(y - x))(t) = 0,$$

i.e.,

$$s(t) = 0 \qquad \text{for a.a. } t \in \mathcal{B}_\delta. \tag{13}$$

Subset $\mathcal{C}_\delta$

Since $x(t) = 0$, by (6) and (10) we get

$$(P_{C^\infty(x)}(y - x))(t) = (P_{C^\infty(x)} y)(t) = (P_{K^\infty} y)(t),$$

i.e.,

$$s(t) = 0 \qquad \text{for a.a. } t \in \mathcal{C}_\delta. \tag{14}$$

Subset $\mathcal{D}_\delta$

On this subset we do not have information on the signs of $y(t)$ and $(y(t) - x(t))$. We only know that

$$\begin{aligned} |s(t)| &\le |(P_{K^\infty} y)(t) - (P_{K^\infty} x)(t)| + |(P_{C^\infty(x)}(y - x))(t)| \\ &\le 2|y(t) - x(t)| \le 2\|y - x\|_\infty. \end{aligned} \tag{15}$$

Using (12)-(15) we find

$$\|s\|_p = \left[ \int_{\mathcal{D}_\delta} |s(t)|^p dt \right]^{\frac{1}{p}} \le 2\ e\, [\text{meas } \mathcal{D}_\delta]^{\frac{1}{p}}\, ||y - x\|_\infty.$$

Since meas $\mathcal{D}_\delta \to 0$ as $\delta \to 0$, for any $p > 0$ and any $\epsilon > 0$ we can find $\delta(p, \epsilon) > 0$ such that

$$\begin{aligned} &\|P_{K^\infty} y - P_{K^\infty} x - P_{C^\infty(x)}(y - x)\|_p \le \epsilon \|y - x\|_\infty, \\ &\text{for all } y \text{ such that } \|y - x\|_\infty < \delta(p, \epsilon). \end{aligned} \tag{16}$$

□

Note that the estimate (16) cannot be obtained for $p = \infty$. The following simple example shows that, in general, the metric projection onto the cone of nonnegative functions is *not* differentiable in $L^\infty(0, 1)$.

**Example.** Let $x(t) = 1-2t,\ y(t) = 1-\tau t$. Hence $y(t)-x(t) = (2-\tau)t$ and $\|y-x\|_\infty = 2-\tau$. By (6) and (10), we have

$$(P_{K^\infty}x)(t) = \begin{cases} 1-2t & \text{for a.a. } t \in [0, \frac{1}{2}], \\ 0 & \text{for a.a. } t \in [\frac{1}{2}, 1], \end{cases}$$
$$(P_{K^\infty}y)(t) = \begin{cases} 1-\tau t & \text{for a.a. } t \in [0, \frac{1}{\tau}], \\ 0 & \text{for a.a. } t \in [\frac{1}{\tau}, 1], \end{cases}$$
$$(P_{C^\infty(x)}(y-x))(t) = \begin{cases} (2-\tau)t & \text{for a.a. } t \in [0, \frac{1}{2}], \\ 0 & \text{for a.a. } t \in [\frac{1}{2}, 1], \end{cases}$$

Hence, by simple calculations we get

$$\|P_{K^\infty}y - P_{K^\infty}x - P_{C^\infty(x)}(y-x)\|_p$$
$$= \begin{cases} \frac{1}{2}[2\tau(p+1)]^{-\frac{1}{p}}|2-\tau|^{\frac{1}{p}}\|y-x\|_\infty & \text{for } p \in [1,\infty), \\ \frac{1}{2}\|y-x\|_\infty, & \text{for } p = \infty. \end{cases}$$

This shows that the projection is Fréchet differentiable at $x$ in $L^p(0,1)$ for all $p \in [1,\infty)$, but not for $p = \infty$.

## 3. Metric projection in $W^{1,p}(0,1)$

In $W^{1,p}(0,1)$ we introduce the norm

$$\begin{aligned} \|x\|_{1,p} &= (|x(0)|^p + \|\dot{x}\|_p^p)^{\frac{1}{p}} \qquad \text{for } p \in [1,\infty), \\ \|x\|_{1,\infty} &= \max\{|x(0)|, \|\dot{x}\|_\infty\}. \end{aligned} \tag{17}$$

By $K^{1,p}$ we denote the cone of nonnegative functions in $W^{1,p}(0,1)$. We will use the geometrical characterization of the metric projection onto the cone of nonnegative functions in $W^{1,p}(0,1)$ given by J. V. Outrata and Z. Schindler [13]. Clearly, $P_{K^{1,p}}x = x$ if $x \in K^{1,p}$. Suppose that $x \notin K^{1,p}$, i.e., $\min_{t\in[0,1]} x(t) < 0$. Let $\widetilde{x}$ be the right-continuous auxiliary function defined on $[-1,1]$ as follows

$$\widetilde{x}(t) = \begin{cases} 0 & \text{for } t \in [-1,0), \\ x(t) & \text{for } t \in [0,1), \\ \min_{t\in[0,1]} x(t) & \text{for } t = 1. \end{cases} \tag{18}$$

Denote by co $\widetilde{x}$ the lower convex envelope of $\widetilde{x}$. Since $\min_{t\in[0,1]} x(t) < 0$, the function $\frac{d}{dt}\text{co}\,\widetilde{x} \in L^p(0,1)$ is nonpositive and nondecreasing (see Fig. 3.1). According to [13], the projection $P_{K^{1,p}}x$ is given by

$$(P_{K^{1,p}}x)(t) = x(t) - \text{co}\,\widetilde{x}(t) \qquad \text{for all } t \in [0,1]. \tag{19}$$

Note that the above construction refers to one-dimensional systems and it seems that it cannot be extended to multidimensional situations.

As in (6), by $P_{K^{1,\infty}}x$ we denote the mapping defined on $W^{1,\infty}(0,1)$ by (19). Introduce the sets

$$\begin{aligned} &N^1_+(x) = \{t \in [0,1] \mid x(t) > \text{co } \widetilde{x}(t)\},\\ &N^1_0(x) = \{t \in [0,1] \mid x(t) = \text{co } \widetilde{x}(t) \text{ and } \dot{x}(t) = \text{const in a neighborhood of } t\},\\ &N^1_-(x) = \{t \in [0,1] \mid x(t) = \text{co } \widetilde{x}(t)\} \setminus N^1_0(x). \end{aligned} \tag{20}$$

As in (8), define the cones

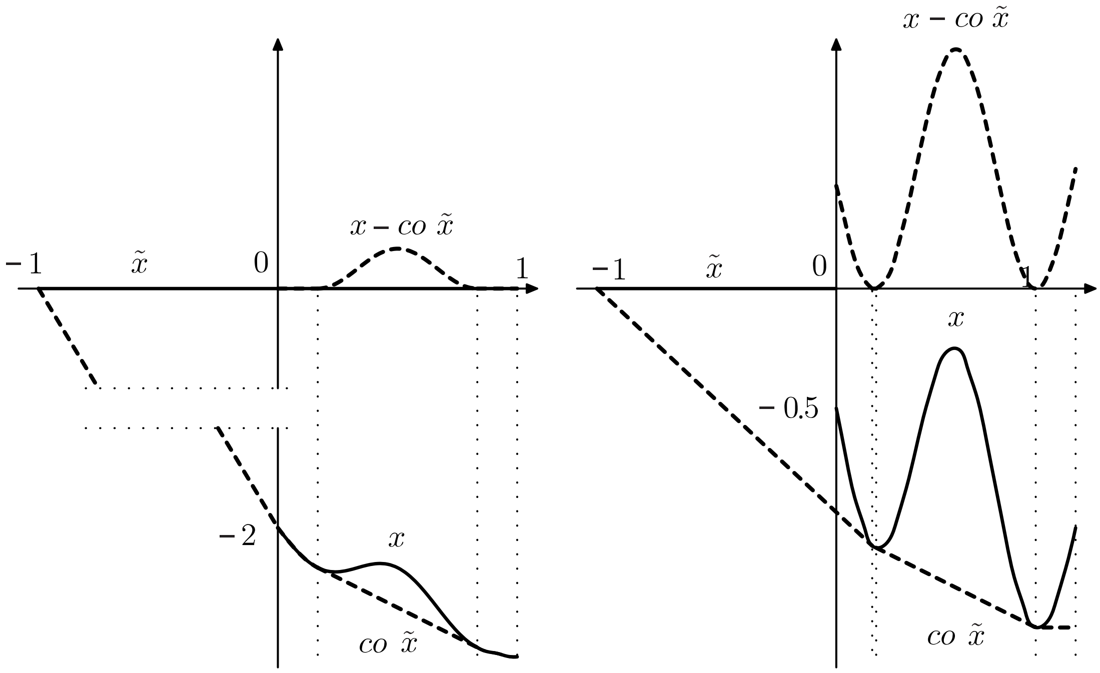


Figure 3.1: Construction of the projection in $W^{1,p}(0,1)$.

$$C^{1,p}(x) = \left\{ y \in W^{1,p}(0,1) \mid y(t) \begin{cases} \text{free} & \text{for all } t \in N^1_+(x), \\ \geq 0 & \text{for all } t \in N^1_0(x), \\ = 0 & \text{for all } t \in N^1_-(x). \end{cases} \right\}. \tag{21}$$

Using (19), one can check that $C^{1,2}(x) = \Xi(x)$, where $\Xi(x)$ corresponds to $K^{1,2}$ according to (3). Hence, by Theorems 1.2 and 1.4, the conical differential of $P_{K^{1,2}}$ at $x$ in the direction $y$ is given by

$$(dP_{K^{1,2}}x; y) = P_{C^{1,2}(x)}y. \tag{22}$$

We would like to check if (22) is the B-differential. The negative answer is given in the following theorem.

**Theorem 3.1.** *The mapping $P_{K^{1,\infty}(0,1)} : W^{1,\infty}(0,1) \to W^{1,p}(0,1)$, in general, is not Bouligand differentiable for any $p \in [1,\infty]$.*

**Proof.** We are going to construct an example, where condition (1) is violated for the mapping $P_{K^{1,\infty}}$. Let $x(t) = \dfrac{t^2}{2} - t - 2$, i.e., $\dot{x} = t - 1$. We have

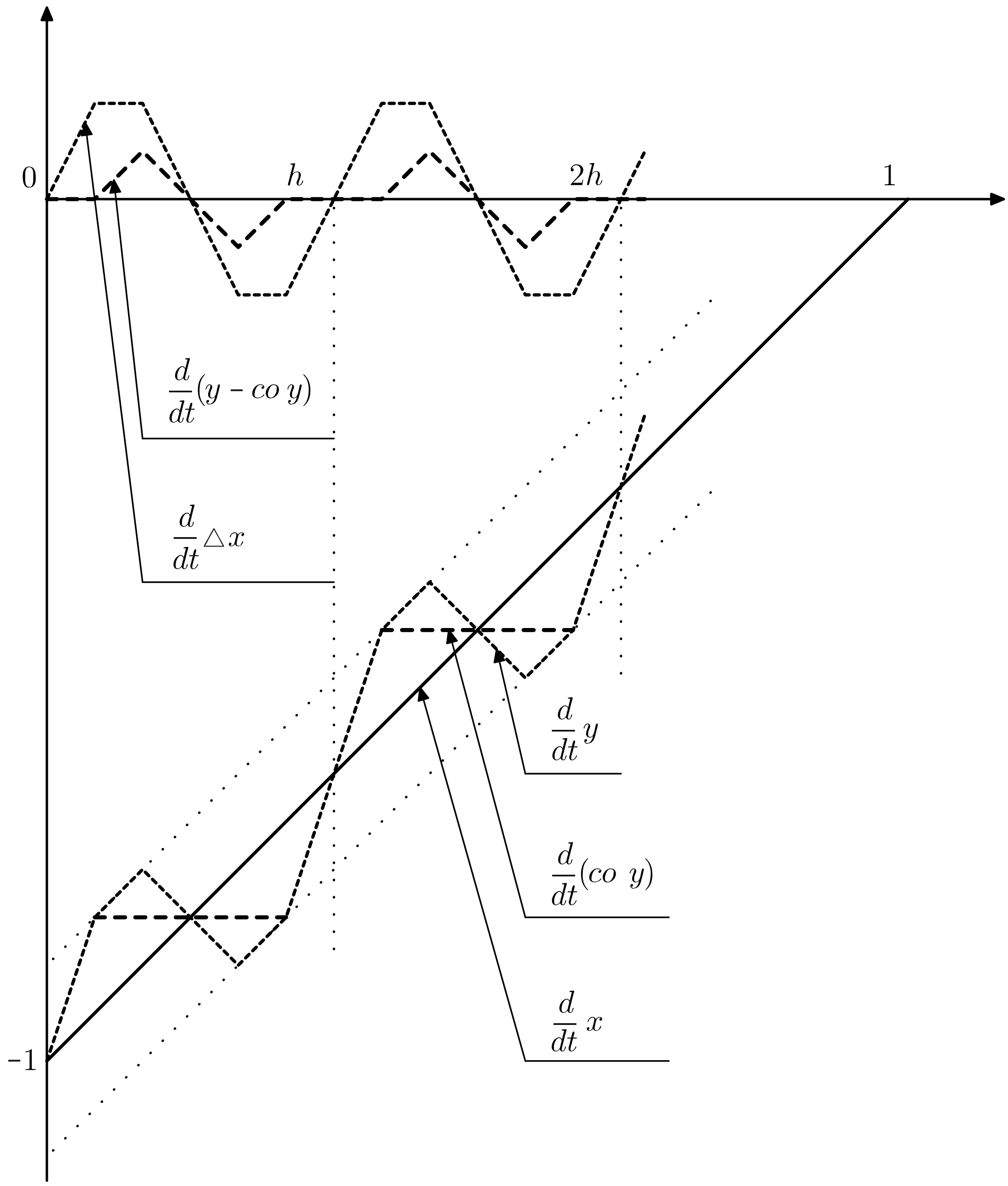


Figure 3.2: Time derivatives of the function and its projection.

$$\text{co}\ \widetilde{x}(t) = \begin{cases} -2-2t & \text{for } t \in [-1,0], \\ x(t) & \text{for } t \in [0,1]. \end{cases}$$

Hence, by (19)

$$P_{K^{1,p}} x \equiv 0, \tag{23}$$

whereas, by (21) we get

$$P_{C^{1,p}} y \equiv 0 \quad \text{for any } y \in W^{1,p}(0,1). \tag{24}$$

Let us construct $y = x + \Delta x$ in the following way. Choose any integer $N$ and divide the interval $[0,1]$ into $\frac{1}{N}$ subintervals of the equal length $h = \frac{1}{N}$. Put $\Delta x(0) = 0$ and define $\frac{d}{dt}\Delta x$ as a periodic function, of period $h$, given on $[0,h]$ by

$$\frac{d}{dt}\Delta x(t) = \begin{cases} 2t & \text{for } t \in [0, \frac{1}{6}h], \\ \frac{1}{3}h & \text{for } t \in [\frac{1}{6}h, \frac{2}{6}h], \\ -2t+h & \text{for } t \in [\frac{2}{6}h, \frac{4}{6}h], \\ -\frac{1}{3}h & \text{for } t \in [\frac{4}{6}h, \frac{5}{6}h], \\ 2(t-h) & \text{for } t \in [\frac{5}{6}h, h], \end{cases} \tag{25}$$

as it is shown in Fig. 3.2, where also $\frac{d}{dt}y(t) = \frac{d}{dt}x(t) + \frac{d}{dt}\Delta x(t)$ is drawn. We have

$$\|y - x\|_{1,\infty} = \frac{1}{3}h. \tag{26}$$

It can be easily checked that the convex envelopes of the functions $y$ and $\widetilde{y}$ coincide on $[0,1]$:

$$(\text{co}\ y)(t) = (\text{co}\ \widetilde{y})(t) \quad \text{for } t \in [0,1], \tag{27}$$

but they do not coincide with the function $y$. We have

$$(\text{co}\ y)(0) = y(0) \tag{28}$$

and, on the interval $[0,h]$ the time derivative $\frac{d}{dt}(\text{co}\ y)$ is given (see, Fig. 3.2) by

$$\frac{d}{dt}(\text{co}\ y)(t) = \begin{cases} -1+3t & \text{for } t \in [0, \frac{1}{6}h], \\ -1+\frac{1}{2}h & \text{for } t \in [\frac{1}{6}h, \frac{5}{6}h], \\ -1-2h+3t & \text{for } t \in [\frac{5}{6}h, h]. \end{cases} \tag{29}$$

Using (25) and (29), we find that

$$\frac{d}{dt}(\text{co}\ y)(t) - \frac{d}{dt}y(t) = \begin{cases} 0 & \text{for } t \in [0, \frac{1}{6}h] \cap [\frac{5}{6}h, h], \\ t - \frac{1}{6}h & \text{for } t \in [\frac{1}{6}h, \frac{2}{6}h], \\ -t+\frac{1}{2}h & \text{for } t \in [\frac{2}{6}h, \frac{4}{6}h], \\ t-\frac{5}{6}h & \text{for } t \in [\frac{4}{6}h, \frac{5}{6}h] \end{cases} \tag{30}$$

and this function is periodic, with the period $h$. On the other hand, (19) and (27) imply

$$P_{K^{1,p}} y = (\text{co } y)(t) - y(t). \tag{31}$$

We will show that the conical differential (22) does not satisfy the estimate (1) for all $y$ constructed above, so it is not the B-differential. In view of (23), (24) and (31) we get

$$s^1 := P_{K^{1,\infty}} y - P_{K^{1,\infty}} x - P_{C^{1,\infty}(x)}(y - x) = P_{K^{1,\infty}} y = \text{co } y - y.$$

By (30), we have

$$\|s^1\|_{1,\infty} = \text{ess sup}_{t\in[0,1]} \left| \frac{d}{dt}(\text{co } y)(t) - \frac{d}{dt} y(t) \right|,$$
$$\|s^1\|_{1,p} = \left[ \int_0^1 \left| \frac{d}{dt}(\text{co } y)(t) - \frac{d}{dt} y(t) \right|^p dt \right]^{\frac{1}{p}}.$$

Using (26) and (30), we obtain

$$\begin{aligned} &\|s^1\|_{1,\infty} = \tfrac{1}{6} h = \tfrac{1}{2} \|y - x\|_{1,\infty}, \\ &\|s^1\|_{1,p} = \left(\tfrac{2}{3(p+1)}\right)^{\frac{1}{p}} \tfrac{1}{6} h = \left(\tfrac{2}{3(p+1)}\right)^{\frac{1}{p}} \tfrac{1}{2} \|y - x\|_{1,\infty} \quad \text{for } p \in [1, \infty). \end{aligned} \tag{32}$$

Equalities (32) show that, for any $h \to 0$ and for any $p \in [1, \infty]$, condition (1) is not satisfied i.e., the mapping of the metric projection onto the cone $K^{1,\infty}$ is not B-differentiable at $x$ in $W^{1,p}(0,1)$, for any $p \in [1, \infty]$. □

**Remark 3.2.** Since the embedding $W^{2,2}(0,1) \subset W^{1,2}(0,1)$ is compact, Theorem 1.3 implies that $P_K$ is B-differentiable as a map from $W^{2,2}(0,1)$ into $W^{1,2}(0,1)$.

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