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Newton's Problem of Minimal Resistance in the Class of Bodies with Prescribed Volume

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The existence of a body of minimal resistance with prescribed volume is proved in the class of profiles which satisfy a two sided bound on the mean curvature.

1. Introduction

One of the oldest problems in the Calculus of Variations is Newton's problem of "Minimal Resistance". Suppose we have a rigid body moving through a fluid with given constant velocity, and assume the body is described as a graph over a given cross section Ω, then we may ask:

What is the Shape of the Body of Minimal Resistance?

Newton gave an explicit expression for the resistance. Under the assumption that the fluid consists of many independent particles hitting the body in a perfectly elastic manner at most once, Newton got the following expression for the resistance F:

$$F(u) = \int_\Omega \frac{1}{1 + |Du(x)|^2}\, dx, \tag{1}$$

where the function $u : \Omega \to [0, +\infty[$ gives the graph describing the body. Newton's problem can then be formulated as a variational problem:

$$\min\{F(u) \ : u \quad \text{is a graph, i.e.} \quad u : \Omega \to [0, +\infty[\}. \tag{2}$$

Standard techniques from the direct method in Calculus of Variations do not give existence of a minimizer, since this problem lacks convexity and coercivity. It is therefore important to choose a proper class of admissible functions.
In [2] an existence theorem in the class of bounded concave profiles and in the class of bounded superharmonic profiles was proved.

In this paper we consider a different class of admissible functions. We will work with

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graphs that enclose a given volume and for which the mean curvature satisfies a bound from above and below in a weak sense. The motivation for this class is given in Chapter 2. The case of concave profiles with prescribed volume was considered by [3]. In one dimension our class of admissible functions is contained in this class (see Chapter 4).
In Chapter 3 we prove the existence of a minimizer in our class of profiles.

2. The Mathematical Model

By one of the authors [8] the resistance of a body surrounded by a Stokes fluid was computed as

$$F_{res} = |v_0|^2 \int\limits_\Omega \frac{1}{1+|Du|^2}\, dx + \frac{2}{(n+1)|v_0|^2} \int\limits_{\partial^+ B} V_0 \cdot\, T(v,p)\, \nu \, dS. \tag{3}$$

The following notation is used:

- $V_0 = (0,\dots,-v_0)$ prescribes the fluid velocity at $x_{n+1} = \pm\infty$ for a given $v_0 > 0$;
- the body is assumed to be of the form

  $$B := \{(x', x_{n+1}) \in I\!R^{n+1} : x' \in \Omega \subset I\!R^n, 0 < x_{n+1} < u(x')\},$$

  where Ω is a given domain;
- $\partial^+ B := \{(x', u(x')) : x' \in \Omega\}$ denotes the upper part of the boundary of B;
- $T(v,p)_{i,j} := -p\delta_{ij} + \mu(\partial_i v^j + \partial_j v^i)$ denote the components of the stress tensor $T(v,p)$ of the fluid with fluid velocity v and pressure p;
- ν is the unit normal vector on $\partial^+ B$ pointing into B;

Roughly spoken this formula tells us, that the resistance of B can be decomposed into two parts.
The first part gives the contribution from "infinity", this part is modelled like the Newton functional for an incoming particle stream of velocity V_0.
The second part collects all surface forces in the V_0 direction. Observe that $T(v,p) = T(v - V_0, p)$.
Now we change our point of view by assuming for a moment, that the body consists of an elastic medium with constant internal pressure (normalized to zero) and constant volume V, separated from the surrounding fluid by an interface with surface tension σ. In such a situation we may consider the interface as a free boundary which may undergo some selfshapening. By Laplace's law the equilibrium is given by the state in which the difference between the surface stresses and the internal pressure of the medium is equal to the mean curvature of the interface:

$$T(v,p)\nu = 2\sigma H \nu \qquad \text{on} \quad \partial^+ B. \tag{4}$$

H denotes the mean curvature of the interface. With this setting the resistance reduces to a purely geometrical functional

$$F_{res} = \int\limits_\Omega \frac{1}{1+|Du|^2}\, dx + \frac{4\sigma}{(n+1)} \int\limits_{\partial^+ B} (V_0 \cdot \nu)\, H \, dS. \tag{5}$$

We consider (5) as a regularisation of the classical Newton functional. We will use it to find a physically reasonable class of constraints. One assumption which will be put onto the model is $H \leq 0$. This assumption has primarily technical reasons, however there is some physical heuristic for it: $V_0 \cdot \nu$ has a sign, since we assume, that ν is the inner normal of a surface which is parametrized as a graph over Ω and V_0 points to the negative e_{n+1} direction. Hence $V_0 \cdot \nu \geq 0$. If we now assume, that $H \leq 0$ this implies (by(4)) that we assume $V_0 \cdot T(v,p)\nu \leq 0$, which is in fact an assumption on the flow surrounding the body. It states, that the forces acting on the body by the fluid don't act against the direction of the incoming fluid from infinity. This should be true for flows which are not far from laminar flows.

A second assumption is, that $|H|$ is bounded from above by some positive constant M. Comparing with (4) this bound implies that all surface terms are bounded.

Assuming $v_0 = 1$ the resistance reads as

$$F_{res} = \int_\Omega \frac{1}{1+|Du|^2}\,dx + \frac{4\sigma}{(n+1)} \int_{\partial^+ B} (-e_{n+1} \cdot \nu)\, H\, dS. \tag{6}$$

where $e_{n+1} = (0, \ldots, 1)$. Rewriting $\partial^+ B$ as a graph over Ω and using

- $-e_{n+1} \cdot \nu = \frac{1}{\sqrt{1+|Du|^2}}$
- $dS = \sqrt{1+|Du|^2}\,dx$

we arrive at

$$F_{res} = \int_\Omega \frac{1}{1+|Du|^2}\,dx - \frac{4\sigma}{n+1} \int_\Omega H(Du)\,dx$$

with $H(Du) = D \cdot \left(\frac{Du}{\sqrt{1+|Du|^2}}\right) \leq 0$. With this notation the ball has constant negative mean curvature.

We will consider bodies of prescribed volume V i.e. $\int_\Omega u\,dx = V$. Hence after letting $\sigma \to 0$ we will be concerned with the following problem:

Minimize

$$F(u) = \int_\Omega \frac{1}{1+|Du|^2}\,dx \tag{7}$$

among all functions

$$u \in K := \{u \in BV_{loc}(\Omega) : u \geq 0, -M \leq H(Du) \leq 0, \int_\Omega u\,dx \leq V\}.$$

In this frame the constraints on the mean curvature are understood in the distributional sense, namely

$$\int_\Omega H\varphi\,dx \leq 0 \qquad \text{and} \qquad 0 \leq \int_\Omega (H+M)\varphi\,dx$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 0$. Observing, that

$$\int_\Omega \frac{Du \cdot D\varphi}{\sqrt{1+|Du|^2}}\,dx + \int_\Omega H\varphi\,dx = 0 \qquad \forall \varphi \in C_0^\infty(\overline{\Omega}),$$

this is equivalent to

$$M\int_\Omega \varphi\,dx \ge \int_\Omega \frac{Du\cdot D\varphi}{\sqrt{1+|Du|^2}}\,dx \ge 0.$$

Remark. The last integral needs an explanation. Let χ_B be the characteristic function of the body B. The gradient $D\chi_B$ is well defined as a measure for every function u in $BV_{loc}(\Omega)$.
The measure $\nu = \frac{D\chi_B}{|D\chi_B|}$ is well defined $\mathcal{H}^{n-1}$ a.e. on $\partial^+ B$ where $\mathcal{H}^{n-1}$ denotes the $n-1$ dimensional Hausdorff measure. Thus

$$\int_\Omega \frac{Du\cdot D\varphi}{\sqrt{1+|Du|^2}}\,dx$$

has to be read as

$$\int_{\Omega\times\boldsymbol{R}} (0, D\varphi)\cdot\nu\,dx\,dt.$$

For a more detailed discussion we refer to [5].

It is worth mentioning, that the set of constraints does not exclude unbounded functions. Indeed it is easy to compute, that for a proper chosen constant c the function $v(x) := c|x-x_0|^\alpha$ is in the set of constraints, if $\alpha > 2-n$ and $x_0\in\Omega$.

3. Existence

The following compactness lemma will be crucial for proving existence.

Lemma 3.1. *Let (u_k) be a sequence in K. Then (u_k) is bounded in $BV_{loc}(\Omega)$ and a subsequence weakly converges to some $u\in K$ in $BV_{loc}(\Omega)$.*

Proof. Consider the equation

$$\int_\Omega \frac{Du_k\cdot D\varphi}{\sqrt{1+|Du_k|^2}}\,dx + \int_\Omega H\varphi\,dx = 0 \qquad \forall\varphi\in C_0^\infty(\overline{\Omega}) \quad \text{with} \quad \varphi\ge 0.$$

Set $\varphi := u_k\eta$, where η is a cut off function $\eta\in C_0^\infty(\overline{\Omega})$ with $0\le\eta\le 1$, $\eta = 1$ in $\Omega_\rho = \{x\in\Omega : \mathrm{dist}(x,\partial\Omega) > \rho\}$ and $|D\eta| < \frac{2}{\rho}$. Hence

$$0 \le -\int_\Omega H u_k\eta\,dx = \int_\Omega \frac{Du_k\cdot D(u_k\eta)}{\sqrt{1+|Du_k|^2}}\,dx \le M\int_\Omega u_k\,dx.$$

Thus we may estimate:

$$\int_\Omega \frac{|Du_k|^2\eta}{\sqrt{1+|Du_k|^2}}\,dx \le M\int_\Omega u_k\,dx - \int_\Omega \frac{Du_k\cdot D\eta u_k}{\sqrt{1+|Du_k|^2}}\,dx.$$

This gives the estimate

$$\begin{aligned}\int_{\Omega_\rho} \sqrt{1+|Du_k|^2}\,dx &= \int_{\Omega_\rho} \frac{1+|Du_k|^2}{\sqrt{1+|Du_k|^2}}\,dx \leq \\ &\leq \int_{\Omega} \frac{1}{\sqrt{1+|Du_k|^2}}\,dx + M\int_{\Omega} u_k\,dx + \int_{\Omega} |D\eta| u_k\,dx.\end{aligned}$$

Consequently we have the bound

$$\int_{\Omega_\rho} \sqrt{1+|Du_k|^2}\,dx \leq |\Omega| + MV + \frac{c}{\rho}V.$$

This proves the boundedness of any minimizing sequence in K. Hence there is a subsequence weakly converging to some $u \in K$. □

Lemma 3.2. *Let (u_k) be a sequence in K and let $u \in BV_{loc}(\Omega)$ be the weak limit which exists by Lemma 3.1. Then $u_k \to u$ in $BV_{loc}(\Omega)$ and $\int_{\Omega'} |Du_k| \to \int_{\Omega'} |Du|$ for all $\Omega' \subset\subset \Omega$.*

Proof. Let Ω_ρ be defined as in Lemma 3.1. Let (u_k) be a sequence in K, then $u_k \to u$ a.e.. Hence by the Theorem of Egoroff, for any $\alpha > 0$, there exists a set A_α with $|A_\alpha| < \alpha$ such that (u_k) converges uniformly on $\Omega_\rho \setminus A_\alpha$. Fix $\epsilon > 0$ and define $v_k := (\epsilon + u - u_k)^+$. Let $g \in K$ and $g \in C_0^\infty(\Omega_\rho \setminus A_\alpha)$. The condition $H(g) \leq 0$ gives

$$0 \leq \int_{\Omega_\rho\setminus A_\alpha} \frac{Dg \cdot Dv_k}{\sqrt{1+|Dg|^2}}\,dx = \int_{\Omega_\rho\setminus A_\alpha} -H(g)v_k\,dx. \tag{8}$$

We observe that $u_k - u \leq \epsilon$ on $\Omega_\rho \setminus A_\alpha$ for sufficiently large k. Hence

$$0 \leq \int_{\Omega_\rho\setminus A_\alpha} \frac{Dg \cdot D(u-u_k)}{\sqrt{1+|Dg|^2}}\,dx = \int_{\Omega_\rho\setminus A_\alpha} -H(g)(\epsilon + u - u_k)\,dx.$$

The right hand side can be estimated on $\Omega_\rho \setminus A_\alpha$

$$\int_{\Omega_\rho\setminus A_\alpha} -H(g)(\epsilon + u - u_k)\,dx \leq M\left(\epsilon|\Omega_\rho \setminus A_\alpha| + \int_{\Omega_\rho\setminus A_\alpha} |u-u_k|\,dx\right).$$

Thus for all $g \in C_0^\infty(\Omega_\rho \setminus A_\alpha)$ we have the estimate

$$0 \leq \int_{\Omega_\rho\setminus A_\alpha} \frac{Dg \cdot D(u-u_k)}{\sqrt{1+|Dg|^2}}\,dx \leq M\left(\epsilon|\Omega_\rho \setminus A_\alpha| + \int_{\Omega_\rho\setminus A_\alpha} |u-u_k|\,dx\right).$$

Taking the supremum over all such g gives

$$\int_{\Omega_\rho\setminus A_\alpha} |D(u-u_k)|\,dx \leq M\left(\epsilon|\Omega_\rho \setminus A_\alpha| + \int_{\Omega_\rho\setminus A_\alpha} |u-u_k|\,dx\right).$$

Since

$$\left| \int_{\Omega_\rho \setminus A_\alpha} (|Du| - |Du_k|) \; dx \right| \leq \int_{\Omega_\rho \setminus A_\alpha} |D(u-u_k)| \, dx$$

we have in the limit, as $k \to \infty$, for every $\alpha > 0$

$$\lim_{k\to\infty} \left| \int_{\Omega_\rho \setminus A_\alpha} (|Du| - |Du_k|) \; dx \right| \leq M\epsilon |\Omega_\rho \setminus A_\alpha|.$$

Here we used the L^1_{loc} convergence of the minimizing sequence (u_k). Since $\epsilon > 0$ is arbitrary, the claim is proved. □

It will not be possible to find a minimizer of (7) in the class K. One reason is, that we cannot exclude the presence of a singular part of the gradient $|Du|$, another reason is the lack of convexity of (7). To overcome the first problem we replace the vector measure Du by its absolutely continuous part $D^a u$. For the second problem we consider the following convexification of $\frac{1}{1+|Du|^2}$, introduced in [2]: consider $\tilde{f}$ defined as

$$\tilde{f}(t) := \begin{cases} 1 - \frac{t}{2} & : \quad 0 \leq t \leq 1 \\ \frac{1}{1+t^2} & : \quad t > 1. \end{cases}$$

The function $\tilde{f}$ is convex for $\mathbb{R}^+$ and hence instead of (7) we consider the following problem:
Minimize

$$\tilde{F}(u) = \int_\Omega \tilde{f}(|D^a u|) \, dx \tag{9}$$

among all functions

$$u \in K := \{u \in BV_{loc}(\Omega) : u \geq 0, -M \leq H(Du) \leq 0, \int_\Omega u \, dx \leq V\}.$$

The following existence theorem is now proved exactly as Theorem 4.5 in [2]. For the convenience of the reader we add the proof.

Theorem 3.3. *The minimum problem*

$$\min\{\tilde{F}(u) : u \in K\}$$

admits at least one solution.

Proof. Let (u_k) be a minimizing sequence in K. By Lemma 3.1 and 3.2 a subsequence it converges strongly to some $u \in K$ in $BV_{loc}(\Omega)$. In particular, if the singular part of Du_k is zero, the same will be true for the limit Du.
We also observe, that the measures $|Du_k|$ converge to the measure $|Du|$ in the weak*

topology of measures (see e.g. [4] Chapter 1.9).
Furthermore $\tilde{f}$ is convex on $\mathbb{R}^n$ and $\lim_{t\to\infty} \frac{\tilde{f}(tr)}{t} = 0$ for all $r > 0$.
As a consequence we can apply a lower semicontinuity result on measures (see e.g. [1]) which implies the lower semicontinuity of $\tilde{F}(u)$. □

Remark 3.4. Theorem 2.3 in [2] shows that the solution u of the Newton problem in the class of bounded concave functions satisfies $|Du(x)| \notin]0,1[$. Thus the Newton functional can be replaced by $\tilde{F}$ without changing the solution of the minimum problem. Here this fact is not even true in the 1-dimensional case (see Section 4).

Remark 3.5. If we replace the class K by the class:

$$J := \{u \in H^1_{loc}(\Omega) : 0 \le u \le M, H(Du) \le 0\},$$

then we get existence of a minimizer as in [2].

4. The 1-d Case

For $\Omega = (-1,1)$ the one dimensional problem reads as
Minimize

$$F(u) = \int_\Omega \frac{1}{1+u'^2}\, dx \tag{10}$$

among all functions

$$u \in K := \{u \in BV_{loc}(\Omega) : u \ge 0, -M \le H(u') \le 0, \int_\Omega u\, dx \le V\}.$$

In one dimension $H(u') \le 0$ is equivalent to $u'' \le 0$ *a.e.*, i.e. u is concave. Also in one dimension we don't need the constraint $H(u') \ge -M$. Thus the 1-d problem is equivalent to
Minimize

$$F(u) = \int_\Omega \frac{1}{1+u'^2}\, dx \tag{11}$$

among all functions

$$u \in K_{concave} := \{u : \Omega \to \mathbb{R} \ :\ u \text{ concave},\ u(x) \ge 0,\ \int_\Omega u\, dx \le V\}.$$

This problem has a solution in $L^p(\Omega)$ for every $p \ge 1$, and solutions $u(x)$ must be concave, piecewise linear and satisfy $u(-1) = u(1) = 0$.
We first consider the convexified Newton's problem. In this case the integrand $\tilde{f}$ is not strictly convex but solutions are, as before, concave, piecewise linear and satisfy $u(-1) = u(1) = 0$.
In fact, suppose u is a minimizer: taking the convex combination

$$v(x) = \frac{1}{2}u(x) + \frac{1}{2}u(-x),$$

then the convexity of $\tilde{F}$ gives us

$$\tilde{F}(v) \leq \frac{1}{2}\tilde{F}(u) + \frac{1}{2}\tilde{F}(u) = \tilde{F}(u),$$

thus we can assume that the minimizer is even.
As a consequence we can restrict all considerations to the interval $[0,1]$: here the minimizer $u(x)$ is monotone decreasing, piecewise affine and concave.
Now we show that the minimizer $u(x)$ is affine (not piecewise affine!) in $[0,1]$. Let us suppose the minimizer $u(x)$ is a piecewise affine function with slopes u_1' in $[0,\xi]$ and u_2' in $]\xi,1]$, where $u_1' > u_2'$ (by concavity), and u_1' and u_2' are two numbers chosen such that the volume constraint $\int_{-1}^{1} u(x)\,dx = V$ is satisfied.
We construct an affine function w such that $w' = (1-\xi^2)u_2' + \xi^2 u_1' \in]u_2', u_1'[$. It is easy to check that

$$\int_\Omega w(x)\,dx = V.$$

Now we have

$$\begin{aligned} \tilde{F}(w) &= 2\int_0^1 \tilde{f}(w')\,dx \\ &\leq 2\left(\xi^2\tilde{f}(u_1') + (1-\xi^2)\tilde{f}(u_2')\right) \\ &< 2\left(\xi\tilde{f}(u_1') + (1-\xi)\tilde{f}(u_2')\right) \\ &= 2\left(\int_0^\xi \tilde{f}(u_1')\,dx + \int_\xi^1 \tilde{f}(u_2')\,dx\right) \\ &= \tilde{F}(u). \end{aligned}$$

Thus the minimizer has the shape of an isosceles triangle with base $[-1,1]$ and height V due to the volume constraint (see Figure 1).

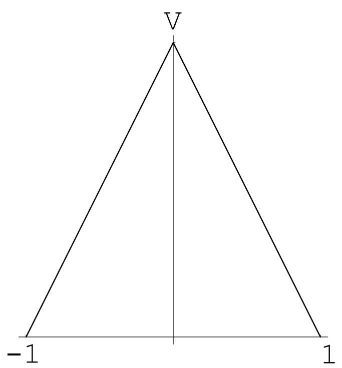

Figure 1: A minimizer for $\tilde{F}$

Hence it is given by

$$u(x) := \begin{cases} V(x+1) & : \quad -1 \leq x \leq 0 \\ -V(x-1) & : \quad 0 < x \leq 1. \end{cases}$$

Let us now consider the Newton problem (11).
As before solutions $u(x)$ must be concave, piecewise linear and satisfy $u(-1) = u(1) = 0$. When $V \geq 1$ there exists a minimizer for the convexified problem as described above. This follows from the fact that the Newton functional and the convexified functional coincide in the region where the slope (in modulus) is greater that 1.
With the same reasoning we can extend this result to the region $\frac{1}{\sqrt{3}} \leq V < 1$. In this case we consider the (convex) functional $\overline{F}(u) = \int_{-1}^{1} \overline{f}(u')\, dx$, where

$$\overline{f}(t) := \begin{cases} \frac{9}{8} - \frac{3\sqrt{3}}{8}t & : \quad 0 \leq t \leq \frac{1}{\sqrt{3}} \\ \frac{1}{1+t^2} & : \quad t > \frac{1}{\sqrt{3}}. \end{cases}$$

When $V < \frac{1}{\sqrt{3}}$ the situation changes. The "triangle solution"

$$u_0(x) := \begin{cases} V(x+1) & : \quad -1 \leq x \leq 0 \\ -V(x-1) & : \quad 0 < x \leq 1 \end{cases}$$

is not a minimizer for F in the class K_{concave}. In fact consider

$$u_\lambda(x) := \begin{cases} \frac{V}{1+\lambda}(x+1) & : \quad -1 \leq x < \lambda \\ \frac{V}{1-\lambda}(1-x) & : \quad \lambda \leq x \leq 1, \end{cases}$$

where $\lambda \in [0, 1]$: a direct calculation shows that

$$F(u_\lambda) < F(u_0) \tag{12}$$

for some sufficiently small positive λ iff $V < \frac{1}{\sqrt{3}}$. Here u_0 denotes the "triangle solution". The calculation is done in two steps. First we compute

$$F(u_\lambda) = \frac{(1+\lambda)^3}{(1+\lambda)^2 + V^2} + \frac{(1-\lambda)^3}{(1-\lambda)^2 + V^2}.$$

Then $\frac{d}{d\lambda}F(u_\lambda) = 0$ for $\lambda = 0$ and the sign of $\frac{d^2}{d\lambda^2}F(u_\lambda)$ for $\lambda = 0$ coincides with the sign of $3V^4 - V^2$.
Hence the minimizer cannot be found in the class of even functions.
However, if we restrict ourselves to the subclass of even function in K_{concave} and $V < \frac{1}{\sqrt{3}}$ it was shown in [7] that two cases have to be considered. Let $\overline{V}$ be the solution to $V^3 + 3V^2 + 3V - 1 = 0$. Then

- u_0 is a minimizer if $\overline{V} \leq V \leq \frac{1}{\sqrt{3}}$;

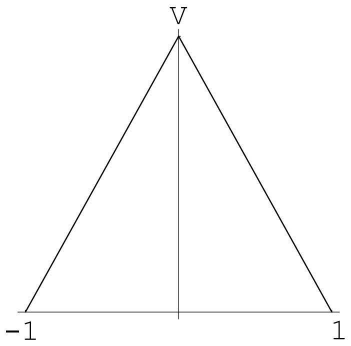

Figure 2: F-minimizer when $V \geq \frac{1}{\sqrt{3}}$

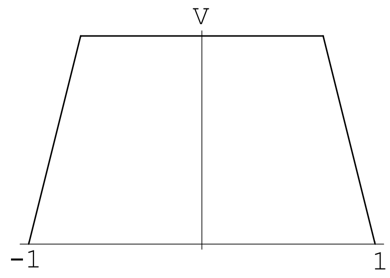

Figure 3: F-even min. when $V < \frac{1}{\sqrt{3}}$

- If $V < \overline{V}$, then the "trapezoidal solution"

$$w(x) := \begin{cases} (1-|x|) & : \quad \sqrt{1-V} \leq |x| \leq 1 \\ 1-\sqrt{1-V} & : \quad 0 \leq |x| \leq \sqrt{1-V}. \end{cases},$$

gives

$$F(w) < F(u_0)$$

see Figure 3.

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