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Positively Convex Modules and Ordered Normed Linear Spaces

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A positively convex module is a non-empty set closed under positively convex combinations but not necessarily a subset of a linear space. Positively convex modules are a natural generalization of positively convex subsets of linear spaces. Any positively convex module has a canonical semimetric and there is a universal positively affine mapping into a regularly ordered normed linear space and a universal completion.

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1. Introduction

In the following all linear spaces will be real. If E is a linear space ordered by a proper cone $C, E = C - C$, and normed by a Riesz norm with respect to C, the positive part of the unit ball $\mathrm{O}(E)$, i.e. $\mathrm{O}(E) \cap C$, is a typical example of a positively convex set or module. Actually any positively convex set or module i.e. any non-empty set P closed under positively convex operations, is an affine quotient of such a positive part of a unit ball. Positively convex modules are natural generalizations of positively convex sets, the postulate that they are subsets of some linear space is dropped.

Special positively convex sets play an important part in the theory of order-unit Banach spaces as so-called "universal caps" (cp. [21], §9) and also as generators of the topology of locally solid ordered topological linear spaces and the theory of Riesz seminorms (cp. [21], §6). In [9] it is proved that there is a functorial connection between the theory of positively (countably) convex sets or modules and the theory of regularly ordered Banach spaces, i.e. ordered by a Riesz norm. To investigate this relation and to give a more explicit description Wickenhäuser in [20] introduced a semimetric for positively convex modules. In [4] Kemper expressed this semimetric by the seminorm of the positively convex module. These results improved the description of the universal regularly ordered normed linear space generated by a positively convex module somewhat but the situation has been far from satisfactory compared with the results for other types of convex sets (cf. [1], [10], [11], [12], [16], [14], [15]).

In §2 of this paper the semimetric of a positively convex module is introduced and discussed. §3 contains the functorial connection between positively convex modules and ordered linear spaces and the characterization of preseparated positively convex modules.

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The positive part of the unit ball of a regularly ordered normed linear space induces a canonical functor to the category of positively convex modules. In §4 it is shown that, vice versa, any positively convex module generates a universal regularly ordered normed linear space and a universal positively affine morphism into the positive part of the unit ball of this space. This leads to a discussion of complete positively convex modules and positively superconvex modules in §5 where also the existence of a universal completion is proved.

2. The semimetric

In the following all linear spaces considered will be real. A *positively convex set* X in a linear space E is a non-empty subset of E closed under positively convex operations i.e. $x_i \in X,\ 1 \le i \le n$, and $\alpha_i \ge 0, \sum_{i=1}^{n} \alpha_i \le 1$ implies $\sum_{i=1}^{n} \alpha_i x_i \in X$. A *positively convex operation* may be described as an element of the set $\Omega_{pc} := \{\hat{\alpha} \mid \hat{\alpha} = (\alpha_1, \dots, \alpha_n),\ n \in \mathbb{N},\ \alpha_i \ge 0,\ 1 \le i \le n, \text{ and } \sum_{i=1}^{n} \alpha_i \le 1\}$. This leads to the following natural generalization.

Definition 2.1. (cf. [4], [5], [9], [13], [16]): A *positively convex module* P is a non-empty set together with a family of mappings $\hat{\alpha}_P : P^n \longrightarrow P,\ \hat{\alpha} = (\alpha_1, \dots, \alpha_n) \in \Omega_{pc}$. In addition, with the notation

$$\sum_{i=1}^{n} \alpha_i p_i := \hat{\alpha}_P(p_1, \dots, p_n)$$

for $p_i \in P,\ 1 \le i \le n$, the following equations have to be satisfied:

(PC1)
$$\sum_{i=1}^{n} \delta_{ik} p_i = p_k,$$

$p_i \in P,\ 1 \le i \le n$, and δ_{ik} the Kronecker symbol, $1 \le i, k \le n$.

(PC2)
$$\sum_{i=1}^{n} \alpha_i \left(\sum_{k \in K_i} \beta_{ik} p_k \right) = \sum_{k=1}^{m} \left(\sum_{\substack{i=1 \\ k \in K_i}}^{n} \alpha_i \beta_{ik} \right) p_k,$$

where $(\alpha_1, \dots, \alpha_n),\ (\beta_{ik} \mid k \in K_i) \in \Omega_{pc},\ 1 \le i \le n,\ p_k \in P$, for $k \in \bigcup_{i=1}^{n} K_i$. Moreover, $\bigcup_{i=1}^{n} K_i = \mathbb{N}_m = \{k \mid 1 \le k \le m\}$ and in the "sum" $\sum_{k \in K_i} \beta_{ik} p_k$ the summands are supposed to be written in the natural oder of the k's.

A number of computational rules follow from these equations (cf. [9], [10]), e.g. the fact that $\sum_{i=1}^{n} \alpha_i p_i$ is not changed by adding or omitting summands with zero coefficients. Hence, (PC2) takes the more simple form

$$\sum_{i=1}^{n} \alpha_i \left(\sum_{k=1}^{m} \beta_{ik} p_k \right) = \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \alpha_i \beta_{ik} \right) p_k.$$

Obviously, any convex set, which will always mean a positively convex subset in some linear space, is a positively convex module. Any positively convex module is a convex module (cp. [16]), the converse does not hold, because any positively convex module contains a zero element $0 := \sum_{i=1}^{n} 0p_i$. However, every absolutely convex module ([10], [14]) is a positively convex module, as is any cone or convex set in a linear space containing the orgin. A familiar example of positively convex sets are the *universal caps* of cones in functional analysis ([21]).

If P_1, P_2 are positively convex modules a mapping $f : P_1 \longrightarrow P_2$ is called a *morphism* or a *positively affine* mapping if

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i f(x_i)$$

holds for any $(\alpha_1, \ldots, \alpha_n) \in \Omega_{pc}$ and any $x_i \in P,\ 1 \leq i \leq n$. This defines the category **PC** of positively convex modules with forgetful functor $\mathrm{U} : \mathbf{PC} \to \mathbf{Set}$. For any $X \in \mathbf{Set}$, $\Delta(\mathbb{R}_+^{(X)}) := \{h \mid h : X \longrightarrow \mathbb{R}_+,$ support of h finite and $\sum_x h(x) \leq 1\}$, $\mathbb{R}_+ := \{t \mid t \in \mathbb{R},\ t \geq 0\}$, is the free positively convex module generated by X. ([9], [10]). $\mathrm{U} : \mathbf{PC} \longrightarrow \mathbf{Set}$ is algebraic. It is the Eilenberg-Moore category of the category $\mathbf{Vec}_1^+$ of regularly ordered normed linear spaces where $\Delta : \mathbf{Vec}_1^+ \longrightarrow \mathbf{Set}$ is the forgetful functor with $\Delta(E) := \{x \mid x \in E,\ x \geq 0, \|x\| \leq 1\}$ ([4], [9]).

Definition 2.2. (cf. [4], [20]): For a positively convex module P and $x, y \in P$ one defines

$d_P(x,y) := \inf\{\frac{\alpha}{2\beta} \mid \beta > 0,\ \alpha > 0$ and there are $a, u, v \in P$ and

$\delta, \varepsilon \in \mathbb{R}_+$ with $\alpha+\beta,\ \delta+\beta,\ \varepsilon+\beta \leq 1$ and $\alpha a+\beta x = \delta u+\beta y,\ \alpha a+\beta y = \varepsilon v+\beta x\}$.

d_P is called the *semimetric* of P (cf. 2.3). If it is obvious to which P the semimetric belongs the index is often omitted. $P \in \mathbf{PC}$ is called a *metric* positively convex module if d_P is a metric.

This semimetric was first introduced by Wickenhäuser in [20], 9.2 in the following form

$d_W(x,y) = \inf\{\alpha \mid 0 \leq \alpha \leq 2$ and there are $a, u, v \in P$ and $n \in \mathbb{N}$
$n \geq 3$ with $\frac{1}{n}x + \frac{\alpha}{n}a = \frac{1}{n}y + \frac{1}{2}v,\ \frac{1}{n}y + \frac{\alpha}{n}a = \frac{1}{n}x + \frac{1}{2}u\}$.

This form proved to be not very well suited for computations. A straightforward argument shows that

$$d_P(x,y) = \frac{1}{2} d_W(x,y).$$

In [4], Kemper expressed $d_W(x,y)$ by the seminorm of positively convex modules (cf. [9]), a form which still presented considerable difficulties in describing the so-called comparison functor ([9]) from **PC** to $\mathbf{Vec}_1^+$. This functor will be investigated in §4.

Proposition 2.3. ([4], [16], [20]):

(i) *For any positively convex module P, d_P is a semimetric with $0 \leq d_P(x,y) \leq 1$ and $d_P(x,0) \leq \|x\|, x, y \in P$, where $\|\Box\|$ denotes the seminorm of P ([9]).*

(ii) *Let* $x_i, y_i \in P,\ 1 \leq i \leq n,\ P \in \mathbf{PC}$ *and* $(\lambda_1, \ldots, \lambda_n) \in \Omega_{pc}$ *then*

$$d_P\left(\sum_{i=1}^{n} \lambda_i x_i, \sum_{i=1}^{n} \lambda_i y_i\right) \leq \sum_{i=1}^{n} \lambda_i d_P(x_i, y_i).$$

(iii) *for* $x, y \in P,\ P \in \mathbf{PC}$ *and* $0 \leq \lambda \leq 1$

$$d_P(\lambda x, \lambda y) = \lambda d_P(x, y).$$

(iv) *If* $f : P_1 \longrightarrow P_2$ *is a morphism in* $\boldsymbol{PC}$ *then*

$$d_2(f(x), f(y)) \leq d_1(x, y),$$

if $d_i(\square, \square)$ *is the semimetric of* $P_i, i = 1, 2$.

Proof. (i): 2.2 implies that $d_P(x, y)$ is symmetric. In [20] $0 \leq d_P(x, y) \leq 1$ was shown. One takes $\alpha = 2^{-1}, \beta = 4^{-1}, \delta = \varepsilon = 2^{-1},\ a = 2^{-1}x + 2^{-1}y,\ u = x$ and $v = y$. In order to show the triangle inequality for $x, y, z \in P$, we may assume $d_P(x, z) + d_P(z, y) < 1$. Hence, $d_P(x, z) < 1$ and $d_P(z, y) < 1$ follows and there are equations

$$\begin{aligned} \alpha_1 a_1 + \beta_1 x &= \delta_1 u_1 + \beta_1 z \\ \alpha_1 a_1 + \beta_1 z &= \varepsilon_1 v_1 + \beta_1 x \\ \alpha_2 a_2 + \beta_2 z &= \delta_2 u_2 + \beta_2 y \\ \alpha_2 a_2 + \beta_2 y &= \varepsilon_2 v_2 + \beta_2 z \end{aligned}$$

with $d_P(x, z) \leq \alpha_1(2\beta_1)^{-1} < 1,\ d_P(z, y) \leq \alpha_2(2\beta_2)^{-1} < 1$. A routine computation using (PC1) and (PC2) yields

$$\frac{1}{2}\alpha_1\beta_2 a_1 + \frac{1}{2}\alpha_2\beta_1 a_2 + \frac{1}{2}\beta_1\beta_2 x = \frac{1}{2}\beta_1\delta_2 u_2 + \frac{1}{2}\beta_2\delta_1 u_1 + \frac{1}{2}\beta_1\beta_2 y,$$

$$\frac{1}{2}\alpha_1\beta_2 a_1 + \frac{1}{2}\alpha_2\beta_1 a_2 + \frac{1}{2}\beta_1\beta_2 y = \frac{1}{2}\beta_1\varepsilon_2 v_2 + \frac{1}{2}\beta_2\delta_1 v_1 + \frac{1}{2}\beta_1\beta_2 y.$$

With $\alpha := 2^{-1}(\alpha_1\beta_2 + \alpha_2\beta_1),\ \beta := 2^{-1}\beta_1\beta_2$

$$d_P(x, y) = \frac{\alpha}{2\beta} = \frac{\alpha_1}{2\beta_1} + \frac{\alpha_2}{2\beta_2}$$

follows (cf. 2.2), which implies the assertion.

In order to show $d_P(x, 0) \leq \|x\|$ we may assume $\|x\| < 1$. Consider ζ with $\|x\| \leq \zeta < 1$ and write $\zeta := \alpha(2\beta)^{-1}$ with $\alpha = 1 - \beta,\ \beta = (1 + 2\zeta)^{-1}$. The definition of the seminorm in P (cf. [9], (4.2.), p. 97) implies the existence of an $e \in P$ with

$$x = \zeta e = \frac{\alpha}{2\beta} e,\ \beta x = \alpha\left(\frac{1}{2}e\right).$$

Hence, with $a := 2^{-1}e,\ \alpha a + \beta x = \alpha e + \beta 0$ and $\alpha a + \beta 0 = 00 + \beta x$ follows, i.e. $d_P(x, 0) \leq \zeta$ and $d_P(x, 0) \leq \|x\|$.

(ii): Let $\alpha_i a_i + \beta_i x_i = \delta_i u_i + \beta_i y,\ \alpha_i a_i + \beta_i y_i = \varepsilon_i v_i + \beta_i x_i, 1 \le i \le n$, be equations from the definition of $d_P(x_i, y_i)$. Put $\beta := \min\{\beta_i \mid 1 \le i \le n\} > 0,\ \mu_i := \alpha_i \beta_i^{-1},\ \nu_i := \delta_i \beta_i^{-1},\ \sigma_i := \varepsilon_i \beta_i^{-1},\ 1 \le i \le n$. (PC1) and (PC2) of 2.1 imply

$$\beta\mu_i a_i + \beta x_i = \beta\nu_i u_i + \beta y_i,\ 1 \le i \le n,$$

and

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\lambda_i\beta\mu_i a_i + \frac{1}{2}\sum_{i=1}^{n}\lambda_i\beta x_i = \sum_{i=1}^{n}\frac{1}{2}\lambda_i\beta(1+\mu_i)\left(\frac{\mu_i}{1+\mu_i}a_i + \frac{1}{1+\mu_i}x_i\right)\\
&= \sum_{i=1}^{n}\frac{1}{2}\lambda_i\frac{\beta}{\beta_i}(\alpha_i+\beta_i)\left(\frac{\alpha_i}{\alpha_i+\beta_i}a_i + \frac{\beta_i}{\alpha_i+\beta_i}x_i\right)\\
&= \sum_{i=1}^{n}\frac{1}{2}\lambda_i\frac{\beta}{\beta_i}(\delta_i+\beta_i)\left(\frac{\delta_i}{\delta_i+\beta_i}u_i + \frac{\beta_i}{\delta_i+\beta_i}y_i\right)\\
&= \frac{1}{2}\sum_{i=1}^{n}\lambda_i\beta\nu_i u_i + \frac{1}{2}\sum_{i=1}^{n}\alpha_i\beta y_i.
\end{aligned}$$

With $\mu := \sum_{k=1}^{n}\lambda_k\mu_k,\ \nu := \sum_{k=1}^{n}\lambda_k\nu_k,\ \sigma := \sum_{k=1}^{n}\lambda_k\sigma_k$ and $a := \sum_{k=1}^{n}\lambda_k\mu_k\mu^{-1}a_k$ this yields

$$\frac{1}{4}\beta\mu a + \frac{1}{4}\beta\sum_{i=1}^{n}\lambda_i x_i = \frac{1}{4}\beta\nu\sum_{i=1}^{n}\lambda_i\nu_i\nu^{-1}u_i + \frac{1}{4}\beta\sum_{i=1}^{n}\lambda_i y_i.$$

By a completely analogous computation one gets

$$\frac{1}{4}\beta\mu a + \frac{1}{4}\beta\sum_{i=1}^{n}\lambda_i y_i = \frac{1}{4}\beta\sigma\sum_{i=1}^{n}\lambda_i\sigma_i\sigma^{-1}v_i + \frac{1}{4}\beta\sum_{i=1}^{n}\lambda_i x_i.$$

Hence

$$d_P\left(\sum_{i=1}^{n}\lambda_i x_i, \sum_{i=1}^{n}\lambda_i y_i\right) \le \frac{1}{2}\mu = \sum_{i=1}^{n}\lambda_i\frac{\alpha_i}{2\beta_i}$$

and

$$d_P\left(\sum_{i=1}^{n}\lambda_i x_i, \sum_{i=1}^{n}\lambda_i y_i\right) \le \sum_{i=1}^{n}\lambda_i d_P(x_i, y_i).$$

(iii): (ii) implies $d_P(\lambda x, \lambda y) \le \lambda d_P(x, y)$ for $0 \le \lambda \le 1$. To show the converse inequality we may assume $0 < \lambda < 1$ and $d_P(\lambda x, \lambda y) < \lambda$. Consider $d_P(\lambda x, \lambda y) < \mu < \lambda$. Then there are equations

$$\begin{aligned}
\alpha a + \beta\lambda x &= \delta u + \beta\lambda y,\\
\alpha a + \beta\lambda y &= \varepsilon v + \beta\lambda x.
\end{aligned}$$

Multiplying these equations by n^{-1}, with some $n > \max\{\varepsilon, \delta, 2\mu\beta\} + \beta\lambda + 1$, one gets

$$\frac{2\beta\mu}{n}(\frac{\alpha}{2\beta\mu}a) + \frac{\beta\lambda}{n}x = \frac{\delta}{n}u + \frac{\beta\lambda}{n}y,$$

$$\frac{2\beta\mu}{n}(\frac{\alpha}{2\beta\mu}a) + \frac{\beta\lambda}{n}y = \frac{\varepsilon}{n}v + \frac{\beta\lambda}{n}x,$$

hence $d_P(x,y) = \mu\lambda^{-1},\ \lambda d_P(x,y) \leq \mu$ and $\lambda d_P(x,y) \leq d_P(\lambda x, \lambda y)$.

(iv) follows immediately from 2.2.

Examples 2.4. (i) If S is a lower semilattice with a smallest element 0, S becomes a positively convex module if one defines

$$\sum_{i=1}^{n} \alpha_i x_i := \bigwedge_{\substack{i=1 \\ \alpha_i \neq 0}}^{n} x_i,$$

for $(\alpha_i, \dots, \alpha_n) \in \Omega_{pc}$ and $x_i \in S, 1 \leq i \leq n$. The semimetric is trivial as an elementary calculation shows.

(ii) Let A be an absolutely convex module or a subset of a real linear space (cf. [10], [11], [14], [20]), then for $x, y \in A$

$$\frac{1}{4}x = \frac{1}{4}y + \frac{1}{2}\left(\frac{1}{2}x - \frac{1}{2}y\right)$$
$$\frac{1}{4}y = \frac{1}{4}x + \frac{1}{2}\left(\frac{1}{2}y - \frac{1}{2}x\right)$$

follows, hence $d_P(x,y) = 0$ i.e. the semimetric of A as a positively convex module is trivial.

(iii) If $P \subset E$ is a positively convex module or, in particular, a positively convex subset in the linear space E and P is not linearly bounded (cf. [1]) there is a non-constant affine mapping ([16])

$$f :]0, \infty[\to P.$$

Put $x := f(\alpha),\ z := f(\beta)$ and $a := f(1)$ for $1 < \alpha < \beta$, then

$$x = \frac{\alpha - 1}{\beta - 1}z + \frac{\beta - \alpha}{\beta - 1}a$$

holds and hence

$$d_P(x,a) \leq \frac{\alpha - 1}{\beta - 1}d_P(z,a) + \frac{\beta - a}{\beta - 1}d_P(a,a) \leq \frac{\alpha - 1}{\beta - 1}.$$

For $\beta \to \infty,\ d_P(x,a) = 0$ follows. As special cases abstract or concrete cones are subsumed under this type.

(iv) In [20] Wickenhäuser gave the following example. The subset

$$P := \{(x,y) \mid -1 \leq x \leq 1,\ x^2 \leq y \leq 1\}$$

of $\mathbb{R}^2$ is obviously positively convex. Moreover, for any $0 \le t \le 1$ and any $n \in \mathbb{N}$,

$$\frac{1}{2n^2}(0,t) + \frac{1}{4n}(-\sqrt{t},t) = \frac{1}{2}\left(-\frac{\sqrt{t}}{n}, \frac{t}{n^2}\right) + \frac{1}{4n}(\sqrt{t},t)$$
$$\frac{1}{2n^2}(0,t) + \frac{1}{4n}(\sqrt{t},t) = \frac{1}{2}\left(\frac{\sqrt{t}}{n}, \frac{t}{n^2}\right) + \frac{1}{4n}(-\sqrt{t},t)$$

hold. Hence, $d_P((\sqrt{t},t),(-\sqrt{t},t)) \le n^{-1}$ and $d_P((\sqrt{t},t),(-\sqrt{t},t)) = 0$ follows. Put

$$P_0 = \{(0,y) \mid 0 \le y \le 1\},$$

then P_0 is a positively convex subset of P and the canonical projection $\pi_2(x,y) := (0,y)$ is a positively affine morphism $\pi_2 : P \to P_0$. Elementary geometric considerations show that

$$d_P(a,b) = d_{P_0}(\pi_2(a), \pi_2(b))$$

for any $a, b \in P$ holds. Moreover, for $x_0, y_0 \in P_0$, one gets $d_{P_0}(x_0,y_0) = \frac{1}{2}\|x_0 - y_0\|_1$ with the l_1-norm $\|\Box\|_1$ on $\mathbb{R}^2$.

This is very interesting, because P is linearly bounded, even compact and the same holds for the absolutely convex set $P - P$. Hence, as a convex set or module, respectively, P is metric (cf. [16], 2.7) while as a positively convex set it is not.

If we denote the *convex semimetric* of any positively convex module P by $d^c(\Box,\Box)$ and consider a defining equation for d^c ([16], 2.2)

$$\alpha a + (1-\alpha)x = \alpha b + (1-\alpha)y,$$

$0 \le \alpha < 1,\ x, y \in P$, an easy computation using (PC1), (PC2) yields

$$\alpha\left(\frac{1}{2}a + \frac{1}{b}b\right) + \frac{1-\alpha}{2}x = \alpha b + \frac{1-\alpha}{2}y$$

$$\alpha\left(\frac{1}{2}a + \frac{1}{2}b\right) + \frac{1-\alpha}{2}y = \alpha a + \frac{1-\alpha}{2}x.$$

Hence, $d_P(x,y) \le (1-\alpha)^{-1}\alpha$ follows for the semimetric of P as a positively convex module, which implies

$$d_P(x,y) \le d^c(x,y), \tag{$*$}$$

for any positively convex module. The above example shows that equality does not hold in general in $(*)$.

3. Positively convex modules and ordered linear spaces

It will be shown that any positively convex module determines an ordered linear space, which is unique up to an isomorphism. This will permit a characterization of different important types of positively convex modules.

Definition 3.1. A *linear space* E is called *ordered* if it is ordered by a generating cone C, i.e. $E = C - C$. A *morphism* $f : E_1 \to E_2$ of ordered linear spaces is a monotone linear mapping, i.e. $f(C_1) \subset C_2, C_i$ the cone of $E_i, i = 1, 2$. The ordering in an ordered linear space is defined by $x \leq y$ iff $y - x \in C$. The *ordering cone* of an ordered linear space will be denoted be Cone (E). **OrdVec** is the category of ordered linear spaces and monotone linear mappings. An ordered linear space E with cone C is called *antisymetrically ordered* if, for $x, y \in E, x \leq y$ and $y \leq x$ implies $x = y$. This is equivalent to $C \cap (-C) = \{0\}$, i.e. to the property of C to be *proper*.

If E is a linear space, E^* is the dual space and if C is any cone in E $C^* := \{\lambda \mid \lambda \in E^*,\ \lambda(c) \geq 0 \text{ for any } c \in C\}$ is called the *dual cone*. Any interval in $\mathbb{R}$ containing 0 is a positively convex set with the canonical structure inherited from $\mathbb{R}$ and will always be considered with this structure in the following if not explicitly stated otherwise. The mapping which maps every $E \in$ **OrdVec** to its cone Cone(E) induces a functor Cone: **OrdVec** $\to$ **PC**. As usual $\mathbf{PC}(P_1, P_2), P_1, P_2 \in \mathbf{PC}$, denotes the set of all morphisms from P_1 to P_2. With the pointwise defined operations and order, $\mathbf{PC}(P, \mathbb{R}), P \in \mathbf{PC}$, is a linear space ordered by the proper cone $\mathbf{PC}(C, \mathbb{R}_+)$; it is in general not an ordered linear space in the sense of 3.1

Definition 3.2. ([16]): For a positively convex module P we define $\widetilde{\rho}_P : P \to \mathbf{PC}(P, \mathbb{R})^*$ by point evaluation, $\widetilde{\rho}_P(x)(f) := f(x),\ x \in P,\ f \in \mathbf{PC}(P, \mathbb{R})$. $\widetilde{\rho}_P$ is a positively affine mapping. $\mathrm{R}(P) := \mathbb{R}_+\widetilde{\rho}_P(P) - \mathbb{R}_+\widetilde{\rho}_P(P)$ is a subspace of $\mathbf{PC}(P, \mathbb{R})^*$ and an ordered linear space with the proper positive cone $\mathbb{R}_+\widetilde{\rho}_P(P) \subset \mathbf{PC}(P, \mathbb{R}_+)^*$. The restriction of $\widetilde{\rho}_P$ to $\mathrm{R}(P)$ is denoted by ρ_P.

Proposition 3.3. (cf. [14], [16]): *The $R(P), P \in$ **PC**, induce a functor* **PC** $\to$ **OrdVec** *which is left adjoint to* Cone: **OrdVec** $\to$ **PC**.

Proof. Let $f : P \to$ Cone (E), $E \in$ **OrdVec**, be positively affine, then, for all $\lambda \in E^*, \lambda f \in \mathbf{PC}(P, \mathbb{R})$ follows. For $x, y \in P$, $\rho_p(x) = \rho_P(y)$ implies $\rho_P(x)(\lambda f) = \rho_P(y)(\lambda f), \lambda \in E^*$, i.e. $\lambda f(x) = \lambda f(y)$ for all $\lambda \in E^*$. This yields $f(x) = f(y)$ and there is a unique positively affine mapping $\varphi : \rho_P(P) \to$ Cone (E) with $\varphi\rho_P = f$. The definition

$$\varphi_0(\alpha\rho_P(x) + \beta\rho_P(y)) := (\alpha + \beta) f \left(\frac{\alpha}{\alpha + \beta} x + \frac{\beta}{\alpha + \beta} y \right)$$

describes the unique extension φ_0 : Cone $(R(P)) \to$ Cone (E) for $x, y \in P$, $\alpha, \beta \geq 0, \alpha + \beta > 0$, of φ to a cone morphism, which induces a unique monotone linear extension $\varphi_1 : \mathrm{R}(C) \to E$ because the functor Cone is full and faithful. Hence, $f =$ Cone $(\varphi_1)\rho_p$ follows.

A positively convex module P is called *preseparated*, if P is a preseparated convex module, i.e. if, for any $x, y, z \in P$ and any $0 < \alpha \leq 1, \alpha x + (1 - \alpha)z = \alpha y + (1 - \alpha)z$ implies $x = y$ (cf. [13], 4.9). This is a well-defined property because any positively convex module is a convex module. Obviously the following lemma holds:

Lemma 3.4. *Let P be a positively convex module, then P is preseparated iff, for any $x, y, z \in P$ and any $\alpha, \beta \geq 0, \alpha + \beta \leq 1, \alpha > 0, \alpha x + \beta z = \alpha y + \beta z$ implies $x = y$.*

Proposition 3.5. (cp. [16], 3.5): *Let P be a positively convex module. Then*

(i) *If P is metric it is preseparated.*
(ii) *P is preseparated if and only if ρ_P is injective.*
(iii) *Let P be metric, $x_i \in P, \alpha_i \geq 0, i = 1, 2,$ and $0 < \alpha_1 + \alpha_2 \leq 1$, then $\alpha_1 x_1 + \alpha_2 x_2 = 0$ and $\alpha_i > 0$ implies $x_i = 0$ for $i = 1, 2$.*

Proof. (i) Let, for $x, y, z \in P$ and some α with $0 < \alpha < 1$,

$$\alpha x + (1 - \alpha)z = \alpha y + (1 - \alpha)z. \qquad (*)$$

Then $(*)$ holds for all $\alpha \in [0, 1[$ (cf. [13], 2.2) and this implies $d_P(x, y) \leq \alpha^{-1}(1 - \alpha)$ for $\alpha \in]0, 1[$, hence $d_P(x, y) = 0$ and $x = y$.

(ii) If ρ_P is injective, P is trivially preseparated. In [9], 4.19 it is shown that there is a cone $\mathrm{K}(P)$ and a positively affine mapping $\kappa_P : P \to \mathrm{K}(P)$. Moreover, if P is preseparated κ_P is injective and $\mathrm{K}(P)$ satisfies the cancellation rule: $x + z = y + z,\ x, y, z \in \mathrm{K}(P)$, implies $x = y$. Hence $\mathrm{K}(P)$ can be embedded as a generating cone into an ordered linear space $\mathrm{E}(P)$. 3.3 yields the existence of a unique $\varphi : \mathrm{R}(P) \to \mathrm{E}(P)$ with $\kappa_P = \mathrm{Cone}\ (\varphi)\rho_P$, hence ρ_P is injective (cp. also [8]).

(iii) Let us first consider the case $\alpha_1 = \alpha_2 = 2^{-1}$, i.e.

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = 0.$$

This implies (cf, [20], 10.5)

$$\frac{1}{4}x_1 = \frac{1}{4}x_2 + \frac{1}{2}\left(\frac{1}{2}x_1 + \frac{1}{2}x_1\right)$$
$$\frac{1}{4}x_2 = \frac{1}{4}x_1 + \frac{1}{2}\left(\frac{1}{2}x_2 + \frac{1}{2}x_2\right),$$

hence $d(x_1, x_2) = 0$ or $x_1 = x_2$, respectively, as P is metric. For arbitrary $\alpha_i > 0,\ 0 < \alpha_1 + \alpha_2 \leq 1$, one has

$$\frac{1}{2}(\alpha_1 x_1) + \frac{1}{2}(\alpha_2 x_2) = 0$$

which implies $\alpha_1 x_1 = \alpha_2 x_2 = 0$ because of the first case. Now, P is preseparated (2.5, (i)) hence $\alpha_i > 0$ yields $x_i = 0,\ i = 1, 2$.

Remark 3.6. (i) ρ_P also induces the epireflection $\mathbf{PC} \to \mathbf{PresepPC}$ onto the full subcategory of preseparated positively convex modules. Put $\mathrm{M}(P) := \rho_P(P)$ and define γ_P as the restriction of ρ_P to $\mathrm{M}(P), P \in \mathbf{PC}$. $\mathrm{M}(P)$ is obviously preseparated and it follows directly from 3.3 that γ_P is the desired epireflection.

(ii) An abstract cone is trivially an object of $\mathbf{PC}$ ([8]). An abstract cone which can be embedded into a linear space is called *vectorial* by Nachbin in [8]. An abstract cone is vectorial iff it satisfies the cancellation rule iff it is preseparated as an object of $\mathbf{PC}$. The vectorial cones generate the full subcategory $\mathbf{VecCone}$ of $\mathbf{PC}$. Define the cone $\mathrm{C}(P) := \mathbb{R}_+\rho_P(P)$ and the morphism in $\mathbf{PC}$ $\varepsilon_P : P \to \mathrm{C}(P),\ P \in \mathbf{PC}$, as the restriction of ρ_P to $\mathrm{C}(P)$. Then a straightforward argument shows that $\mathrm{C}(P)$ is the reflection $\mathbf{PC} \to \mathbf{VecCone}$ with reflection morphism ε_P.

4. Positively convex modules and regularly ordered normed linear spaces

Definition 4.1. ([6], [9], [21]): *A normed linear space* E is called *regularly ordered* if it is ordered by a cone C and the norm $\|\Box\|$ is a *Riesz norm* with respect to C, i.e.

$$\|x\| = \inf \{\|c\| \mid -c \leq x \leq c\},$$

$x \in E$. One may assume that the cone C is closed, because $\|\Box\|$ is also a Riesz norm with respect to the closure $\overline{C}$ as a straightforward argument shows. The closed cone of E will be denoted by Cone (E), the closed unit ball by $\mathrm{O}(E)$ and the open unit ball by $\overset{\circ}{\mathrm{O}}(E)$. Moreover, one defines $\Delta(E) := \mathrm{O}(E) \cap \mathrm{Cone}(E)$.

A mapping between regularly ordered normed linear spaces $f : E_1 \to E_2$ is called a morphism if it is a *positive, linear contraction*. These morphisms and the regularly ordered normed linear spaces constitute the category $\mathbf{Vec}_1^+$ and the $\Delta(E), E \in \mathbf{Vec}_1^+$, induce a functor $\Delta : \mathbf{Vec}_1^+ \to \mathbf{Set}$ with left adjoint $\mathrm{l}_1 : \mathbf{Set} \to \mathbf{Vec}_1^+$ (cf. [9]). The full subcategory of regularly ordered Banach spaces is denoted by $\mathbf{Ban}_1^+$.

In [9] it is shown that $\mathbf{PC}$ is the category of Eilenberg-Moore algebras for the functor $\Delta : \mathbf{Vec}_1^+ \to \mathbf{Set}$. Actually only the Eilenberg-Moore algebras of $\Delta : \mathbf{Ban}_1^+ \to \mathbf{Set}$ were investigated which are the positively superconvex modules (cp. §5 in [9]). But the results of [9] carry over verbatim to $\Delta : \mathbf{Vec}_1^+ \to \mathbf{Set}$. There is a canonical so-called *comparison functor* $\widehat{\Delta} : \mathbf{Vec}_1^+ \to \mathbf{PC}$ induced by the canonical positively convex structure on $\Delta(E), E \in \mathbf{Vec}_1^+$ and there is a left adjoint $\mathrm{S} : \mathbf{PC} \to \mathbf{Vec}_1^+$ of $\widehat{\Delta}$ assigning to any $P \in \mathbf{PC}$ a universal $\mathrm{S}(P) \in \mathbf{Vec}_1^+$. The constructions of S in [5], [9] and [20] are not very satisfying because they do not give much information about the relation between P and $\mathrm{S}(P) \in \mathbf{PC}$. In the following, $\mathrm{S}(P)$ will be constructed in a different way.

Definition 4.2. ([21]): Let E be an ordered linear space. $A \subset E$ is called *absolutely dominated* if, for any $x \in A$, there is an $a \in A$ with $-a \leq x \leq a$. A is called *absolutely order convex*, if, for any $a \in A \cap C$, the interval $[-a, a] = \{x \mid x \in E, -a \leq x \leq a\}$ is contained in A. A is called *solid*, if it is absolutely dominated and absolutely order convex. For any $A \subset E$

$$\mathrm{sol}(A) := \bigcup_{a \in A \cap C} [-a, a]$$

is called the *solid hull* of A. It is the smallest solid set in E containing A, if A is absolutely dominated.

Solid sets play a fundamental role in the theory of regularly ordered linear spaces, in characterizing and constructing them, analogous to the role absolutely convex sets play for normed linear spaces (cf. [14]) and convex base sets for base normed linear spaces (cf. [2], [15], [16]). An ordered normed linear space E is a regularly ordered normed space iff the open unit ball $\overset{\circ}{\mathrm{O}}(E)$ is solid (cf. [21], 6.12).

As mentioned already a regularly ordered normed linear space E supplies a paradigmatic case of a metric positively convex set, namely $\Delta(E)$ with its canonical positively convex structure. If $x, y \in \Delta(E)$ and two equations as in 2.2 $\alpha a + \beta x = \varepsilon u + \beta y, \alpha a + \beta y = \delta v + \beta x$ are given, one gets

$$-\frac{\alpha}{2\beta} a \leq \frac{1}{2}(x - y) \leq \frac{\alpha}{2\beta} a,$$

hence $2^{-1}\|x-y\| \le d_{\Delta(E)}(x,y)$. To show the converse inequality we use the representation $\|z\| = \inf\{\alpha > 0 \mid -\alpha e \le z \le \alpha e,\ e \in \Delta(E)\}$ for the norm $\|z\|,\ z \in E$. This follows at once from the chain of inclusions

$$\overset{\circ}{\mathrm{O}}(E) \subset \mathrm{sol}(\Delta(E)) \subset \mathrm{O}(E).$$

$-\alpha a \le 2^{-1}(x-y) \le \alpha a,\ \alpha > 0,\ x,y,a \in \Delta(E)$ now implies

$$\alpha a + \frac{1}{2}x = \delta u + \frac{1}{2}y$$

$$\alpha a + \frac{1}{2}y = \varepsilon v + \frac{1}{2}x$$

with suitable $u,v \in \Delta(E), \varepsilon, \delta > 0$. By dividing these equations by a sufficiently large $M > 0,\ d_{\Delta(E)}(x,y) \le \alpha$ results or $d_{\Delta(E)}(x,y) \le 2^{-1}\|x-y\|$.

Lemma 4.3. (cf. [21], 6.3): *If E is an ordered linear space and $A \subset E$ is absorbing, convex and solid, then the Minkowski functional $\|\Box\|$ of A is a Riesz seminorm.*

Proof. Put $|x| := \inf\{\|c\| \mid -c \le x \le c\}$ for $x \in E$. If $-c \le x \le c$ and $\|c\| < \alpha$, then $-\alpha^{-1}c \le \alpha^{-1}x \le \alpha^{-1}c$. As $\alpha^{-1}c \in \overset{\circ}{\mathrm{O}}(E) \subset A,\ \alpha^{-1}x \in A$ follows, i.e. $\|x\| \le \alpha$. This yields $\|x\| \le \|c\|$ and $\|x\| \le |x|$. Conversely, $x = \alpha a \in \alpha A,\ a \in A$, implies $-a_0 \le a \le a_0$ with an $a_0 \in A$, hence $-\alpha a_0 \le x \le \alpha a_0$ and $|x| \le \|\alpha a_0\| \le \alpha$ follows, which implies $|x| \le \|x\|$ and proves the assertion.

Theorem 4.4. (cp. [12], [14], [16], [20]): *$\widehat{\Delta} : \mathbf{Vec}_1^+ \to \mathbf{PC}$ has a left adjoint functor $S : \mathbf{PC} \to \mathbf{Vec}_1^+$.*

Proof. Let, for $P \in \mathbf{PC}$,

$$\mathrm{Aff}_b^+(P) := \{f \mid f \in \mathbf{PC}(P,\mathbb{R}), f \text{ bounded}\},$$
$$\mathrm{C}_0(P) := \{f \mid f \in \mathrm{Aff}_b^+(P) \text{ and } f(x) \ge 0,\ x \in P\}.$$

$\mathrm{Aff}_b^+(P)$ is a Banach space with the supremum norm $\|\Box\|_\infty$ and is pointwise ordered. $\mathrm{C}_0(P)$ is the proper cone of positive elements of $\mathrm{Aff}_b^+(P)$ and a straight forward argument shows that $\mathrm{C}_0(P)$ is $\|\Box\|_\infty$-closed. For the subspace $\mathrm{Q}_0(P) := \mathrm{C}_0(P) - \mathrm{C}_0 P$ and $f \in \mathrm{Q}_0(P)$

$$\|f\| := \inf\{\|g\|_\infty \mid -g \le f \le g,\ g \in \mathrm{C}_0(P)\}$$

defines a Riesz seminorm on $\mathrm{Q}_0(P)$. As $\|f\|_\infty \le \|f\|$ holds for $f \in \mathrm{Q}_0(P), \|\Box\|$ is even a norm. If $g \in \mathrm{C}_0(P)$, then obviously also $\|g\| \le \|g\|_\infty$ is satisfied, i.e. $\|g\| = \|g\|_\infty$ and this shows that $\|\Box\|$ is a Riesz norm. Hence, $\mathrm{Q}_0(P)$ is a regularly ordered normed linear space with cone $\mathrm{C}_0(P)$. For a set $M \subset \mathrm{Q}_0(P)$, let $\overline{M}$ denote the closure with respect to $\|\Box\|$ and $\overline{M}_\infty$ the closure with respect to $\|\Box\|_\infty$. As $\mathrm{C}_0(P) \subset \overline{\mathrm{C}_0(P)} \subset \overline{\mathrm{C}_0(P)}_\infty = \mathrm{C}_0(P)$ holds, $\mathrm{C}_0(P)$ is also $\|\Box\|$-closed.

Because of [21], (6.12) the topological dual $\mathrm{Q}_0'(P)$ of $\mathrm{Q}_0(P)$ is a regularly ordered Banach space with the dual cone $\mathrm{C}_0'(P)$ and the Riesz norm $\|\Box\|'$ determined by $\mathrm{C}_0'(P)$. The mapping $\hat{\sigma}_P : P \to \mathrm{Q}_0'(P)$ defined by point evaluation, $\hat{\sigma}_P(x)(f) := f(x), x \in P, f \in \mathrm{Q}_0(P)$, is positively affine. For $f \in \mathrm{C}_0(P), x \in P, \hat{\sigma}_P(x)(f) \geq 0$, hence $\hat{\sigma}_P(P) \subset \mathrm{C}_0'(P)$ and $\mathrm{C}(P) := \mathbb{R}_+\hat{\sigma}_P(P)$ is a proper cone. If $\|\Box\|_\infty$ denotes the supremum norm in $\mathrm{Q}_0'(C)$ $\|\lambda\|' = \|\lambda\|_\infty$ holds for $\lambda \in \mathrm{C}_0'(P)$, hence, for $x \in P$,

$$\|\hat{\sigma}_P(x)\|' = \|\hat{\sigma}_P(x)\|_\infty = \sup\{|\hat{\sigma}_P(x)(f)| \mid \|f\| \leq 1\} \leq 1$$

follows. The restriction of $\hat{\sigma}_P$ to the subspace $\mathrm{S}(P) := \mathrm{C}(P) - \mathrm{C}(P)$, ordered by $\mathrm{C}(P)$, is denoted by $\sigma_P : P \to \mathrm{S}(P)$. For $\lambda \in \mathrm{sol}(\sigma_P(P))$, there is an $x \in P$ with $-\sigma_P(x) \leq \lambda \leq \sigma_P(x)$. This implies $\|\lambda\|' \leq \|\sigma_P(x)\|' \leq 1$, i.e. $\mathrm{sol}(\sigma_P(P))$ is $\|\Box\|'$-bounded and hence linearly bounded (cf. [1]). If $x_i \in \mathrm{sol}(\sigma_P(P)),\ i = 1, 2$, then there are $p_i \in P$ with $-\sigma_P(p_i) \leq x_i \leq \sigma_P(p_i),\ i = 1, 2$. For $\alpha, \beta \geq 0, \alpha + \beta \leq 1$, this implies

$$-\sigma_P(\alpha p_1 + \beta p_2) \leq \alpha x_1 + \beta x_2 \leq \sigma_P(\alpha p_1 + \beta p_2),$$

i.e. $\alpha x_1 + \beta x_2 \in \mathrm{sol}(\sigma_P(P))$. Also, $x \in \mathrm{sol}(\sigma_P(P))$ obviously implies $-x \in \mathrm{sol}(\sigma_P(P))$, hence $\mathrm{sol}(\sigma_P(P))$ is absolutely convex. Any $x \in \mathrm{S}(P)$ can be written as $x = \alpha\sigma_P(P))$, i.e. $\mathrm{sol}(\sigma_P(P))$ is absorbing. An application of 4.3 now yields that the Minkowski functional a $\|\Box\|$ of $\mathrm{sol}(\sigma_P(P))$ is a Riesz norm and $\mathrm{S}(P) \in \mathbf{Vec}_1^+$.

To simplify notation, let us denote the restriction of σ_P to $\hat{\Delta}(\mathrm{S}(P))$ again by σ_P. If $\sigma_P(x) = \sigma_P(y),\ x, y \in P$, and $g : P \to \hat{\Delta}(E), E \in \mathbf{Vec}_1^+$, is a positively affine mapping, $\lambda g \in \mathrm{Q}_0(C)$ holds for any $\lambda \in E'$. This implies $\sigma_P(x)(\lambda g) = \sigma_P(y)(\lambda g)$ or $\lambda(g(x)) = \lambda(g(y))$ for all $\lambda \in E'$ and we get $g(x) = g(y)$. Hence, there is a unique positively affine $\varphi : \sigma_P(P) \to \hat{\Delta}(E)$ with $g = \varphi\sigma_P$. In order to extend φ to $\mathrm{S}(P)$ take a $z \in \mathrm{S}(P), z \neq 0$, and two representations

$$z = \alpha_1\sigma_P(x_1) - \beta_1\sigma_P(y_1) = \alpha_2\sigma_P(x_2) - \beta_2\sigma_P(y_2),$$

$\alpha_i, \beta_i \geq 0, \alpha_i + \beta_i > 0, x_i, y_i \in P, i = 1, 2$. This yields

$$(\alpha_1 + \beta_2)\sigma_P\left(\frac{\alpha_1}{\alpha_1 + \beta_2}x_1 + \frac{\beta_2}{\alpha_1 + \beta_2}y_2\right) = (\alpha_2 + \beta_1)\sigma_P\left(\frac{\alpha_2}{\alpha_2 + \beta_1}x_2 + \frac{\beta_1}{\alpha_2 + \beta_1}y_1\right).$$

We may assume $\alpha_1 + \beta_2 > \alpha_2 + \beta_1$ and by dividing by $\alpha_1 + \beta_2$ one gets

$$\sigma_P\left(\frac{\alpha_1}{\alpha_1 + \beta_2}x_1 + \frac{\beta_2}{\alpha_1 + \beta_2}y_2\right) = \sigma_P\left(\frac{\alpha_2}{\alpha_1 + \beta_2}x_2 + \frac{\beta_1}{\alpha_1 + \beta_2}y_1\right)$$

and

$$g\left(\frac{\alpha_1}{\alpha_1 + \beta_2}x_1 + \frac{\beta_2}{\alpha_1 + \beta_2}y_2\right) = g\left(\frac{\alpha_2}{\alpha_1 + \beta_2}x_2 + \frac{\beta_1}{\alpha_1 + \beta_2}y_1\right).$$

Hence, $\varphi(z) := \alpha_1 g(x_1) - \beta_1 g(y_1)$ yields a well-defined mapping and a routine argument shows that $\varphi : \mathrm{S}(P) \to E$ is a morphism in $\mathbf{Vec}_1^+$ with $g = \widehat{\Delta}(\varphi)\sigma_P$ uniquely determined by g. This proves the assertion.

We are now in the position to give a characterization of metric positively convex modules analogous to the result 2.7 in [16] for convex modules. First, separated objects of **PC** are introduced.

Definition 4.5. (cf. [4], [13], [16], [20]): A positively convex module P is called *separated* if $\mathbf{PC}(P,[0,1])$ separates the points of P. This notion is analogous to the corresponding notion for convex, superconvex ([1], [12], [13]), absolutely and totally convex ([11]) modules.

Proposition 4.6. (cp. [4], [9], [16], [20]): *The following statements are equivalent for a positively convex module P:*

(i) P *is metric.*
(ii) P *is preseparated and* $\mathrm{sol}(\rho_{\mathrm{P}}(\mathrm{P}))$ *is linearly bounded.*
(iii) P *is separated.*
(iv) σ_P *is injective.*

Proof. (i) $\Rightarrow$ (ii): To simplify the notation we will omit the index P in ρ_P in the following. ρ is injective because of (i) and 3.5, (i) and (ii). To show the linear boundedness of $\mathrm{sol}(\rho(P))$ we take an $a \in \mathrm{sol}(\rho(P))$ and assume $ta \in \mathrm{sol}(\rho(P))$, for all $t \in \mathbb{R}$. Hence, for any $t > 0$, we have an inequality

$$-\rho(p_t) \le ta \le \rho(p_t) \tag{$*$}$$

in $\mathrm{R}(P)$ with $p_t \in P$. This yields $-t^{-1}\rho(p_t) \le a \le t\rho(p_t)$. a can be written as $a = \alpha\rho(p) - \beta\rho(q), \alpha, \beta \ge 0,\ p, q \in P$. As a and $M^{-1}a,\ M > 0$, span the same line, we may assume $0 \le \alpha, \beta \le 1$ and get $a = \rho(p_0) - \rho(q_0)$, with $p_0 = \alpha p,\ q_0 := \beta q$, both in P. ($*$) implies the existence of $\mu_t, \nu_t \ge 0$ and $u_t, v_t \in P$, such that

$$\rho(p_0) - \rho(q_0) = -t^{-1}\rho(p_t) + \mu_t\rho(u_t) = t^{-1}\rho(p_t) - \nu_t\rho(v_t).$$

By an elementary computation and by dividing the equations by an $n > \max\{t^{-1}, \nu_t, \mu_t\} + 1$ one gets

$$\rho\left(\frac{1}{nt}p_t + \frac{1}{n}p_0\right) = \rho\left(\frac{\mu_t}{n}u_t + \frac{1}{n}q_0\right),$$

$$\rho\left(\frac{1}{nt}p_t + \frac{1}{n}q_0\right) = \rho\left(\frac{\nu_t}{n}v_t + \frac{1}{n}p_0\right).$$

As ρ is injective, these equations also hold in P and this yields

$$d_P(p_0, q_0) \le (2t)^{-1}$$

for any $t > 0$, i.e. $d_P(p_0, q_0) = 0$ and $p_0 = q_0$ because of (i). Hence $a = 0$ follows and $\mathrm{sol}(\rho(P))$ is linearly bounded.

(ii) $\Rightarrow$ (iii): The same reasoning as in the proof of 4.4 shows that $\mathrm{sol}(\rho(P))$ is absolutely convex and absorbing. As it is also linearly bounded, 3.3 implies that its Minkowski functional $\|\Box\|$ is a Riesz norm on $\mathrm{R}(P)$ and $\mathrm{R}(P)$ is a regularly ordered normed space. ρ is injective, hence the $\lambda\rho, \lambda \in \mathrm{R}'(P)$, separate the points of P i.e. (iii) is proved.

(iii) $\Rightarrow$ (iv): Let $\sigma_P(x) = \sigma_P(y)$, for $x, y \in P$. For any $\lambda \in \mathbf{PC}(P,[0,1])$ there is a unique $\lambda_0 : \mathrm{S}(P) \to \mathbb{R}$ in $\mathbf{Vec}_1^+$ with $\lambda = \widehat{\Delta}(\lambda_0)\sigma_P$, i.e. $\lambda(x) = \lambda(y)$ follows for $\lambda \in \mathbf{PC}(P,[0,1])$. (iv) follows now from 4.5.

(iv) ⇒ (i): If $E \in \mathbf{Vec}_1^+$ the semimetric in the positively convex subset $\widehat{\Delta}(E)$ is $2^{-1}\|x - y\|, x, y \in \widehat{\Delta}(E)$. Hence, if $d_0(\Box, \Box)$ now denotes the semimetric of the positively convex module $\sigma(P) \subset \widehat{\Delta}(\mathrm{S}(C))$ and $\|\Box\|$ the norm of $\widehat{\Delta}(\mathrm{S}(C))$,

$$\frac{1}{2}\|\sigma(x) - \sigma(y)\| \leq d_0(\sigma(x), \sigma(y)) \leq d_P(x, y)$$

follows, for $x, y \in P$. If $d_P(x, y) = 0$ then $\sigma_P(x) = \sigma_P(y)$ or $x = y$ holds, i.e. $d_P(\Box, \Box)$ and also $d_0(\Box, \Box)$ are metrics.

Corollary 4.7. *If $P \in \mathbf{PC}$ is metric $\sigma_P : P \to \widehat{\Delta}(S(P))$ is an isometry.*

Proof. We know already that $2^{-1}\|\sigma(x) - \sigma(y)\| \leq d_P(x, y)$, $x, y \in P$, hence, we may assume $2^{-1}\|\sigma(x) - \sigma(y)\| < 1$. Now, it is an easy computation to show that, for $z \in \mathrm{S}(P)$,

$$\|z\| = \inf\{\alpha > 0 \mid -\alpha\sigma(p) \leq z < \alpha\sigma(p),\ p \in P\}$$

holds. Let $2^{-1}\|x - y\| < \alpha < 1$ and

$$-\alpha\sigma(a) \leq \frac{1}{2}(\sigma(x) - \sigma(y)) \leq \alpha\sigma(a),$$

with $a \in P$. By the same argument as in the passage above using the fact that σ is injective this yields $d_P(x, y) \leq \alpha$ and, hence, $d_P(x, y) \leq 2^{-1}\|x - y\|$.

For metric P, σ_P and ρ_P are essentially the same up to an isomorphism. If P is metric ρ_P is injective and $\mathrm{R}(P)$ is a regularly ordered normed space (cf. 4.6). If $i_R : \widehat{\Delta}(\mathrm{R}(P)) \subset \mathrm{Cone}(\mathrm{R}(P))$, $i_S : \widehat{\Delta}(\mathrm{S}(P)) \subset \mathrm{Cone}(\mathrm{S}(P))$ are the inclusions and $\rho_P = i_R\rho_P^0$, one gets uniquely determined positive linear contractions $\varphi : \mathrm{S}(P) \to \mathrm{R}(P)$ and $\psi : \mathrm{R}(P) \to \mathrm{S}(P)$ with $\rho_P^0 = \widehat{\Delta}(\varphi)\sigma_P$, $i_S\sigma_P = \mathrm{Cone}(\psi)\rho_P$. This implies that φ and ψ are isomorphisms.

5. Complete positively convex modules

A positively convex module P is called *complete* if it is metric and complete with respect to $d_P(\Box, \Box)$. There is a close connection between complete positively convex and *positively superconvex* modules.

Definition 5.1. ([1], [9], [16]): $\Omega_{psc} := \{\hat{\alpha} \mid \hat{\alpha} = (\alpha_i \mid i \in \mathbb{N}\},\ \alpha_i \geq 0,\ i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} \alpha_i \leq 1\}$ is the set of *formal positively superconvex combinations.* A *positively superconvex module* P is a non-empty set together with a family of mappings $\hat{\alpha}_P : P^{\mathbb{N}} \to P,\ \hat{\alpha} \in \Omega_{psc}$. Moreover, with the notation

$$\sum_{i=1}^{\infty} \alpha_i x_i := \hat{\alpha}_P(x_1, x_2, \ldots),$$

$x_i \in P,\ i \in \mathbb{N}$, the following equations are satisfied:

$$\sum_{i=1}^{\infty} \delta_{ik} x_i = x_k, \qquad \text{(PSC1)}$$

$x_i \in P,\ i \in \mathbb{N},\ \delta_{ik}$ the Kronecker symbol, and

$$\sum_{i=1}^{\infty} \alpha_i \left(\sum_{k=1}^{\infty} \beta_{ik} x_k \right) = \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} \alpha_i \beta_{ik} \right) x_k, \tag{PSC2}$$

for $x_k \in P,\ k \in \mathbb{N},\ (\alpha_i \mid i \in \mathbb{N}),\ (\beta_{ik} \mid k \in \mathbb{N}) \in \Omega_{psc}, i \in \mathbb{N}$.

As $\Omega_{pc} \subset \Omega_{psc}$ (by identifying $(\alpha_1, \dots, \alpha_n) \in \Omega_{pc}$ with $\hat{\alpha} := (\overline{\alpha}_i \mid i \in \mathbb{N}),\ \overline{\alpha}_i := \alpha_i,\ 1 \leq i \leq n,\ \overline{\alpha}_i = 0$, for $i > n$) any positively superconvex module is in **PC**. A mapping $f : P_1 \to P_2$ between positively superconvex modules is called *positively superaffine* or a morphism if

$$f\left(\sum_{i=1}^{\infty} \alpha_i x_i \right) = \sum_{i=1}^{\infty} \alpha_i f(x_i)$$

holds for $\hat{\alpha} \in \Omega_{psc}$ and $x_i \in P_1,\ i \in \mathbb{N}$. The positively superconvex modules and these morphisms form the subcategory **PSC** $\subset$ **PC**. As in the case of **PC** we have a canonical so-called "comparison" functor $\widehat{\Delta} : \mathbf{Ban}_1^+ \to \mathbf{PSC}$. In [9] it was proved that **PSC** is the Eilenberg-Moore category for the functor $\Delta : \mathbf{Ban}_1^+ \to \mathbf{Set}$ with its left adjoint $\mathrm{l}_1 : \mathbf{Set} \to \mathbf{Ban}_1^+$. The construction of the left adjoint $\mathrm{S}_1 : \mathbf{PSC} \to \mathbf{Ban}_1^+$ of $\widehat{\Delta}$ in [9] is rather indirect and uses the Adjoint Functor Theorem.

A subset P of a linear space E is called a *positively superconvex set* if it is a positively superconvex module and the positively superconvex sums extend the usual positively convex sums in E, i.e. if $\hat{\alpha} \in \Omega_{psc},\ p_i \in P,\ i \in \mathbb{N}$, and $\alpha_i = 0$ for $i > n_0$, then $\sum\limits_{i=1}^{\infty} \alpha_i p_i = \sum\limits_{i=1}^{n_0} \alpha_i p_i$ (cf. [3], [16]).

Examples of positively superconvex modules abound. All bounded positively convex subsets of finite dimensional linear spaces are positively superconvex as are all bounded closed positively convex subsets of Banach spaces. If L is a σ-complete lower semilattice one defines

$$\sum_{i=1}^{\infty} \alpha_i x_i := \bigwedge_{\substack{i=1 \\ \alpha_i \neq 0}}^{\infty} x_i$$

for $x_i \in L,\ i \in \mathbb{N}$, and $(\alpha_i \mid i \in \mathbb{N}) \in \Omega_{psc}$. Then L is a positively superconvex module (cf. (2.4), (i)). Also any complete positively convex module is positively superconvex (see 5.2). As **PSC** is a subcategory of **PC** 2.3 also holds for **PSC**. But 2.3, (ii), is also true for infinite combinations. To see this let $x_i, y_i \in P,\ i \in \mathbb{N}$, for a positively superconvex module P and $\hat{\alpha} \in \Omega_{psc}$. We may assume that $\alpha_i > 0$ for all $i \in \mathbb{N}$. Put $A_n := \sum\limits_{i=1}^{n} \alpha_i > 0,\ n \in \mathbb{N}$, and $A := \sum\limits_{i=1}^{\infty} \alpha_i > 0$. Then (PSC2) implies

$$\sum_{i=1}^{\infty} \alpha_i x_i = A_n \sum_{i=1}^{n} \frac{\alpha_i}{A_n} x_i + (A - A_n) \sum_{i=n+1}^{\infty} \frac{\alpha_i}{A - A_n} x_i,$$

and analogously for $\sum\limits_{i=1}^{\infty} \alpha_i y_i$. 2.3, (ii), now yields

$$d_P\left(\sum_{i=1}^{\infty} \alpha_i x_i, \sum_{i=1}^{\infty} \alpha_i y_i \right) \leq \sum_{i=1}^{n} \alpha_i d_P(x_i, y_i) + (A - A_n),$$

which proves the assertion.

Proposition 5.2. *A complete positively convex module P is positively superconvex.*

Proof. Let $x_i \in P,\ i \in \mathbb{N}$, and $(\alpha_i \mid i \in \mathbb{N}) \in \Omega_{psc}$. Moreover, we may assume that $\alpha_i > 0$ for all $i \in \mathbb{N}$. With $A_n := \sum\limits_{i=1}^{n} \alpha_i > 0,\ n \in \mathbb{N},\ A := \sum\limits_{i=1}^{\infty} \alpha_i > 0$ and

$$s_n := \sum_{i=1}^{n} \frac{\alpha_i}{A_n} x_i,$$

$n \in \mathbb{N}$, for $m > n$

$$\left(1 - \frac{A_n}{A_m}\right) \sum_{i=n+1}^{m} \frac{\alpha_i}{A_m - A_n} x_i + \frac{A_n}{A_m} s_n = \left(1 - \frac{A_n}{A_m}\right) s_m + \frac{A_n}{A_m} s_m$$

follows. If we denote the convex semimetric (cf. [16], 2.2), of P considered as as convex module by $d^c(\Box, \Box)$ this implies

$$d^c(s_m, s_n) \leq A_n^{-1}(A_m - A_n) \leq \alpha_1^{-1}(A_m - A_n).$$

Now $d_P(s_m, s_n) \leq d^c(s_m, s_n)$ holds because of (2.4), (iv), $(*)$. Hence s_n is a $d_P(\Box, \Box)$-Cauchy sequence which has a limit in P and one defines

$$\sum_{i=1}^{\infty} \alpha_i x_i := \lim_{n \to \infty} s_n.$$

A routine computation shows that this definition makes P a positively superconvex module and that the infinite combinations extend the finite ones. That the converse implication of 5.2 does not holds is shown by the positively superconvex set $\mathrm{Cone}(E) \cap \mathring{\mathrm{O}}(E)$ in any regularly ordered Banach space E, which in general is not complete.

Lemma 5.3. ([14], 1.5, [16], 4.2): *Let E be a linear space ordered by the proper cone $C = \mathbb{R}_+ P$, where P is a positively convex set, and let $\|\Box\|$ denote the Minkowski functional of* $\mathrm{sol}(P)$, $\|x\| = \inf\{\alpha > 0 \mid -\alpha p \leq x \leq \alpha p,\ p \in P\},\ x \in E$. *If P is positively superconvex, in particular, if it is complete, E with $\|\Box\|$ is a regularly ordered Banach space.*

Proof. If $x \in 2^{-1}P - 2^{-1}P$ holds, $x \in \mathrm{sol}(P)$ follows. $x \in \mathrm{sol}(P)$, i.e. $-p \leq x \leq p$ with $p \in P$, implies $\|p \pm x\| < 4$. Hence, there are $\lambda_\pm$ with $\|p \pm x\| \leq \lambda_\pm < 4$ and $q_\pm \in P$ with $p \pm x = \lambda_\pm q_\pm$. As $\lambda_\pm q_\pm \in 4P$ and $x = 2^{-1}\lambda_+ q_+ - 2^{-1}\lambda_- q_-$ holds, we get $x \in 2P - 2P$ and

$$\frac{1}{2}P - \frac{1}{2}P \subset \mathrm{sol}(P) \subset 2P - 2P. \qquad (*)$$

If the Minkowski functional of the absolutely convex set $P - P$ is denoted by $\|\Box\|_0$, $(*)$ yields

$$\frac{1}{2}\|\Box\|_0 \leq \|\Box\| \leq 2\|\Box\|_0.$$

Now, $P - P$ is linearly bounded ([1], 4.2), and superconvex ([1]), hence totally convex and E with $\|\Box\|_0$ is a Banach space (cf. [14], 1.5). The assertion now follows from the last inequality.

There is a universal completion of a positively convex module, which is presented in two forms, 5.4 and 5.5.

Theorem 5.4. (cp. [16], 3.3): *$\widehat{\Delta} : \mathbf{Ban}_1^+ \to \mathbf{PC}$ has a left adjoint $\mathrm{S}_1 : \mathbf{PC} \to \mathbf{Ban}_1^+$.*

Proof. In the following the notations and results in the proof of 4.4 are used. We will also omit the index P in $\hat{\sigma}_P$. $\hat{\sigma}(P) \subset \mathrm{C}_0'(P) \cap \overset{\circ}{\mathrm{O}}(\mathrm{Q}_0'(P))$ holds and, because of [21], 6.12, $\mathrm{C}_0'(P)$ is complete i.e. $\Delta(\mathrm{Q}_0'(P))$ is complete and hence positively superconvex. Therefore the positively superconvex hull of $\hat{\sigma}(P)$ $\mathrm{psconv}(\hat{\sigma}(P)) = \{\sum_{i=1}^{\infty} \alpha_i \hat{\sigma}(p_i) \mid (\alpha_i \mid i \in \mathbb{N}) \in \Omega_{psc},\ p_i \in P,\ i \in \mathbb{N}\}$ exists in $\Delta(\mathrm{Q}_0'(P))$. Define $\mathrm{C}_1(P) := \mathbb{R}_+\mathrm{psconv}(\hat{\sigma}(P))$ and $\mathrm{S}_1(P) := \mathrm{C}_1(P) - \mathrm{C}_1(P)$. Then 5.3 implies $\mathrm{S}_1(P) \in \mathbf{Ban}_1^+$ and the restriction of $\hat{\sigma}_P$ to $\widehat{\Delta}(\mathrm{S}_1(P))$ is denoted by τ_P or simply by τ in the following.

Now, consider $E \in \mathbf{Ban}_1^+$ and a morphism $f : P \to \widehat{\Delta}(E)$ in $\mathbf{PC}$ and $x, y \in P$ with $\tau(x) = \tau(y)$. For any $\lambda \in E'$ $\lambda f \in \mathrm{Q}_0(P)$ holds hence $\tau(x)(\lambda f) = \tau(y)(\lambda f)$ or $\lambda(f(x)) = \lambda(f(y))$ follows. This yields $f(x) = f(y)$ and therefore a unique $\mathbf{PC}$-morphism $\varphi_0 : \tau(P) \to \widehat{\Delta}(E)$ with $f = \varphi_0\tau$. If $\sum_i \alpha_i \tau(p_i) = \sum_i \beta_i \tau(q_i)$ in $\mathrm{psconv}(\hat{\sigma}(P))$, application ..... $\lambda f, \lambda \in E'$, yields $\sum_i \alpha_i \lambda f(p_i) = \sum_i \beta_i \lambda f(q_i)$ i.e. $\sum_i \alpha_i f(p_i) = \sum_i \beta_i f(q_i)$. Hence, $\varphi_1\left(\sum_i \alpha_i \tau(p_i)\right) := \sum_i \alpha_i f(p_i)$ is a well-defined positively superaffine mapping $\varphi_1 : \mathrm{psconv}(\hat{\sigma}(P)) \to \widehat{\Delta}(E)$, which, in turn, can be uniquely extended to a cone morphism $\varphi_2 : \mathrm{C}_1(P) \to \mathrm{Cone}(E)$. To see this, consider $\alpha u = \beta v,\ \alpha, \beta > 0,\ u, v \in \mathrm{psconv}(\hat{\sigma}(P))$ and any $M > \alpha, \beta$. Then $(M^{-1}\alpha)u = (M^{-1}\beta)v \in \mathrm{psconv}(\hat{\sigma}(P))$ holds and $M^{-1}\alpha\varphi_1(u) = M^{-1}\beta\varphi_1(v)$ or $\alpha\varphi_1(u) = \beta\varphi_1(v)$ follows. Hence, $\varphi_2(\alpha u) := \alpha\varphi_1(u)$ yields the unique extension of φ_1. An analogous argument shows that for $z \in \mathrm{S}_1(P),\ z = c_1 - c_2,\ c_i \in \mathrm{C}_1(P),\ i = 1, 2,\ \varphi(z) := \varphi_2(c_1) - \varphi_2(c_2)$ defines a unique extension $\varphi : \mathrm{S}_1(P) \to E$ of φ_2. φ satisfies the equation $f = \widehat{\Delta}(\varphi)\tau$ and is uniquely determined by it as retracing the above argument step by step shows.

This proof also shows that, for a positively superconvex module P, $\mathrm{S}_1(P) = \mathrm{S}(P)$ and $\tau_P = \sigma_P$ hold because $\hat{\sigma}_P(P)$ is already positively superconvex. Moreover, the fact that $\mathrm{S}_1(P)$ is the completion of $\mathrm{S}(P)$ as shown in the proof of 5.4 is only a special case of a general method. If E is a regularly ordered normed linear space let $e : E \to E''$ be the canonical isometric and isotonic embedding and put $P := \mathrm{psconv}(e(\Delta(E)))$. Then it can be shown as in the proof of 5.4 that $E_1 := \mathbb{R}_+P - \mathbb{R}_+P$ together with $e_1 : E \to E_1$, the restriction of e, is the completion of E.

Proposition 5.5. *A positively convex module P is metric if and only if τ_P is an injective isometry.*

Proof. If P is metric, 4.6 implies that $\tau = \tau_P$ is injective because $\tau_P = \mathrm{in}\ \sigma_P$ with the inclusion in: $\Delta(\mathrm{S}(P)) \to \Delta(\mathrm{S}_1(P))$. Let $d(\square,\square) = d_P(\square,\square)$, then, for $x, y \in P$,

$$\frac{1}{2}\|\tau(x) - \tau(y)\| \leq d(x,y)$$

follows from 2.3, (iv). We may assume $\|\tau(x) - \tau(y)\| < 2$ and $2^{-1}\|\tau(x) - \tau(y)\| < \alpha < 1$. Then there is $p \in P$ with $-\alpha\tau(p) \leq 2^{-1}(\tau(x) - \tau(y)) \leq \alpha\tau(p)$ and there exist $u, v \in P, \varepsilon, \delta \geq 0$, such that

$$\frac{1}{2}(\tau(x) - \tau(y)) = -\alpha\tau(p) + \delta\tau(u),$$

$$\frac{1}{2}(\tau(x) - \tau(y)) = \alpha\tau(p) - \varepsilon\tau(v).$$

By dividing by an $M > \max\{1, \varepsilon, \delta\}$ and rearranging one gets

$$\frac{\alpha}{M}\tau(p) + \frac{1}{2M}\tau(x) = \frac{\delta}{M}\tau(u) + \frac{1}{2M}\tau(y),$$

$$\frac{\alpha}{M}\tau(p) + \frac{1}{2M}\tau(y) = \frac{\varepsilon}{M}\tau(v) + \frac{1}{2M}\tau(x).$$

As τ is injective the above equations also hold in P. Hence $d(x,y) \leq \alpha$ and $d(x,y) = 2^{-1}\|\tau(x) - \tau(y)\|$ follow. The converse statement is trivial.

Theorem 5.6. (cp. [16]): *The full subcategory* **ComplPC** *of* **PC** *of complete positively convex modules is a reflective subcategory with reflection functor* $C(P) := \overline{\tau_p(P)}$ *and reflection morphism* $\gamma_P : P \to C(P)$ *the restriction of* τ_P.

Proof. For $P \in \mathbf{PC}$ denote by γ_P the restriction of τ_P to $\mathrm{C}(P) := \overline{\tau_P(P)}$, the closure of $\tau_P(P)$ in $\mathrm{S}_1(P)$. $\mathrm{C}(P)$ is complete and we consider a morphism $f : P \to Q$ in **PC**, $Q \in$ **ComplPC**. Because of 5.4 there is a unique morphism $\varphi : \mathrm{S}_1(P) \to \mathrm{S}_1(Q)$ with $\widehat{\Delta}(\varphi)\tau_P = \tau_Q f$. φ can be restricted to a morphism $\varphi_1 : \overline{\tau_P(P)} \to \overline{\tau_Q(Q)}$. For any $z \in \mathrm{C}(Q) = \overline{\tau_Q(Q)}$ there is a Cauchy sequence $\tau_Q(q_n), q_n \in Q, n \in \mathbb{N}$, converging to z. As τ_Q is an injective isometry, q_n is a Cauchy sequence, hence convergent in Q and $z = \tau_Q(\lim_{n\to\infty} q_n) \in \tau_Q(Q)$. Hence, $\tau_Q(Q) = \overline{\tau_Q(Q)}$ follows, γ_Q is an isomorphism and $\varphi_0 := \gamma_Q^{-1}\varphi_1$ is the unique morphism with $f = \varphi_0\gamma_Q$.

If **B** is the full subcategory of **ComplPC** spanned by the $P \in$ **ComplPC** such that τ_P is an isomorphism, it can be easily verified that $\widehat{\Delta} : \mathbf{Ban}_1^+ \to \mathbf{B}$ is an equivalence. The restriction K of $\widehat{\Delta} \circ \mathrm{S}_1$ to **ComplPC** together with $\tau_P : P \in$ **ComplPC**, is a mono-reflection K : **ComplPC** $\to$ **B**, $\tau_P : P \to \mathrm{K}(P)$. If $f\tau_P = g\tau_P$ in **ComplPC**, $f : \mathrm{K}(P) \to Q$, $\tau_Q f = \widehat{\Delta}(f_1)$ and $\tau_Q g = \widehat{\Delta}(g_1)$ follows with unique $f_1, g_1 : \mathrm{S}_1(P) \to \mathrm{S}_1(Q)$. This implies $\widehat{\Delta}(f_1)\tau_P = \widehat{\Delta}(g_1)\tau_P$ or $f_1 = g_1$, i.e. $\tau_Q g = \tau_Q f$ which yields $g = f$. Hence, τ_P is a bimorphism and **B** a bireflective subcategory.

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