Journal of Convex Analysis
Volume 10 (2003), No. 1, 89–107

Smoothness of Absolute Norms on $\mathbb{C}^n$

Ken-Ichi Mitani
Department of Mathematics and Information Science,
Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
mitani@toki.gs.niigata-u.ac.jp

Kichi-Suke Saito*
Department of Mathematics, Faculty of Science,
Niigata University, Niigata 950-2181, Japan
saito@math.sc.niigata-u.ac.jp

Tomonari Suzuki*
Department of Mathematics and Information Science,
Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
tomonari@math.sc.niigata-u.ac.jp

Received June 10, 2001

In this paper, we study norming functionals of absolute normalized norms on $\mathbb{C}^n$. We also prove the characterization of smoothness of absolute normalized norms on $\mathbb{C}^n$.

Keywords: Absolute normalized norm, norming functional, smoothness

1. Introduction

A norm $\|\cdot\|$ on $\mathbb{C}^n$ is said to be absolute if

$$\|(x_0, x_1, \cdots, x_{n-1})\| = \big\|\big(|x_0|, |x_1|, \cdots, |x_{n-1}|\big)\big\|$$

for all $(x_0, x_1, \cdots, x_{n-1}) \in \mathbb{C}^n$, and normalized if

$$\|(1, 0, \cdots, 0)\| = \|(0, 1, 0, \cdots, 0)\| = \cdots = \|(0, \cdots, 0, 1)\| = 1.$$

The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(x_0, x_1, \cdots, x_{n-1})\|_p = \begin{cases} \big(|x_0|^p + |x_1|^p + \cdots + |x_{n-1}|^p\big)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\big\{|x_0|, |x_1|, \cdots, |x_{n-1}|\big\} & \text{if } p = \infty. \end{cases}$$

Let AN_n be the family of all absolute normalized norms on $\mathbb{C}^n$. Bonsall and Duncan in [3] showed the following characterization of absolute normalized norms on $\mathbb{C}^2$ (cf. [6]). Namely, the set AN_2 of all absolute normalized norms on $\mathbb{C}^2$ is in one-to-one correspondence with the set Ψ_2 of all (continuous) convex functions on $[0, 1]$ satisfying $\psi(0) = \psi(1) = 1$ and $\max\{1 - t, t\} \le \psi(t) \le 1$ for $t \in [0, 1]$. The correspondence is given by

$$\psi(t) = \|(1 - t, t)\| \quad \text{for } t \in [0, 1]. \tag{1}$$

*The authors are supported in part by Grants-in-Aid for Scientific Research, Japan Society for the Promotion of Science.

ISSN 0944-6532 / \$ 2.50 © Heldermann Verlag

Indeed, for any $\psi \in \Psi_2$, the norm $\|\cdot\|_\psi$ on $\mathbb{C}^2$ defined as

$$\|(x_0, x_1)\|_\psi = \begin{cases} \big(|x_0| + |x_1|\big)\psi\left(\dfrac{|x_1|}{|x_0| + |x_1|}\right), & \text{if } (x_0, x_1) \neq (0, 0), \\ 0, & \text{if } (x_0, x_1) = (0, 0) \end{cases}$$

belongs to AN_2 and satisfies (1). From this result, we have a plenty of concrete absolute normalized norms on $\mathbb{C}^2$ which are not ℓ_p-type. Recently, Saito, Kato and Takahashi in [7] generalized this result to $\mathbb{C}^n$. Before stating it, we give some notations. For each $n \in \mathbb{N}$ with $n \geq 2$, we put

$$\Delta_n = \left\{ (t_1, t_2, t_3, \cdots, t_{n-1}) \in \mathbb{R}^{n-1} : t_j \geq 0, \sum_{j=1}^{n-1} t_j \leq 1 \right\}$$

and define the set Ψ_n of all (continuous) convex functions on Δ_n satisfying the following conditions:

$$\begin{aligned} &\psi(0, 0, \cdots, 0) = \psi(1, 0, 0, \cdots, 0) = \psi(0, 1, 0, \cdots, 0) \\ &\quad = \cdots = \psi(0, \cdots, 0, 1) = 1, \end{aligned} \tag{A_0}$$

$$\begin{aligned} &\psi(t_1, \ldots, t_{n-1}) \geq \\ &\quad (t_1 + \cdots + t_{n-1})\psi\left(\frac{t_1}{t_1 + \cdots + t_{n-1}}, \cdots, \frac{t_{n-1}}{t_1 + \cdots + t_{n-1}}\right), \\ &\qquad\qquad\qquad \text{if } t_1 + \cdots + t_{n-1} \neq 0, \end{aligned} \tag{A_1}$$

$$\psi(t_1, \ldots, t_{n-1}) \geq (1 - t_1)\psi\left(0, \frac{t_2}{1 - t_1}, \cdots, \frac{t_{n-1}}{1 - t_1}\right), \quad \text{if } t_1 \neq 1, \tag{A_2}$$

$$\psi(t_1, \ldots, t_{n-1}) \geq (1 - t_2)\psi\left(\frac{t_1}{1 - t_2}, 0, \frac{t_3}{1 - t_2}, \cdots, \frac{t_{n-1}}{1 - t_2}\right), \quad \text{if } t_2 \neq 1, \tag{A_3}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\psi(t_1, \ldots, t_{n-1}) \geq (1 - t_{n-1})\psi\left(\frac{t_1}{1 - t_{n-1}}, \cdots, \frac{t_{n-2}}{1 - t_{n-1}}, 0\right), \quad \text{if } t_{n-1} \neq 1. \tag{A_n}$$

Saito, Kato and Takahashi in [7] showed that, for each $n \in \mathbb{N}$ with $n \geq 2$, AN_n and Ψ_n are in one-to-one correspondence under the following equation:

$$\psi(t_1, \cdots, t_{n-1}) = \left\| \left(1 - \sum_{j=1}^{n-1} t_j, t_1, \cdots, t_{n-1}\right) \right\| \tag{2}$$

for $(t_1, \cdots, t_{n-1}) \in \Delta_n$. Indeed, for any $\psi \in \Psi_n$, the norm $\|\cdot\|_\psi$ on $\mathbb{C}^n$ defined as

$$\begin{aligned} &\big\|(x_0, x_1, \cdots, x_{n-1})\big\|_\psi = \\ &\begin{cases} \big(|x_0| + \cdots + |x_{n-1}|\big)\psi\left(\dfrac{|x_1|}{|x_0| + \cdots + |x_{n-1}|}, \cdots, \dfrac{|x_{n-1}|}{|x_0| + \cdots + |x_{n-1}|}\right), \\ \qquad\qquad\qquad \text{if } (x_0, \cdots, x_{n-1}) \neq (0, \cdots, 0), \\ 0, \qquad\qquad\quad \text{if } (x_0, \cdots, x_{n-1}) = (0, \cdots, 0) \end{cases} \end{aligned}$$

belongs to AN_n and satisfies (2). For $1 \leq p \leq \infty$, the ℓ_p-norm $\|\cdot\|_p$ on $\mathbb{C}^n$ is an absolute normalized norm, and so the associated function ψ_p is defined by

$$\psi_p(t_1, t_2, \cdots, t_{n-1}) =$$
$$\begin{cases} \left(\left(1 - \sum_{j=1}^{n-1} t_j\right)^p + t_1^p + \cdots + t_{n-1}^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\left\{1 - \sum_{j=1}^{n-1} t_j, t_1, \cdots, t_{n-1}\right\} & \text{if } p = \infty. \end{cases}$$

In [7, 8], we proved that, if $\psi \in \Psi_n$, then $(\mathbb{C}^n, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex on Δ_n.

Our main purpose of this paper is to give the necessary and sufficient condition of ψ that $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth. Namely, we shall show that the space $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth if and only if the associated convex function ψ satisfies that, for each $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$, the following equalities hold:

1. $\psi'_-(t; p_j - t) = \psi'_+(t; p_j - t)$ for all $j \in I_n$ with $t_j > 0$;
2. $\psi'_+(t; p_j - t) = -\psi(t)$ for all $j \in I_n$ with $t_j = 0$

(see the notations of ψ'_-, ψ'_+, p_j and I_n in Section 2). In Section 3 and Section 4, we calculate all norming functionals of absolute normalized norms on $\mathbb{C}^2$ and $\mathbb{C}^n$, respectively. In Section 5, we prove the characterization of smoothness of absolute normalized norms on $\mathbb{C}^n$.

2. Preliminaries

Throughout of this paper, we denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ the set of positive integers, real numbers and complex numbers, respectively. Let X be a Banach space with norm $\|\cdot\|$ and let X^* be the dual space of X. $\alpha \in X^*$ is said be a norming functional of $x \in X$ with $x \neq 0$ if $\|\alpha\| = 1$ and $\langle \alpha, x \rangle = \|x\|$ (see [1]). We denote by $D(X, x)$ the set of all norming functionals of x. The Hahn-Banach theorem yields that, for every $x \in X$ with $x \neq 0$, there exists at least one norming functional of x. A Banach space X is said to be smooth if for every $x \in X$ with $x \neq 0$, there exists a unique norming functional of x. We know that X is smooth if and only if $\|\cdot\|$ is Gâteaux differentiable at any $x \in X \setminus \{0\}$, that is,

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for every $x, y \in X$ with $x \neq 0$ (cf. [1]). Let f be a continuous convex function from a convex subset C of a real Banach space X into $\mathbb{R}$. As in [4], we denote by $\partial f(x)$ the subdifferential of f at $x \in C$;

$$\partial f(x) = \{a \in X^* : f(y) \geq f(x) + \langle a, y - x \rangle \text{ for } y \in C\}.$$

It is clear that $\partial f(x)$ is a closed convex subset of X^*. We know $\partial f(x) \neq \varnothing$ at every $x \in \overset{\circ}{C}$, where $\overset{\circ}{C}$ is the set of interior points of C. In particular, if C is the closed interval $[0, 1]$ of

$\mathbb{R}$, then the following equation holds:

$$\partial f(t) = \begin{cases} \big(-\infty, f'_R(t)\big], & \text{if } t = 0, \\ \big[f'_L(t), f'_R(t)\big], & \text{if } 0 < t < 1, \\ \big[f'_L(t), \infty\big), & \text{if } t = 1, \end{cases}$$

where $f'_L(t)$ is the left derivative of f at t and $f'_R(t)$ is the right derivative of f at t, respectively.

In this paper, we use the following notations. For $n \in \mathbb{N}$ with $n \geq 2$, we put $I_n = \{0, 1, 2, \cdots, n-1\}$. We also put

$$p_0 = (0, 0, 0, \cdots, 0) \in \Delta_n$$

and

$$p_j = (0, 0, \cdots, 0, \overset{(j)}{1}, 0, 0, \cdots, 0) \in \Delta_n$$

for $j = 1, 2, \ldots, n-1$. For $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$, we put $t_0 \in [0, 1]$ as

$$t_0 = 1 - \sum_{j=1}^{n-1} t_j,$$

and $q_j(t) \in \Delta_n$ as

$$q_j(t) = \begin{cases} \dfrac{1}{1 - t_j}(t - t_j p_j), & \text{if } t \neq p_j, \\ p_j, & \text{if } t = p_j \end{cases}$$

for $j \in I_n$. Note that, for each $t \in \Delta_n$, t is on the line segment between p_j and $q_j(t)$ for $j \in I_n$. From the conditions (A_0)–(A_n) in Section 1, it is clear that a (continuous) convex function ψ on Δ_n belongs to Ψ_n if and only if

$$\psi(p_j) = 1 \quad \text{and} \quad \psi(t) \geq (1 - t_j)\psi\big(q_j(t)\big)$$

for all $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ and $j \in I_n$. We denote by $\overset{\circ}{\Delta}_n$ the set of interior points of Δ_n. It is clear that

$$\overset{\circ}{\Delta}_n = \left\{ (t_1, t_2, t_3, \cdots, t_{n-1}) \in \mathbb{R}^{n-1} : t_j > 0 (\text{for } j = 1, \cdots, n-1), \sum_{j=1}^{n-1} t_j < 1 \right\}.$$

We define the directional derivative $\psi'_+(t; s)$ of ψ at $t \in \Delta_n$ with respect to $s \in \mathbb{R}^{n-1}$ which satisfies $t + \lambda s \in \Delta_n$ for some $\lambda > 0$,

$$\psi'_+(t; s) = \lim_{\lambda \to +0} \frac{\psi(t + \lambda s) - \psi(t)}{\lambda}.$$

Similarly, if $t \in \Delta_n$ and $s \in \mathbb{R}^{n-1}$ satisfy $t + \lambda s \in \Delta_n$ for some $\lambda < 0$, we define $\psi'_-(t; s)$ by

$$\psi'_-(t; s) = \lim_{\lambda \to -0} \frac{\psi(t + \lambda s) - \psi(t)}{\lambda}.$$

It is clear that $\psi'_-(t;s) = -\psi'_+(t;-s)$ if there exists $\lambda > 0$ such that $t - \lambda s$ belong to Δ_n. We also denote by $\|\cdot\|_*$ the norm of the dual space of $(\mathbb{C}^n, \|\cdot\|_\psi)$. That is, for $(\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) \in \mathbb{C}^n$,

$$\begin{aligned}
&\big\|(\alpha_0, \alpha_1, \cdots, \alpha_{n-1})\big\|_* \\
&= \sup\Big\{\big|\big\langle(\alpha_0, \alpha_1, \cdots, \alpha_{n-1}), (x_0, x_1, \cdots, x_{n-1})\big\rangle\big| : \big\|(x_0, x_1, \cdots, x_{n-1})\big\|_\psi = 1\Big\} \\
&= \sup\left\{\left|\sum_{j=0}^{n-1} \alpha_j x_j\right| : \big\|(x_0, x_1, \cdots, x_{n-1})\big\|_\psi = 1\right\}.
\end{aligned}$$

3. Norming functionals on $(\mathbb{C}^2, \|\cdot\|_\psi)$

In this section, we describe the set $D(\mathbb{C}^2, x)$ of all norming functionals of x in $\mathbb{C}^2$ (cf. [3]). The reason for this is that the result for $\mathbb{C}^2$ illustrates all the mechanisms involved in the induction to follow. Fix $\psi \in \Psi_2$. For each $t \in (0, 1]$, we denote by $\psi'_L(t)$ the left derivative of ψ at t. Similarly for each $t \in [0, 1)$, we denote by $\psi'_R(t)$ the right derivative of ψ at t. Since $\psi(0) = 1$ and $\psi(t) \geq 1 - t$ for $t \in [0, 1]$, we have

$$\psi'_R(0) = \lim_{t \to +0} \frac{\psi(t) - \psi(0)}{t} \geq \lim_{t \to +0} \frac{1 - t - 1}{t} = -1.$$

Similarly, since $\psi(1) = 1$ and $\psi(t) \geq t$ for $t \in [0, 1]$, we have

$$\psi'_L(1) = \lim_{t \to -0} \frac{\psi(1 + t) - \psi(1)}{t} \leq \lim_{t \to -0} \frac{1 + t - 1}{t} = 1.$$

Thus, if $s, t \in (0, 1)$ with $s < t$, then we have

$$-1 \leq \psi'_R(0) \leq \psi'_R(s) \leq \psi'_L(t) \leq \psi'_R(t) \leq \psi'_L(1) \leq 1.$$

We define a mapping G from $[0, 1]$ into the set of subintervals of $[-1, 1]$ as

$$G(t) = \begin{cases} [-1, \psi'_R(0)], & \text{if } t = 0, \\ [\psi'_L(t), \psi'_R(t)], & \text{if } 0 < t < 1, \\ [\psi'_L(1), 1], & \text{if } t = 1. \end{cases}$$

For each $x = (x_0, x_1) \in \mathbb{C}^2$ with $\|x\|_\psi = 1$, we put

$$t = \frac{|x_1|}{|x_0| + |x_1|} \quad \text{and} \quad x(t) = \frac{1}{\psi(t)}(1 - t, t).$$

Then we write

$$x = \frac{1}{\psi(t)}\left(e^{i\rho_0}(1 - t), e^{i\rho_1} t\right),$$

where $x_k = e^{i\rho_k}|x_k|$ $(k = 0, 1)$. Since $\|\cdot\|_\psi$ is absolute on $\mathbb{C}^2$, it is clear to prove that $\alpha = (\alpha_0, \alpha_1) \in \mathbb{C}^2$ is a norming functional of $x(t)$ if and only if $(e^{-i\rho_0}\alpha_0, e^{-i\rho_1}\alpha_1)$ is a norming functional of x. Thus, we only describe the set $D\big(\mathbb{C}^2, x(t)\big)$ of all norming functionals of $x(t)$ for any $t \in [0, 1]$ (cf. Theorem 3.2). The following theorem is proved by Bonsall and Duncan [3]. For the convenience of the reader, we rewrite the proof in our setting.

Theorem 3.1 ([3]). *Let $\psi \in \Psi_2$ be fixed. Then*

$$D\big(\mathbb{C}^2, x(t)\big) = \begin{cases} \left\{ \begin{pmatrix} 1 \\ c(1+a) \end{pmatrix} : a \in G(0), |c| = 1 \right\}, & \text{if } t = 0, \\ \left\{ \begin{pmatrix} \psi(t) - at \\ \psi(t) + a(1-t) \end{pmatrix} : a \in G(t) \right\}, & \text{if } 0 < t < 1, \\ \left\{ \begin{pmatrix} c(1-a) \\ 1 \end{pmatrix} : a \in G(1), |c| = 1 \right\}, & \text{if } t = 1 \end{cases}$$

holds for each $t \in [0, 1]$.

Proof. We put B_0 as

$$B_0 = \left\{ \begin{pmatrix} 1 \\ c(1+a) \end{pmatrix} : a \in G(0), |c| = 1 \right\}.$$

We first show that $D\big(\mathbb{C}^2, x(0)\big) \subset B_0$. Fix $(\alpha_0, \alpha_1) \in D\big(\mathbb{C}^2, x(0)\big)$. From the definition of $D\big(\mathbb{C}^2, x(0)\big)$, $\|(\alpha_0, \alpha_1)\|_* = 1$ and

$$\alpha_0 = \big\langle (\alpha_0, \alpha_1), x(0) \big\rangle = 1.$$

We put $\theta = \arg \alpha_1 \in [0, 2\pi)$, where $\arg 0 = 0$. For each $s \in (0, 1]$, we have

$$\begin{aligned} \psi(s) &= \|(1-s, s)\|_\psi = \big\|\big(1-s, e^{-i\theta}s\big)\big\|_\psi \\ &\geq \big|\big\langle (\alpha_0, \alpha_1), \big(1-s, e^{-i\theta}s\big)\big\rangle\big| = \big|\alpha_0(1-s) + \alpha_1 e^{-i\theta}s\big| \\ &= 1 - s + |\alpha_1|s. \end{aligned}$$

So,

$$\begin{aligned} \psi_R'(0) &= \lim_{s \to +0} \frac{\psi(s) - \psi(0)}{s} \geq \lim_{s \to +0} \frac{1 - s + |\alpha_1|s - 1}{s} \\ &= -1 + |\alpha_1| \geq -1 \end{aligned}$$

and hence $|\alpha_1| - 1 \in G(0)$. We put $a = |\alpha_1| - 1$. Then

$$\alpha_1 = e^{i\theta}|\alpha_1| = e^{i\theta}(1+a).$$

So, we obtain $(\alpha_0, \alpha_1) \in B_0$ and hence $D\big(\mathbb{C}^2, x(0)\big) \subset B_0$. We next show $D\big(\mathbb{C}^2, x(t)\big) \supset B_0$. Fix $a \in G(0)$ and $c \in \mathbb{C}$ with $|c| = 1$. Then

$$\big\langle \big(1, c(1+a)\big), (1, 0) \big\rangle = 1.$$

Since

$$\frac{\psi(s) - \psi(0)}{s} \leq \frac{\psi(t) - \psi(0)}{t}$$

for $s, t \in (0, 1]$ with $s \leq t$, we have

$$a \leq \psi_R'(0) \leq \frac{\psi(s) - \psi(0)}{s} = \frac{\psi(s) - 1}{s}$$

and hence $\psi(s) \geq 1 + as$ for $s \in (0,1]$. Fix $(z_0, z_1) \in \mathbb{C}^2$ with $\|(z_0, z_1)\|_\psi = 1$. Let us prove

$$\left|\left\langle \big(1, c(1+a)\big), (z_0, z_1) \right\rangle\right| \leq 1.$$

Put

$$s = \frac{|z_1|}{|z_0| + |z_1|}.$$

Note that

$$1 = \big\|(z_0, z_1)\big\|_\psi = \big(|z_0| + |z_1|\big)\psi(s).$$

So we have

$$\begin{aligned}\left|\left\langle \big(1, c(1+a)\big), (z_0, z_1) \right\rangle\right| &= \big|1 \cdot z_0 + c(1+a) \cdot z_1\big| \\ \leq |z_0| + (1+a)|z_1| &= \frac{|z_0| + (1+a)|z_1|}{\big(|z_0| + |z_1|\big)\psi(s)} = \frac{1+as}{\psi(s)} \leq 1.\end{aligned}$$

Thus, we have $\big\|\big(1, c(1+a)\big)\big\|_* = 1$. These imply $\big(1, c(1+a)\big) \in D\big(\mathbb{C}^2, x(0)\big)$. So $D\big(\mathbb{C}^2, x(0)\big) \supset B_0$ and hence $D\big(\mathbb{C}^2, x(0)\big) = B_0$. Fix $t \in (0,1)$ and put B_t as

$$B_t = \left\{ \begin{pmatrix} \psi(t) - at \\ \psi(t) + a(1-t) \end{pmatrix} : a \in G(t) \right\}.$$

We shall show that $D\big(\mathbb{C}^2, x(t)\big) \subset B_t$. Fix $(\alpha_0, \alpha_1) \in D\big(\mathbb{C}^2, x(t)\big)$. We put

$$\theta_0 = \arg \alpha_0 \in [0, 2\pi) \quad \text{and} \quad \theta_1 = \arg \alpha_1 \in [0, 2\pi),$$

where $\arg 0 = 0$. From the definition of $D\big(\mathbb{C}^2, x(t)\big)$, $\|(\alpha_0, \alpha_1)\|_* = 1$ and

$$1 = \big\langle (\alpha_0, \alpha_1), x(t) \big\rangle = \frac{\alpha_0 \cdot (1-t) + \alpha_1 \cdot t}{\psi(t)}.$$

Hence

$$\psi(t) = \alpha_0(1-t) + \alpha_1 t.$$

Then we have

$$\begin{aligned}\psi(t) &= \mathrm{Re}(\alpha_0)(1-t) + \mathrm{Re}(\alpha_1)t \leq |\alpha_0|(1-t) + |\alpha_1|t \\ &= \alpha_0 e^{-i\theta_0}(1-t) + \alpha_1 e^{-i\theta_1}t = \big\langle (\alpha_0, \alpha_1), \big(e^{-i\theta_0}(1-t), e^{-i\theta_1}t\big) \big\rangle \\ &\leq \big\|\big(e^{-i\theta_0}(1-t), e^{-i\theta_1}t\big)\big\|_\psi = \big\|(1-t, t)\big\|_\psi = \psi(t).\end{aligned}$$

Thus, we obtain $\mathrm{Re}(\alpha_0) = |\alpha_0|$ and $\mathrm{Re}(\alpha_1) = |\alpha_1|$. Therefore $\alpha_0 \geq 0$ and $\alpha_1 \geq 0$. For $s \in (0,1)$, we have

$$\begin{aligned}1 = \big\|x(s)\big\|_\psi &\geq \big|\big\langle (\alpha_0, \alpha_1), x(s) \big\rangle\big| \\ &= \frac{\big|\alpha_0(1-s) + \alpha_1 s\big|}{\psi(s)} = \frac{\alpha_0(1-s) + \alpha_1 s}{\psi(s)}.\end{aligned}$$

Then we have

$$\psi(s) \geq \alpha_0(1-s) + \alpha_1 s.$$

So,

$$\begin{aligned}\psi'_R(t) &= \lim_{s\to t+0}\frac{\psi(s)-\psi(t)}{s-t}\\ &\geq \lim_{s\to t+0}\frac{\alpha_0(1-s)+\alpha_1 s-\alpha_0(1-t)-\alpha_1 t}{s-t}\\ &= \alpha_1-\alpha_0.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\psi'_L(t) &= \lim_{s\to t-0}\frac{\psi(s)-\psi(t)}{s-t}\\ &\leq \lim_{s\to t-0}\frac{\alpha_0(1-s)+\alpha_1 s-\alpha_0(1-t)-\alpha_1 t}{s-t}\\ &= \alpha_1-\alpha_0\end{aligned}$$

and hence $\alpha_1-\alpha_0\in G(t)$. From

$$\psi(t)-(\alpha_1-\alpha_0)t=\alpha_0(1-t)+\alpha_1 t-(\alpha_1-\alpha_0)t=\alpha_0$$

and

$$\psi(t)+(\alpha_1-\alpha_0)(1-t)=\alpha_0(1-t)+\alpha_1 t+(\alpha_1-\alpha_0)(1-t)=\alpha_1,$$

we obtain $(\alpha_0,\alpha_1)\in B_t$. Hence, $D\big(\mathbb{C}^2,x(t)\big)\subset B_t$. Let us show $D\big(\mathbb{C}^2,x(t)\big)\supset B_t$. Fix $a\in G(t)$, and put

$$\alpha_0=\psi(t)-at \quad\text{and}\quad \alpha_1=\psi(t)+a(1-t).$$

Then we have

$$\big\langle(\alpha_0,\alpha_1),x(t)\big\rangle=\big(\psi(t)-at\big)\frac{1-t}{\psi(t)}+\big(\psi(t)+a(1-t)\big)\frac{t}{\psi(t)}=1.$$

Since $G(t)\subset[-1,1]$, we have

$$\alpha_0=\psi(t)-at\geq\psi(t)-t\geq 0$$

and

$$\alpha_1=\psi(t)+a(1-t)\geq\psi(t)-(1-t)\geq 0.$$

Fix $(z_0,z_1)\in\mathbb{C}^2$ with $\|(z_0,z_1)\|_\psi=1$, and put

$$s=\frac{|z_1|}{|z_0|+|z_1|}.$$

In the case of $s<t$,

$$a\geq\psi'_L(t)\geq\frac{\psi(s)-\psi(t)}{s-t}.$$

In the case of $s>t$,

$$a\leq\psi'_R(t)\leq\frac{\psi(s)-\psi(t)}{s-t}.$$

Therefore we have

$$\psi(s) \geq \psi(t) + a(s-t).$$

Since $(|z_0| + |z_1|)\psi(s) = 1$, we have

$$\begin{aligned} \left|\left\langle (\alpha_0, \alpha_1), (z_0, z_1) \right\rangle\right| &= |\alpha_0 z_0 + \alpha_1 z_1| \leq \alpha_0 |z_0| + \alpha_1 |z_1| \\ &= \frac{\alpha_0 + (\alpha_1 - \alpha_0)s}{\psi(s)} = \frac{\psi(t) - at + as}{\psi(s)} \leq 1. \end{aligned}$$

These imply $(\alpha_0, \alpha_1) \in D\big(\mathbb{C}^2, x(t)\big)$. So $D\big(\mathbb{C}^2, x(t)\big) \supset B_t$ and hence $D\big(\mathbb{C}^2, x(t)\big) = B_t$. Similarly, we can show that

$$D\big(\mathbb{C}^2, x(1)\big) = \left\{ \begin{pmatrix} c(1-a) \\ 1 \end{pmatrix} : a \in G(1), |c| = 1 \right\}.$$

This completes the proof. □

From Theorem 3.1, we obtain the following.

Theorem 3.2. *Let $\psi \in \Psi_2$ be fixed. Let $(x_0, x_1) \in \mathbb{C}^2$ with $\big\|(x_0, x_1)\big\|_\psi = 1$. Put*

$$t = \frac{|x_1|}{|x_0| + |x_1|},$$

and

$$\rho_0 = \arg x_0 \in [0, 2\pi) \quad \text{and} \quad \rho_1 = \arg x_1 \in [0, 2\pi),$$

where $\arg 0 = 0$. *Then*

$$D\big(\mathbb{C}^2, (x_0, x_1)\big) = \begin{cases} \left\{ \begin{pmatrix} e^{-i\rho_0} \\ c(1+a) \end{pmatrix} : a \in G(0), |c| = 1 \right\}, & \text{if } x_1 = 0, \\ \left\{ \begin{pmatrix} e^{-i\rho_0}\big(\psi(t) - at\big) \\ e^{-i\rho_1}\big(\psi(t) + a(1-t)\big) \end{pmatrix} : a \in G(t) \right\}, & \text{if } x_0 \cdot x_1 \neq 0, \\ \left\{ \begin{pmatrix} c(1-a) \\ e^{-i\rho_1} \end{pmatrix} : a \in G(1), |c| = 1 \right\}, & \text{if } x_0 = 0 \end{cases}$$

holds.

As a direct consequence of Theorem 3.2, we obtain the following.

Theorem 3.3. *Fix $\psi \in \Psi_2$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is smooth if and only if ψ is differentiable at any $t \in (0,1)$, $\psi_R'(0) = -1$ and $\psi_L'(1) = 1$.*

4. Norming functionals on $(\mathbb{C}^n, \|\cdot\|_\psi)$

In this section, we discuss norming functionals on $\mathbb{C}^n$ for $n \geq 2$. We put $I_n = \{0, 1, 2, \cdots, n-1\}$ and

$$x(t) = \frac{(t_0, t_1, \cdots, t_{n-1})}{\psi(t)} \in \mathbb{C}^n$$

for $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$, where $t_0 = 1 - \sum_{j=1}^{n-1} t_j$.

Lemma 4.1. *For every $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ and $a = (a_1, a_2, \cdots, a_{n-1}) \in \partial\psi(t)$, the inequality*

$$\psi(t) + \langle a, p_j - t\rangle \geq 0 \tag{3}$$

holds for every $j \in I_n$ with $t_j > 0$.

Proof. Fix $j \in I_n$ with $t_j > 0$. In the case of $t_j = 1$, i.e., $t = p_j$, (3) clearly holds. If $0 < t_j < 1$, then we have, by the properties of ψ as in Section 2,

$$\begin{aligned} & t_j\big\{\psi(t) + \langle a, p_j - t\rangle\big\} \\ &= \psi(t) - (1-t_j)\left\{\psi(t) + \left\langle a, \frac{1}{1-t_j}(t - t_j p_j) - t\right\rangle\right\} \\ &= \psi(t) - (1-t_j)\big\{\psi(t) + \big\langle a, q_j(t) - t\big\rangle\big\} \\ &\geq \psi(t) - (1-t_j)\psi\big(q_j(t)\big) \geq 0. \end{aligned}$$

Thus, we have this lemma. □

As a direct consequence of Lemma 4.1, we obtain the following.

Corollary 4.2. *For every $t = (t_1, t_2, \cdots, t_{n-1}) \in \overset{\circ}{\Delta}_n$ and $a = (a_1, a_2, \cdots, a_{n-1}) \in \partial\psi(t)$, the inequality*

$$\psi(t) + \langle a, p_j - t\rangle \geq 0$$

holds for every $j \in I_n$.

Using Lemma 4.1, we obtain the following.

Theorem 4.3.

$$\begin{aligned} & D\big(\mathbb{C}^n, x(t)\big) \qquad\qquad\qquad (4) \\ &= \left\{ \begin{pmatrix} e^{i\theta_0}\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ e^{i\theta_1}\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ e^{i\theta_2}\big(\psi(t) + \langle a, p_2 - t\rangle\big) \\ \vdots \\ e^{i\theta_{n-1}}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} a \in \partial\psi(t), \\ \psi(t) + \langle a, p_j - t\rangle \geq 0 \\ \quad \textit{for } j \in I_n \textit{ with } t_j = 0, \\ \theta_j \in [0, 2\pi) \\ \quad \textit{for } j \in I_n \textit{ with } t_j = 0, \\ \theta_j = 0 \\ \quad \textit{for } j \in I_n \textit{ with } t_j > 0 \end{array} \right\} \end{aligned}$$

for all $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$.

Proof. We put B as the right hand side of (4). We first show that $D\big(\mathbb{C}^n, x(t)\big) \subset B$. Fix $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) \in D\big(\mathbb{C}^n, x(t)\big)$. We put

$$\theta_j = \arg \alpha_j \in [0, 2\pi)$$

for $j \in I_n$, where $\arg 0 = 0$. From the definition of $D\big(\mathbb{C}^n, x(t)\big)$, $\|\alpha\|_* = 1$ and

$$1 = \langle \alpha, x(t) \rangle = \frac{1}{\psi(t)} \sum_{j=0}^{n-1} \alpha_j t_j.$$

Hence

$$\psi(t) = \sum_{j=0}^{n-1} \alpha_j t_j.$$

From

$$\begin{aligned} \psi(t) &= \sum_{j=0}^{n-1} \mathrm{Re}(\alpha_j) t_j \le \sum_{j=0}^{n-1} |\alpha_j| t_j = \sum_{j=0}^{n-1} \alpha_j e^{-i\theta_j} t_j \qquad (5) \\ &= \big\langle \alpha, \big(e^{-i\theta_0} t_0, e^{-i\theta_1} t_1, \cdots, e^{-i\theta_{n-1}} t_{n-1}\big)\big\rangle \\ &\le \|\alpha\|_* \big\|(e^{-i\theta_0} t_0, e^{-i\theta_1} t_1, \cdots, e^{-i\theta_{n-1}} t_{n-1})\big\|_\psi \\ &= \big\|(t_0, t_1, \cdots, t_{n-1})\big\|_\psi = \psi(t), \end{aligned}$$

we obtain $\mathrm{Re}(\alpha_j) = |\alpha_j|$ for $j \in I_n$ with $t_j > 0$. Hence $\alpha_j \ge 0$ and $\theta_j = 0$ for $j \in I_n$ with $t_j > 0$. From (5), we also obtain

$$\psi(t) = \sum_{j=0}^{n-1} |\alpha_j| t_j.$$

We put a as

$$a = \begin{pmatrix} |\alpha_1| - |\alpha_0| \\ |\alpha_2| - |\alpha_0| \\ \vdots \\ |\alpha_{n-1}| - |\alpha_0| \end{pmatrix} \in \mathbb{R}^{n-1}.$$

We fix $s = (s_1, s_2, \cdots, s_{n-1}) \in \Delta_n$ and put $s_0 = 1 - \sum_{j=1}^{n-1} s_j$. From

$$\begin{aligned} \psi(s) &= \big\|(s_0, s_1, \cdots, s_{n-1})\big\|_\psi \\ &= \|\alpha\|_* \cdot \big\|(e^{-i\theta_0} s_0, e^{-i\theta_1} s_1, \cdots, e^{-i\theta_{n-1}} s_{n-1})\big\|_\psi \\ &\ge \big|\big\langle \alpha, \big(e^{-i\theta_0} s_0, e^{-i\theta_1} s_1, \cdots, e^{-i\theta_{n-1}} s_{n-1}\big)\big\rangle\big| \\ &= \left| \sum_{j=0}^{n-1} \alpha_j e^{-i\theta_j} s_j \right| = \sum_{j=0}^{n-1} |\alpha_j| s_j = \sum_{j=0}^{n-1} |\alpha_j| s_j + \psi(t) - \sum_{j=0}^{n-1} |\alpha_j| t_j \\ &= \psi(t) + |\alpha_0| \left(1 - \sum_{j=1}^{n-1} s_j\right) + \sum_{j=1}^{n-1} |\alpha_j| s_j - |\alpha_0| \left(1 - \sum_{j=1}^{n-1} t_j\right) - \sum_{j=1}^{n-1} |\alpha_j| t_j \\ &= \psi(t) + \sum_{j=1}^{n-1} \big(|\alpha_j| - |\alpha_0|\big)(s_j - t_j) \\ &= \psi(t) + \langle a, s - t \rangle, \end{aligned}$$

we have $a \in \partial\psi(t)$. We also obtain

$$\begin{aligned}\psi(t) + \langle a, p_0 - t\rangle &= \psi(t) + \langle a, -t\rangle \\ &= \psi(t) + \sum_{j=1}^{n-1} \big(|\alpha_j| - |\alpha_0|\big)(-t_j) \\ &= \psi(t) - \sum_{j=1}^{n-1} |\alpha_j| t_j - |\alpha_0| \sum_{j=1}^{n-1}(-t_j) \\ &= \psi(t) - \sum_{j=0}^{n-1} |\alpha_j| t_j + |\alpha_0| = |\alpha_0|\end{aligned}$$

and hence

$$\alpha_0 = \begin{cases} e^{i\theta_0}\big(\psi(t) + \langle a, p_0 - t\rangle\big), & \text{if } t_0 = 0, \\ \psi(t) + \langle a, p_0 - t\rangle, & \text{if } t_0 > 0. \end{cases}$$

For each $j \in I_n$ with $j \neq 0$, we have

$$\begin{aligned}&\psi(t) + \langle a, p_j - t\rangle = \psi(t) + \langle a, -t\rangle + \langle a, p_j\rangle = |\alpha_0| + \langle a, p_j\rangle \\ &\quad = |\alpha_0| + |\alpha_j| - |\alpha_0| = |\alpha_j|\end{aligned}$$

and hence

$$\alpha_j = \begin{cases} e^{i\theta_j}\big(\psi(t) + \langle a, p_j - t\rangle\big), & \text{if } t_j = 0, \\ \psi(t) + \langle a, p_j - t\rangle, & \text{if } t_j > 0. \end{cases}$$

Therefore $\alpha \in B$ and hence $D\big(\mathbb{C}^n, x(t)\big) \subset B$. We next show $D\big(\mathbb{C}^n, x(t)\big) \supset B$. Fix $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) \in B$. Then there exist $a = (a_1, a_2, \cdots, a_{n-1}) \in \partial\psi(t)$ and $\theta_j \in [0, 2\pi)$ for $j \in I_n$ with $t_j = 0$ which satisfy

$$\psi(t) + \langle a, p_j - t\rangle \geq 0$$

for $j \in I_n$ with $t_j = 0$ and

$$\alpha_j = \begin{cases} e^{i\theta_j}\big(\psi(t) + \langle a, p_j - t\rangle\big), & \text{if } t_j = 0, \\ \psi(t) + \langle a, p_j - t\rangle, & \text{if } t_j > 0. \end{cases}$$

From Lemma 4.1, for each $j \in I_n$ with $t_j > 0$, we have

$$\psi(t) + \langle a, p_j - t\rangle \geq 0$$

and hence $\alpha_j \geq 0$ holds. We also have

$$|\alpha_j| = \psi(t) + \langle a, p_j - t\rangle$$

for $j \in I_n$. For each $j \in I_n$ with $j \neq 0$, from

$$|\alpha_j| - |\alpha_0| = \langle a, p_j\rangle = a_j,$$

we have

$$|\alpha_j| = |\alpha_0| + a_j.$$

Since $\alpha_j \geq 0$ for $j \in I_n$ with $t_j > 0$, we have

$$\begin{aligned}
\psi(t)\langle \alpha, x(t)\rangle &= \sum_{j=0}^{n-1} \alpha_j t_j = \sum_{j=0}^{n-1} |\alpha_j| t_j = |\alpha_0| t_0 + \sum_{j=1}^{n-1} |\alpha_j| t_j \\
&= |\alpha_0| t_0 + \sum_{j=1}^{n-1} (|\alpha_0| + a_j) t_j = |\alpha_0| + \sum_{j=1}^{n-1} a_j t_j \\
&= |\alpha_0| + \langle a, t\rangle = |\alpha_0| - \langle a, p_0 - t\rangle = \psi(t)
\end{aligned}$$

and hence

$$\langle \alpha, x(t)\rangle = 1.$$

Fix $z = (z_0, z_1, \cdots, z_{n-1}) \in \mathbb{C}^n$ with $\|z\|_\psi = 1$. Let us prove $|\langle \alpha, z\rangle| \leq 1$. Put

$$s_j = \frac{|z_j|}{\sum_{k=0}^{n-1} |z_k|}$$

for $j \in I_n$, and $s = (s_1, s_2, \cdots, s_{n-1}) \in \Delta_n$. Note that $\sum_{j=0}^{n-1} s_j = 1$ and

$$1 = \|z\|_\psi = \left(\sum_{k=0}^{n-1} |z_k|\right) \psi(s).$$

So we have

$$\begin{aligned}
\psi(s)|\langle \alpha, z\rangle| &= \psi(s) \left|\sum_{j=0}^{n-1} \alpha_j z_j\right| \leq \psi(s) \left(\sum_{j=0}^{n-1} |\alpha_j| \cdot |z_j|\right) \\
&= \frac{\left(\sum_{j=0}^{n-1} |\alpha_j| \cdot |z_j|\right)}{\sum_{k=0}^{n-1} |z_k|} = \sum_{j=0}^{n-1} |\alpha_j| s_j = |\alpha_0| s_0 + \sum_{j=1}^{n-1} (|\alpha_0| + a_j) s_j \\
&= |\alpha_0| + \sum_{j=1}^{n-1} a_j s_j = |\alpha_0| + \langle a, s\rangle = \psi(t) + \langle a, p_0 - t\rangle + \langle a, s\rangle \\
&= \psi(t) + \langle a, s - t\rangle \leq \psi(s).
\end{aligned}$$

Thus we have $|\langle \alpha, z\rangle| \leq 1$ and so $\|\alpha\|_* = 1$. These imply $\alpha \in D(\mathbb{C}^n, x(t))$ and hence $D\big(\mathbb{C}^n, x(t)\big) \supset B$. This completes the proof. □

As a direct consequence of Theorem 4.3 we have

Corollary 4.4. *For all* $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$,

$$D\big(\mathbb{C}^n, x(t)\big) = \left\{ \begin{pmatrix} e^{i\theta_0}\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ e^{i\theta_1}\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ e^{i\theta_2}\big(\psi(t) + \langle a, p_2 - t\rangle\big) \\ \vdots \\ e^{i\theta_{n-1}}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} a \in \partial\psi(t), \\ \psi(t) + \langle a, p_j - t\rangle \geq 0, \\ \quad \text{for } j \in I_n, \\ \theta_j \in [0, 2\pi) \\ \quad \text{for } j \in I_n \text{ with } t_j = 0, \\ \theta_j = 0 \\ \quad \text{for } j \in I_n \text{ with } t_j > 0 \end{array} \right\}.$$

In particular, for all $t = (t_1, t_2, \cdots, t_{n-1}) \in \overset{\circ}{\Delta}_n$,

$$D\big(\mathbb{C}^n, x(t)\big) = \left\{ \begin{pmatrix} \psi(t) + \langle a, p_0 - t\rangle \\ \psi(t) + \langle a, p_1 - t\rangle \\ \psi(t) + \langle a, p_2 - t\rangle \\ \vdots \\ \psi(t) + \langle a, p_{n-1} - t\rangle \end{pmatrix} : a \in \partial\psi(t) \right\}.$$

Corollary 4.5. *Let* $\psi \in \Psi_n$ *be fixed. Let* $x = (x_0, x_1, x_2, \cdots, x_{n-1}) \in \mathbb{C}^n$ *with* $\|x\|_\psi = 1$. *Put*

$$t_j = \frac{|x_j|}{\sum_{k=0}^{n-1} |x_k|}$$

for $j \in I_n$, *and*

$$t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n.$$

Put $\rho_j = \arg x_j \in [0, 2\pi)$ *for* $j \in I_n$, *where* $\arg 0 = 0$. *Then*

$$D(\mathbb{C}^n, x) = \left\{ \begin{pmatrix} c_0\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ c_1\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ c_2\big(\psi(t) + \langle a, p_2 - t\rangle\big) \\ \vdots \\ c_{n-1}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} a \in \partial\psi(t), \\ \psi(t) + \langle a, p_j - t\rangle \geq 0, \\ \quad \text{for } j \in I_n, \\ |c_j| = 1 \\ \quad \text{for } j \in I_n \text{ with } t_j = 0, \\ c_j = e^{-i\rho_j} \\ \quad \text{for } j \in I_n \text{ with } t_j > 0 \end{array} \right\}.$$

Proof. Since

$$\|x\|_\psi = \left(\sum_{j=0}^{n-1} |x_j| \right) \psi(t) = 1,$$

we can write

$$x = \frac{1}{\psi(t)} \left(e^{i\rho_0} t_0, e^{i\rho_1} t_1, \cdots, e^{i\rho_{n-1}} t_{n-1} \right).$$

Since $\| \cdot \|_\psi$ is absolute on $\mathbb{C}^n$, it is clear that $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) \in \mathbb{C}^n$ is a norming functional of $x(t)$ if and only if $(e^{-i\rho_0}\alpha_0, e^{-i\rho_1}\alpha_1, \cdots, e^{-i\rho_{n-1}}\alpha_{n-1})$ is a norming functional of x as in Section 3. This completes the proof. $\square$

5. Smoothness of $(\mathbb{C}^n, \|\cdot\|_\psi)$

In this section, we discuss the smoothness of absolute norms on $\mathbb{C}^n$ for $n \geq 2$. We put $I_n = \{0, 1, 2, \cdots, n-1\}$.

Theorem 5.1. *Let $\psi \in \Psi_n$. Then $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth if and only if for each $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$, the following equalities hold:*

1. $\psi'_-(t; p_j - t) = \psi'_+(t; p_j - t)$ *for all $j \in I_n$ with $t_j > 0$;*
2. $\psi'_+(t; p_j - t) = -\psi(t)$ *for all $j \in I_n$ with $t_j = 0$.*

To prove Theorem 5.1, we need some preliminaries. We define a function φ on $\mathbb{R}^{n-1}$ by

$$\varphi(t) = \sup \left\{ \psi(s) + \langle a, t-s \rangle : \begin{array}{l} s = (s_1, s_2, \cdots, s_{n-1}) \in \Delta_n, \\ a \in \partial\psi(s), \\ \psi(s) + \langle a, p_j - s \rangle \geq 0 \text{ for } j \in I_n \end{array} \right\}$$

for every $t \in \mathbb{R}^{n-1}$. In fact, φ is an extension of ψ on Δ_n to $\mathbb{R}^{n-1}$ from Lemma 5.3.

Remark 5.2. If $\psi \in \Psi_2$, we have

$$\varphi(t) = \begin{cases} 1-t, & \text{if } t < 0, \\ \psi(t), & \text{if } 0 \leq t \leq 1, \\ t, & \text{if } t > 1, \end{cases}$$

and $\partial\varphi(t) = G(t)$ for $t \in [0,1]$ (see the definition of $G(t)$ as in Section 3).

Lemma 5.3. *The function φ has the following properties:*

1. *φ is a convex function on $\mathbb{R}^{n-1}$ such that $\varphi(t) < \infty$ for all $t \in \mathbb{R}^{n-1}$;*
2. *$\varphi(t) = \psi(t)$ for $t \in \Delta_n$;*
3. *for each $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ with $t_\ell = 0$ for some $\ell \in I_n$, the equality*
$$\varphi\big(\lambda(t - p_\ell) + p_\ell\big) = \lambda\psi(t)$$
holds for all $\lambda > 1$, and the equality
$$\varphi'_-(t; p_\ell - t) = -\psi(t)$$
holds;
4. *$\varphi(\lambda p_j) \leq |\lambda| + 1$ for all $\lambda \in \mathbb{R}$ and j $(1 \leq j \leq n-1)$.*

Proof. By Corollary 4.4, for each $s = (s_1, s_2, \cdots, s_{n-1}) \in \Delta_n$, there exists $a \in \psi(s)$ satisfying $\psi(s) + \langle a, p_j - s\rangle \geq 0$ for all $j \in I_n$. So $\varphi(t) > -\infty$ for all $t \in \mathbb{R}^{n-1}$. Since $\psi(s) + \langle a, t-s\rangle$ is linear about t, it is clear that φ is convex on $\mathbb{R}^{n-1}$. We next show (2). Fix $t \in \Delta_n$. By the definition of $\partial\psi(t)$, we have $\varphi(t) \leq \psi(t)$. By Corollary 4.4, there exists $b \in \partial\psi(t)$ satisfying $\psi(t) + \langle b, p_j - t\rangle \geq 0$ for all $j \in I_n$. So

$$\psi(t) = \psi(t) + \langle b, t-t\rangle \leq \varphi(t).$$

Therefore $\varphi(t) = \psi(t)$ for $t \in \Delta_n$. Let us show (3). We fix $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ with $t_\ell = 0$ for some $\ell \in I_n$. Assume that there exists $\lambda > 1$ such that

$$\varphi\big(\lambda(t - p_\ell) + p_\ell\big) > \lambda\psi(t).$$

Then there exist $u \in \Delta_n$ and $a \in \partial\psi(u)$ satisfying $\psi(u) + \langle a, p_j - u\rangle \geq 0$ for all $j \in I_n$, and

$$\psi(u) + \langle a, \lambda(t - p_\ell) + p_\ell - u\rangle > \lambda\psi(t).$$

We have

$$\begin{aligned}&\psi(u) + \langle a, t - u\rangle \\ &= \frac{\lambda - 1}{\lambda}\big(\psi(u) + \langle a, p_\ell - u\rangle\big) + \frac{1}{\lambda}\big(\psi(u) + \langle a, \lambda(t - p_\ell) + p_\ell - u\rangle\big) \\ &\geq \frac{1}{\lambda}\big(\psi(u) + \langle a, \lambda(t - p_\ell) + p_\ell - u\rangle\big) > \frac{1}{\lambda}\big(\lambda\psi(t)\big) = \psi(t).\end{aligned}$$

This contradicts to $a \in \partial\psi(u)$. Therefore

$$\varphi\big(\lambda(t - p_\ell) + p_\ell\big) \leq \lambda\psi(t)$$

for $\lambda > 1$. We next show

$$\varphi\big(\lambda(t - p_\ell) + p_\ell\big) \geq \lambda\psi(t)$$

for $\lambda > 1$. By Corollary 4.4, there exists $a \in \partial\psi(t)$ satisfying $\psi(t) + \langle a, p_j - t\rangle \geq 0$ for all $j \in I_n$. From $t_\ell = 0$, we have

$$0 = \sum_{j=0}^{n-1} t_j p_j - t = \sum_{j=0}^{n-1} t_j(p_j - t) = \sum_{j \neq \ell} t_j(p_j - t)$$

and hence $\{p_j - t : j \in I_n, j \neq \ell\}$ is linearly dependent. On the other hand, the linear span of $\{p_j - t : j \in I_n\}$ equals to $\mathbb{R}^{n-1}$ because

$$(p_j - t) - (p_0 - t) = p_j = (0, 0, \cdots, 0, \overset{(j)}{1}, 0, 0, \cdots, 0)$$

for j $(1 \leq j \leq n-1)$. So, $p_\ell - t$ does not belong to the linear span of $\{p_j - t : j \in I_n, j \neq \ell\}$. Therefore we can choose $b \in \mathbb{R}^{n-1}$ satisfying

$$\langle b, p_\ell - t\rangle = -\psi(t)$$

and

$$\langle b, p_j - t\rangle = \langle a, p_j - t\rangle$$

for $j \in I_n$ with $j \neq \ell$. Note that

$$\langle b, p_\ell - t\rangle = -\psi(t) \leq -\psi(t) + \psi(t) + \langle a, p_\ell - t\rangle = \langle a, p_\ell - t\rangle.$$

For any $u = (u_1, u_2, \cdots, u_{n-1}) \in \Delta_n$, putting $u_0 = 1 - \sum_{j=1}^{n-1} u_j$, we have

$$\begin{aligned}\psi(u) \geq \psi(t) + \langle a, u - t\rangle &= \psi(t) + \sum_{j=0}^{n-1} u_j\langle a, p_j - t\rangle \\ &\geq \psi(t) + \sum_{j=0}^{n-1} u_j\langle b, p_j - t\rangle = \psi(t) + \langle b, u - t\rangle.\end{aligned}$$

This shows $b \in \partial\psi(t)$. Since

$$\psi(t) + \langle b, p_\ell - t\rangle = \psi(t) - \psi(t) = 0$$

and

$$\psi(t) + \langle b, p_j - t\rangle = \psi(t) + \langle a, p_j - t\rangle \geq 0$$

for $j \in I_n$ with $j \neq \ell$, we obtain

$$\begin{aligned}\varphi\big(\lambda(t-p_\ell)+p_\ell\big) &\geq \psi(t) + \big\langle b, \lambda(t-p_\ell)+p_\ell - t\big\rangle \\ &= \psi(t) + (1-\lambda)\langle b, p_\ell - t\rangle = \psi(t) - (1-\lambda)\psi(t) = \lambda\psi(t)\end{aligned}$$

for $\lambda > 1$. Therefore

$$\varphi\big(\lambda(t-p_\ell)+p_\ell\big) = \lambda\psi(t)$$

for $\lambda > 1$. From this equality, we have

$$\begin{aligned}\varphi'_-(t; p_\ell - t) &= -\varphi'_+(t; t-p_\ell) = -\lim_{\lambda\to+0}\frac{\varphi\big(t+\lambda(t-p_\ell)\big)-\varphi(t)}{\lambda} \\ &= -\lim_{\lambda\to+0}\frac{\varphi\big((1+\lambda)(t-p_\ell)+p_\ell\big)-\varphi(t)}{\lambda} = -\lim_{\lambda\to+0}\frac{(1+\lambda)\psi(t)-\psi(t)}{\lambda} \\ &= -\psi(t).\end{aligned}$$

We have (3). We use (3) in order to show (4). Fix j $(1 \leq j \leq n-1)$ and $\lambda \in \mathbb{R}$. In the case of $\lambda > 1$, we have

$$\varphi(\lambda p_j) = \varphi\big(\lambda(p_j - p_0) + p_0\big) = \lambda\psi(p_j) = \lambda \leq |\lambda| + 1.$$

In the case of $0 \leq \lambda \leq 1$, from $\lambda p_j \in \Delta_n$, we have

$$\varphi(\lambda p_j) = \psi(\lambda p_j) \leq 1 \leq |\lambda| + 1.$$

In the case of $\lambda < 0$, we have

$$\varphi(\lambda p_j) = \varphi\big((1-\lambda)(p_0 - p_j) + p_j\big) = (1-\lambda)\psi(p_0) = |\lambda| + 1.$$

These imply (4). Fix $t = (t_1, t_2, \cdots, t_{n-1}) \in \mathbb{R}^{n-1}$. In the case of $t = 0$, we have

$$\varphi(0) = \psi(0) = 1 < \infty.$$

In the case of $t \neq 0$, by using (4), we have

$$\begin{aligned}\varphi(t) = \varphi\left(\sum_{j=1}^{n-1} t_j p_j\right) &= \varphi\left(\frac{\sum_{j=1}^{n-1}|t_j|(\mathrm{sgn}\ t_j)\left(\sum_{k=1}^{n-1}|t_k|\right)p_j}{\sum_{k=1}^{n-1}|t_k|}\right) \\ &\leq \sum_{j=1}^{n-1}\frac{|t_j|}{\sum_{k=1}^{n-1}|t_k|}\varphi\left((\mathrm{sgn}\ t_j)\left(\sum_{k=1}^{n-1}|t_k|\right)p_j\right) \\ &\leq \sum_{j=1}^{n-1}\frac{|t_j|}{\sum_{k=1}^{n-1}|t_k|}\left(\sum_{k=1}^{n-1}|t_k| + 1\right) = \sum_{k=1}^{n-1}|t_k| + 1 < \infty.\end{aligned}$$

Hence we have (1). This completes the proof. □

Proposition 5.4. *Let $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$ and $a \in \mathbb{R}^{n-1}$. Then $a \in \partial\varphi(t)$ if and only if $a \in \partial\psi(t)$ and $\psi(t) + \langle a, p_j - t\rangle \geq 0$ for $j \in I_n$.*

Proof. Assume that $a \in \partial\varphi(t)$. For each $u \in \Delta_n$, we have

$$\psi(u) = \varphi(u) \geq \varphi(t) + \langle a, u - t\rangle = \psi(t) + \langle a, u - t\rangle$$

and hence $a \in \partial\psi(t)$. For each $j \in I_n$ with $t_j > 0$, by Lemma 4.1, we have $\psi(t) + \langle a, p_j - t\rangle \geq 0$. For each $j \in I_n$ with $t_j = 0$, by Lemma 5.3 (3), we have

$$\psi(t) + \langle a, p_j - t\rangle \geq \psi(t) + \varphi'_-(t; p_j - t) = \psi(t) - \psi(t) = 0,$$

because $\varphi'_-(t; p_j - t) \leq \langle a, p_j - t\rangle$. Therefore $\psi(t) + \langle a, p_j - t\rangle \geq 0$ for all $j \in I_n$.

Conversely, we assume that $a \in \partial\psi(t)$ and $\psi(t) + \langle a, p_j - t\rangle \geq 0$ for $j \in I_n$. For each $u \in \mathbb{R}^{n-1}$, from the definition of φ, we obtain

$$\varphi(u) \geq \psi(t) + \langle a, u - t\rangle = \varphi(t) + \langle a, u - t\rangle.$$

This shows $a \in \partial\varphi(t)$. This completes the proof. □

Remark 5.5. We rewrite Theorem 4.3 by using φ in place of ψ. By Proposition 5.4, we have

$$D\big(\mathbb{C}^n, x(t)\big) = \left\{ \begin{pmatrix} e^{i\theta_0}\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ e^{i\theta_1}\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ e^{i\theta_2}\big(\psi(t) + \langle a, p_2 - t\rangle\big) \\ \vdots \\ e^{i\theta_{n-1}}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} a \in \partial\varphi(t), \\ \theta_j \in [0, 2\pi) \\ \quad \text{for } j \in I_n \text{ with } t_j = 0, \\ \theta_j = 0 \\ \quad \text{for } j \in I_n \text{ with } t_j > 0 \end{array} \right\}$$

for all $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$.

Proof of Theorem 5.1. We first assume that $(\mathbb{C}^n, \|\cdot\|_\psi)$ is smooth. Fix $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$. Since the linear span of $\{p_j - t : j \in I_n\}$ equals to $\mathbb{R}^{n-1}$, and $\sharp D\big(\mathbb{C}^n, x(t)\big) = 1$, we have $\sharp\partial\varphi(t) = 1$ and hence φ is differentiable at t. Therefore

$$\varphi'_-(t; p_j - t) = \varphi'_+(t; p_j - t)$$

for $j \in I_n$. In the case of $t_j = 1$, i.e., $t = p_j$, we have

$$\psi'_-(t; p_j - t) = 0 = \psi'_+(t; p_j - t).$$

In the case of $t_j = 0$, by Lemma 5.3 (3), we have

$$\psi'_+(t; p_j - t) = \varphi'_+(t; p_j - t) = \varphi'_-(t; p_j - t) = -\psi(t).$$

In the case of $0 < t_j < 1$, we have

$$\psi'_+(t; p_j - t) = \varphi'_+(t; p_j - t) = \varphi'_-(t; p_j - t) = \psi'_-(t; p_j - t).$$

Conversely, we assume that for each $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$, the following equalities hold:

1. $\psi'_-(t; p_j - t) = \psi'_+(t; p_j - t)$ for all $j \in I_n$ with $t_j > 0$;
2. $\psi'_+(t; p_j - t) = -\psi(t)$ for all $j \in I_n$ with $t_j = 0$.

Fix $t = (t_1, t_2, \cdots, t_{n-1}) \in \Delta_n$. In the case of $t_j = 1$, i.e., $t = p_j$, we have

$$\varphi'_-(t; p_j - t) = 0 = \varphi'_+(t; p_j - t).$$

In the case of $t_j = 0$, by Lemma 5.3 (3), we have

$$\varphi'_+(t; p_j - t) = \psi'_+(t; p_j - t) = -\psi(t) = \varphi'_-(t; p_j - t).$$

In the case of $0 < t_j < 1$, we have

$$\varphi'_+(t; p_j - t) = \psi'_+(t; p_j - t) = \psi'_-(t; p_j - t) = \varphi'_-(t; p_j - t).$$

Therefore

$$\varphi'_-(t; p_j - t) = \varphi'_+(t; p_j - t)$$

for $j \in I_n$. Since the linear span of $\{p_j - t : j \in I_n\}$ equals to $\mathbb{R}^{n-1}$, we have φ is differentiable at t and hence $\sharp\partial\varphi(t) = 1$. Then we write $\partial\varphi(t) = \{a\}$. For each $j \in I_n$ with $t_j = 0$, by Lemma 5.3 (3), we have

$$\psi(t) + \langle a, p_j - t\rangle = \psi(t) + \varphi'_-(t; p_j - t) = \psi(t) - \psi(t) = 0.$$

So, we obtain

$$D\big(\mathbb{C}^n, x(t)\big) = \left\{ \begin{pmatrix} c_0\big(\psi(t) + \langle a, p_0 - t\rangle\big) \\ c_1\big(\psi(t) + \langle a, p_1 - t\rangle\big) \\ c_2\big(\psi(t) + \langle a, p_2 - t\rangle\big) \\ \vdots \\ c_{n-1}\big(\psi(t) + \langle a, p_{n-1} - t\rangle\big) \end{pmatrix} : \begin{array}{l} c_j = 0 \\ \quad \text{for } j \in I_n \text{ with } t_j = 0, \\ c_j = 1 \\ \quad \text{for } j \in I_n \text{ with } t_j > 0 \end{array} \right\}.$$

Therefore $\sharp D\big(\mathbb{C}^n, x(t)\big) = 1$. □

From the proof of Theorem 5.1, we obtain the following.

Corollary 5.6. $(\mathbb{C}^n, \|\cdot\|_\psi)$ *is smooth if and only if* φ *is differentiable at any* $t \in \Delta_n$.

References

[1] B. Beauzamy: Introduction to Banach Spaces and their Geometry, 2nd ed., North-Holland, Amsterdam et al. (1985).

[2] R. Bhatia: Matrix Analysis, Springer (1997).

[3] F. F. Bonsall, J. Duncan: Numerical Ranges II, London Math. Soc. Lecture Note Series 10 (1973).

[4] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[5] K.-S. Saito, M. Kato: Uniform convexity of ψ-direct sums of Banach spaces, preprint.

[6] K.-S. Saito, M. Kato, Y. Takahashi: Von Neumann-Jordan constant of absolute norms on $\mathbb{C}^2$, J. Math. Anal. Appl. 244 (2000) 515–532.

[7] K.-S. Saito, M. Kato, Y. Takahashi: Absolute norms on $\mathbb{C}^n$, J. Math. Anal. Appl. 252 (2000) 879–905.

[8] Y. Takahashi, M. Kato, K.-S. Saito: Strict convexity of absolute norms on $\mathbb{C}^2$ and direct sums of Banach spaces, J. Inequal. Appl. 7 (2002) 179–186.