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Contracting the Maximal Points of an Ordered Convex Set

G. R. Burton
Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, U.K.

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The maximal points of a nonempty closed bounded convex set in a reflexive Banach space, relative to an ordering defined by a locally uniformly convex cone, are studied. The set of maximal points is proved to be contractible, and sufficient conditions are found for it to be contractible by a homotopy with the semigroup property, or by the flow of an ordinary differential equation.

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Our purpose here is to find sufficient conditions for contractibility of the set of maximal points of a closed convex set in a Banach space, relative to an ordering defined by a cone.

To introduce terminology, let X be a Banach space. A *cone* in X will mean a convex set $C \subset X$ containing a nonzero vector, such that for all $x \in C$ and $\lambda > 0$ we have $\lambda x \in C$. A cone may thus be topologically open, closed, or neither, and may or may not contain 0. A cone $C \subset X$ defines a partial ordering on X by $x \prec y \Leftrightarrow y \in x + C$; notice that if $0 \notin C$ then the ordering is strict, in the sense that $x \not\prec x$ always. Given a cone $C \subset X$ and a set $K \subset X$, we say $x \in K$ is *maximal* if there is no $y \in K \setminus \{x\}$ with $x \prec y$ (which is equivalent to requiring $(x + C) \cap K \subset \{x\}$); we call the set of maximal points of K the *maximal set*.

Although in sufficiently regular finite-dimensional examples a maximal set must be a topological ball (of dimension 1 less) and therefore be contractible trivially, in general the situation is rather worse; a 3-dimensional example in which the maximal set is not locally connected will be given below.

A set Q in a Banach space Y is said to be *locally uniformly convex* if Q is convex, bounded, has nonempty interior, and whenever $\xi \in Y^*$ attains its supremum relative to Q, every maximising sequence for ξ relative to Q is (strongly) convergent. This definition extends standard usage; notice that a Banach space is locally uniformly convex, in the terminology of the geometry of Banach spaces, if and only if its unit ball is a locally uniformly convex set according to our definition. For convex bodies in Euclidean space, local uniform convexity is equivalent to strict convexity. A *locally uniformly convex cone* C in a Banach space X is a cone such that, for some $\xi \in X^*$, the section $\xi^{-1}(1) \cap C$ is a translate of a locally uniformly convex set in the Banach space $\xi^{-1}(0)$.

We prove three main theorems. Firstly we prove contractibility of the maximal set of a nonempty closed bounded convex subset of a reflexive Banach space, with an ordering

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defined by a locally uniformly convex cone. Secondly, for an ordering defined by an elliptical open cone in Hilbert space, we prove that the maximal set of a nonempty compact convex set can be contracted to a point through a homotopy having the semigroup property. Thirdly for an elliptical open cone and a smooth convex body in Euclidean space, we show that the homotopy of the preceding result can be used to construct a flow for a certain tangent vector field on the surface of the body.

To set these results in context, let us note that, in the case of a smooth (i.e. C^1) strictly convex body in $\mathbb{R}^d$ and any cone, the outward unit normal vector field to the body maps the compact maximal set, S, homeomorphically onto a topological $(d-1)$-ball in the unit sphere of $\mathbb{R}^d$, so S is trivially contractible. Consider however the following construction in $\mathbb{R}^3$. Let $e = (0,0,1) \in \mathbb{R}^3$, let $\Gamma = \{(x,y,0) \in \mathbb{R}^3 | x^2 + y^2 = 1\}$, and let $C = \{(x,y,z) \in \mathbb{R}^3 | z > (x^2+y^2)^{1/2}\}$. Choose any compact set $\Delta \subset \Gamma$ having at least three points, and let $K = \operatorname{conv}(\Delta \cup \{e\})$. The set of maximal points of K under the ordering defined by C, is $S := \bigcup_{u \in \Delta} [u, e]$. Thus S may comprise a highly complicated compact union of line segments having a common endpoint, and S can fail to be locally connected. Hybrids of these two kinds of behaviour are also possible. The theorems proved here are nevertheless sufficiently general to encompass the above example.

Our results are motivated by work of Hofer and Toland [2, 4], who used degree theory to construct periodic or homoclinic solutions for certain indefinite Hamiltonian systems, where the Hamiltonian was the sum of an indefinite quadratic "kinetic" term, and a "potential" term that had a convex level surface. An important step in the degree calculation was the contractibility of the set on this level surface comprising the maximal points with respect to an ordering defined by an elliptical cone, which together with its reflection in the origin forms the negative region of the quadratic. Assuming the surface was smooth enough, the flow of a certain tangent vector field was used to contract the maximal set.

Our theorems weaken very considerably the hypotheses imposed in the Hofer and Toland theory, and are based on a different method, using nearest-point projections and related maps. Under strong enough hypotheses, the result of our construction is indeed a flow for a tangent vector field; by contrast a tangent flow was the basis for Hofer and Toland's construction.

Theorem 1. *Let C be a locally uniformly convex cone in a reflexive Banach space X, let $K \subset X$ be a nonempty closed bounded convex set, and let S be the set of maximal points of K with respect to the ordering defined by C. Then S is contractible.*

Proof of Theorem 1. We may write C in the form $C \setminus \{0\} = \bigcup_{\lambda > 0} \lambda(e + Q)$ for some $\xi \in X^*$, $e \in X$, where $\xi(e) = 1$ and Q is a locally uniformly convex set in $\xi^{-1}(0)$ with $0 \in Q^{\mathrm{o}}$ (the interior of Q relative to $\xi^{-1}(0)$). We can assume that $\inf \xi(S) = 0$ and $\sup \xi(S) = 1$. Reflexivity ensures the set S_1 of maximisers of ξ on K is nonempty, and since ξ is positive on C we have $S_1 \subset S$.

For $x \in S$ and $0 \le \alpha \le 1 - \xi(S)$ let $\lambda(x,\alpha) = \inf\{\lambda > 0 | (x + \alpha e + \lambda Q) \cap K \ne \emptyset\}$. Then reflexivity and local uniform convexity ensure that $(x + \alpha e + \lambda(x,\alpha)\overline{Q}) \cap K$ comprises exactly one point $f(x,\alpha)$, while $(x + \alpha e + \lambda(x,\alpha)Q^{\mathrm{o}}) \cap K = \emptyset$ (thus $f(x,\alpha)$ is a "nearest-point" in a "metric" with "ball" Q).

We now prove continuity of $f(\cdot,\cdot)$. Suppose, to seek a contradiction, that for some $\varepsilon > 0$ we can choose a sequence $\{x_n\}_{n=1}^{\infty}$ in S, converging say to $x \in S$, and a sequence $\{\alpha_n\}_{n=1}^{\infty}$

in $[0,1]$ satisfying $\alpha_n \le 1-\xi(x_n)$ for each $n \in \mathbb{N}$, and converging say to α, such that $||f(x_n,\alpha_n) - f(x,\alpha)|| \ge \varepsilon$ for each n. Let $z = f(x,\alpha)$, let $\lambda_n = \lambda(x_n,\alpha_n)$ and let $z_n = f(x_n,\alpha_n)$ for each n. We assume $\xi(x_n)+\alpha_n < \xi(x)+\alpha$ for each n; a similar argument applies to the terms with $\xi(x_n)+\alpha_n > \xi(x)+\alpha$. We choose a point $b \in K$ with $\xi(b) < \xi(x)+\alpha$; then $[b,z] \subset K$, and when n is large, $[b,z]$ contains a point b_n with $\xi(b_n) = \xi(x_n)+\alpha_n$. This leads to an upper bound for λ_n, and hence to the conclusion that $\limsup_{n\to\infty} \lambda_n \le \lambda(x,\alpha)$. The sequence $\{z_n\}_{n=1}^{\infty}$ is bounded, and therefore has a weakly convergent subsequence, which we assume to be the whole sequence, with weak limit z^* say. Then $\xi(z^*) = \xi(x)+\alpha$ and $z^* \in (x+\alpha e+\lambda^*\overline{Q}) \cap K$, where $\lambda^* := \liminf_{n\to\infty} \lambda_n$, hence $\lambda(x,\alpha) \le \lambda^*$. These two inequalities show that $\lambda_n \to \lambda(x,\alpha) = \lambda^*$ as $n \to \infty$, and hence $z^* = z$. If $\lambda(x,\alpha) \ne 0$ then $\{\lambda_n^{-1}(z_n - x_n - \alpha_n e)\}_{n=1}^{\infty}$ is a sequence in $\overline{Q}$ converging weakly to $\lambda(x,\alpha)^{-1}(z-x-\alpha e) \in \partial Q$, so an application of the Hahn-Banach Theorem, together with local uniform convexity, ensures the convergence is strong. If $\lambda(x,\alpha) = 0$ then $\alpha = 0$ and $z = x$; since $z_n \in x_n + \alpha_n e + \lambda_n \overline{Q}$ we now have $z_n \to x = z$ strongly. This contradiction proves the continuity of $f(\cdot,\cdot)$.

We now show that for $x \in S$ and $0 \le \alpha \le 1-\xi(S)$ we have $f(x,\alpha) \in S$. We can assume $\alpha > 0$. We write $\lambda = \lambda(x,\alpha)$, $z = f(x,\alpha)$ and $\beta = \alpha^{-1}\lambda$. Suppose $z \notin S$, so we can choose $u \in (z+C) \cap K \setminus \{z\}$; say $u = z + \gamma(e+c)$ where $c \in Q$ and $\gamma > 0$. Choose $0 < \theta < 1$ such that $w := (1-\theta)x + \theta u$ satisfies $\xi(w) = \xi(z)$; thus $\theta(\gamma+\alpha) = \alpha$. Say $z = x + \alpha(e+\beta q)$ where $q \in \overline{Q}$. Now $w \in K$ and

$$\begin{aligned} w &= x + \theta(u-x) \\ &= x + \theta(z + \gamma(e+c) - x) \\ &= x + \theta(x + \alpha(e+\beta q) + \gamma(e+c) - x) \\ &= x + \theta((\alpha+\gamma)e + \alpha\beta q + \gamma c) \\ &= x + \alpha e + (\theta\alpha\beta q + \theta\gamma c) \\ &= x + \alpha e + \alpha\beta(\theta q + (1-\theta)\beta^{-1}c). \end{aligned}$$

If $\beta > 1$ (in which case $\beta^{-1}c \in Q^{\circ}$) or $q \ne \beta^{-1}c$, then $\theta q + (1-\theta)\beta^{-1}c \in Q^{\circ}$, contradicting $(x+\alpha(e+\beta Q^{\circ})) \cap K = \emptyset$. The remaining possibility is that $\beta = 1$ and $q = c$, and then $z = x + \alpha(e+q) = x + \alpha(e+c) \in x + C$ contrary to the supposition that $x \in S$. This contradiction shows that $z \in S$ as desired.

The map $g : S \times [0,1] \to S$ defined by $g(x,s) = f(x, \min\{s, 1-\xi(x)\})$ now provides a deformation retraction of S onto S_1. Since S_1 is contractible, being a nonempty convex set, it follows that S is contractible. □

Lemma 2. *Suppose the real Hilbert space H is separable, and let $K \subset H$ be a nonempty closed bounded convex subset. Then the set of $z \in H$ for which the functional $\langle \cdot, z\rangle$ achieves its supremum relative to K at a single point, is dense in H.*

Proof of Lemma 2. Let χ_K denote the characteristic function of K defined by

$$\chi_K(x) = \begin{cases} 0, & x \in K, \\ \infty, & x \in H \setminus K, \end{cases}$$

let χ_K^* denote the conjugate convex function of χ_K, so that

$$\chi_K^*(z) = \sup_{x\in K} \langle x, z\rangle, \quad z \in H,$$

and let $\partial\chi_K : H \to 2^H$ and $\partial\chi_K^* : H \to 2^H$ denote the subdifferentials of χ_K and χ_K^*.

Now χ_K^* is convex, and continuous since K is nonempty and bounded. Hence by Mazur's Theorem (see [3, Thm. 1.20]) χ_K^* is Gâteaux differentiable on a dense $\mathcal{G}_\delta$-set $A \subset H$; thus $\partial\chi_K^*(z)$ is a singleton for all $z \in A$.

Observe that χ_K is convex, lower semicontinuous, and non-negative. It follows that for $x, z \in H$ we have $z \in \partial\chi_K(x)$ if and only if $x \in \partial\chi_K^*(z)$ (see [1, Ch. 1 Sect. 5]), which shows that the set of maximisers of $\langle\cdot, z\rangle$ relative to K is precisely $\partial\chi_K^*(z)$. Hence for $z \in A$ the functional $\langle\cdot, z\rangle$ attains its supremum relative to K at exactly one point. □

An *elliptical cone* in a Hilbert space H is a cone of the form $\{x \in H | \langle x, e\rangle > ||Tx||\}$ where $T : H \to H$ is a linear homeomorphism and $0 \neq e \in H$.

Theorem 3. *Let C be an elliptical open cone in the real Hilbert space H, let $K \subset H$ be a nonempty compact convex set, and let S be the set of all points of K that are maximal with respect to the ordering induced by C. Then there is a continuous map $h : S \times [0, 1] \to S$ and a point $a \in S$ with the properties*

$$h(x, 0) = x \ \forall x \in S,$$

$$h(x, 1) = a \ \forall x \in S,$$

$$h(h(x, s), t) = h(x, s + t) \ \forall x \in S, \ s, t, s + t \in [0, 1].$$

Proof of Theorem 3. Notice that S is a closed subset of K since C is open.

We begin by choosing a bounded linear functional ξ on H having the following four convenient additional properties:

$\mathcal{A}1$: ξ assumes its supremum relative to K at exactly one point, a say;
$\mathcal{A}2$: $C = \{x \in H | \ \xi(x) > \kappa||x||\}$ for some $\kappa > 0$;
$\mathcal{A}3$: $\xi(S) = [0, 1]$;
$\mathcal{A}4$: $\eta < 0$, where $\eta := \inf \xi(K)$.

$\mathcal{A}1$ holds for a dense set of $\xi \in H^*$ by Lemma 2, since K lies in a separable subspace of H. Additionally, ξ may be assumed to lie in the nonempty open set of functionals in H^* satisfying $\inf_{x \in C} \xi(x)/||x|| > 0$; applying a suitable linear homeomorphism to C and K then ensures $\mathcal{A}2$. A suitable dilation and translation of K now ensure $\mathcal{A}3$, for if ξ were constant on S then $S = \{a\}$ and S would be contractible trivially.

$\mathcal{A}4$ requires more justification; if it fails we show that K can be replaced by another convex body satisfying $\mathcal{A}4$, without changing the maximal set S. Suppose $\eta = 0$ and choose $e \in H$ such that $\xi(\cdot) = \langle\cdot, e\rangle$, so $e \in C$, and choose $c \in H$ such that $\xi(c) < 0$ and $c + te \in K$ for some $t > 0$. Then $c \notin K$, so define $K_1 = \mathrm{conv}(\{c\} \cup K)$ and let S_1 denote the set of maximal points of K_1. We show $S_1 = S$. Suppose $x \in S$. Then, by Hahn-Banach, a bounded linear functional φ can be chosen such that $C \subset \varphi^{-1}((0, \infty))$ (hence $\varphi(e) > 0$) and $K \subset x + \varphi^{-1}((-\infty, 0])$. Now a general point of K_1 has the form $(1-\alpha)y + \alpha c$ for some $y \in K$ and $0 \le \alpha \le 1$. Then $\varphi((1-\alpha)y + \alpha c) = \varphi((1-\alpha)y + \alpha(c + te)) - \alpha t\varphi(e) \le \varphi(x)$ since $(1-\alpha)y + \alpha(c + te) \in K$. Thus $(x + C) \cap K_1 = \emptyset$. This shows $S \subset S_1$.

Conversely suppose $x \in S_1$, so $(x+C) \cap K_1 = \emptyset$. Then $x = (1-\alpha)y + \alpha c$ for some $y \in K$, $0 \leq \alpha \leq 1$. Also there exists a bounded linear functional φ such that $C \subset \varphi^{-1}((0,\infty))$ (so $\varphi(e) > 0$) and $K_1 \subset x + \varphi^{-1}((-\infty, 0])$. Again $\varphi((1-\alpha)y+\alpha c) = \varphi((1-\alpha)y+\alpha(c+te)) - \alpha t\varphi(e) < 0$ unless $\alpha = 0$. Thus $\alpha = 0$, so $x \in K$ and $(x+C) \cap K \subset (x+C) \cap K_1 = \emptyset$, hence $x \in S$. This shows $S_1 \subset S$, hence $S_1 = S$. Moreover $\mathcal{A}1, 2, 3$ have not been disturbed. This completes the justification of $\mathcal{A}4$.

Thus $\xi(a) = 1$ and $\eta := \inf \xi(K) < 0$. For $0 \leq t \leq 1$ let

$$K(t) = \{x \in K \,|\, \xi(x) = t\}$$

and for $x \in S$ and $0 \leq t \leq 1 - \xi(x)$ let $f(x,t)$ be the nearest-point to x of $K(\xi(x)+t)$. This is the same construction employed in the proof of Theorem 1, and yields $f(x,t) \in S$ that is a (strongly) continuous function of (x,t).

For $0 \leq \gamma \leq 1$ and $x \in S$ define

$$g(x,\gamma) = f(x, \min\{\gamma, 1-\xi(x)\}),$$

so $g(\cdot,\cdot)$ maps $S \times [0,1]$ continuously into S. In particular

$$g(x,\gamma) \to x \text{ as } \gamma \downarrow 0. \tag{1}$$

Let D denote the set of dyadic rationals in $[0,1]$, let $F = S \cap \xi^{-1}(D)$, and for natural numbers m define

$$\begin{aligned} D_m &= \{k2^{-m} | 0 \leq k \leq 2^m\}, \\ F_m &= S \cap \xi^{-1}(D_m). \end{aligned}$$

It follows from (1) that F is dense in S. For $\alpha = k2^{-m} \in D_m$ let $g_m(\cdot,\alpha)$ denote the k-th iterate of $g(\cdot, 2^{-m})$. Notice that $g_m(\cdot,\alpha)$ preserves F_m. Moreover g_m has the properties

$$g_m(x,0) = x \;\; \forall x \in F_m, \tag{2}$$

$$g_m(x,1) = a \;\; \forall x \in F_m, \tag{3}$$

$$g_m(x,\alpha+\beta) = g_m(g_m(x,\alpha),\beta) \;\; \forall x \in F_m \;\; \forall \alpha,\beta,\alpha+\beta \in D_m; \tag{4}$$

(4) holds because $g_m(\cdot,\alpha)$, $g_m(\cdot,\beta)$ and $g_m(\cdot,\alpha+\beta)$ are respectively the $\alpha 2^m$-th, $\beta 2^m$-th and $(\alpha+\beta)2^m$-th iterates of $g(\cdot, 2^{-m})$. We now prove the following "equicontinuity" property for the g_m:

> ($\mathcal{P}$) *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for every $m \geq 1$ and for every (x,α) and (y,β) in $F_m \times D_m$ satisfying $||x-y|| + ||\alpha-\beta|| < \delta$ there holds $||g_m(x,\alpha) - g_m(y,\beta)|| < \varepsilon$.*

Let $\varepsilon > 0$. Since K is compact and $K \cap \xi^{-1}(1) = \{a\}$ we can choose $0 < \theta < 1$ such that $K \cap \xi^{-1}[1-2\theta, 1]$ has diameter less than ε. Since $\eta < 0$ the map $t \mapsto K(t)$ is Lipschitz on $[0, 1-\theta]$, say with rank ρ, with respect to the Hausdorff metric. Recall that the nearest-point map, of any nonempty closed convex set in a Hilbert space, is Lipschitz of rank 1. Observe that if $u \in S$ and $0 \leq \xi(u) < \xi(u) + 2^{-m} \leq 1-\theta$ then

$$||g(u, 2^{-m}) - u|| \leq \rho 2^{-m}.$$

Iteration of this inequality yields

$$||g_m(u,\alpha)-u|| \le \rho\alpha \text{ if } \alpha \in D_m \text{ and } 0 \le \xi(u) < \xi(u)+\alpha \le 1-\theta. \tag{5}$$

Consider the case when $\alpha \in D_m$, $x, y \in S$ and $\xi(x) = \xi(y)$. Then $g_m(x,\alpha)$ and $g_m(y,\alpha)$ are obtained by composing the same family of m nearest-point maps. Hence

$$||g_m(x,\alpha)-g_m(y,\alpha)|| \le ||x-y|| \text{ if } \xi(x)=\xi(y) \text{ and } \alpha \in D_m. \tag{6}$$

Consider the more general case when $x, y \in F_m$, with $\xi(x) = \lambda \le \mu = \xi(y)$, and $\alpha \in D_m$ such that $\mu + \alpha \le 1-\theta$. Then $\mu - \lambda \in D_m$, and $z = g_m(x, \mu-\lambda)$ satisfies $\xi(z) = \mu$, so by (6) we have

$$||g_m(z,\alpha)-g_m(y,\alpha)|| \le ||z-y||. \tag{7}$$

From (5) we obtain

$$\mu-\lambda \le ||z-x|| \le \rho(\mu-\lambda)$$

and therefore

$$||z-y|| \le ||z-x||+||x-y|| \le \rho(\mu-\lambda)+||x-y||. \tag{8}$$

Writing $w = g_m(x,\alpha)$, we have by (5)

$$||g_m(z,\alpha)-g_m(x,\alpha)|| = ||g_m(w,\mu-\lambda)-w|| \le \rho(\mu-\lambda). \tag{9}$$

Hence, by (9), (7) and (8),

$$\begin{aligned} ||g_m(x,\alpha)-g_m(y,\alpha)|| &\le ||g_m(x,\alpha)-g_m(z,\alpha)||+||g_m(z,\alpha)-g_m(y,\alpha)|| \\ &\le 2\rho(\mu-\lambda)+||x-y|| \le (2\rho+1)||x-y||. \end{aligned} \tag{10}$$

Now consider the still more general case when (x,α), $(y,\beta) \in F_m \times D_m$, with $\xi(x)+\alpha \le 1-\theta$ and $\xi(y)+\beta \le 1-\theta$; suppose $\alpha \le \beta$. Then, using (5) and (10),

$$\begin{aligned} & ||g_m(x,\alpha)-g_m(y,\beta)|| \\ \le\ & ||g_m(y,\alpha)-g_m(y,\beta)||+||g_m(x,\alpha)-g_m(y,\alpha)|| \\ =\ & ||g_m(y,\alpha)-g_m(g_m(y,\alpha),\beta-\alpha)||+||g_m(x,\alpha)-g_m(y,\alpha)|| \\ \le\ & \rho(\beta-\alpha)+(2\rho+1)||x-y|| \\ \le\ & (2\rho+1)((\beta-\alpha)+||x-y||) \\ <\ & \varepsilon \end{aligned}$$

provided $||\beta-\alpha||+||x-y|| < \varepsilon/(2\rho+1)$.

The final case to be considered in the verification of $(\mathcal{P})$ is when (x,α), $(y,\beta) \in F_m \times D_m$ and, say, $\xi(x)+\alpha > 1-\theta$. If $||x-y||+||\alpha-\beta|| < \theta$ then $\xi(y)+\beta > 1-2\theta$, and it follows from the choice of θ that $||g_m(x,\alpha)-g_m(y,\beta)|| < \varepsilon$. The proof of $(\mathcal{P})$ is completed by taking $\delta = \min\{\theta, \varepsilon/(2\rho+1)\}$.

We now take the limit as $m \to \infty$. It follows from $(\mathcal{P})$ that for each l the family $\{g_m\}_{m=l}^{\infty}$ is equicontinuous on $F_l \times D_l$, and is therefore precompact in $C(F_l \times D_l, K)$ by Ascoli's theorem. We can now use a diagonal sequence argument to choose a subsequence $\{g_{m_j}\}_{j=1}^{\infty}$ that is uniformly convergent on every $F_l \times D_l$, and therefore converges pointwise on $F \times D$. The limit function h is uniformly continuous on $F \times D$ by $(\mathcal{P})$, hence h extends to a continuous function (also denoted h) on $S \times [0,1]$. From (2), (3) we deduce

$$h(x,0) = x \tag{11}$$

$$h(x,1) = a \tag{12}$$

for all $x \in F$; hence by continuity (11), (12) hold for all $x \in S$.

To take the limit in (4), consider $x \in F$ and $\alpha, \beta, \alpha+\beta \in D$. Then $x \in F_l$ and $\alpha, \beta, \alpha+\beta \in D_l$ for some l; for all $m \geq l$ we then have $g_m(x,\alpha) \in F_l$, and hence $h(x,\alpha) \in F_l$. Given $\varepsilon > 0$, we can take the number $\delta > 0$ provided by $(\mathcal{P})$, and then for all large j we have $||g_{m_j}(x,\alpha) - h(x,\alpha)|| < \delta$, hence $||g_{m_j}(h(x,\alpha),\beta) - g_{m_j}(g_{m_j}(x,\alpha),\beta)|| < \varepsilon$. This enables us to pass to the limit in (4) yielding

$$h(x,\alpha+\beta) = h(h(x,\alpha),\beta) \tag{13}$$

for all $x \in F$ and $\alpha, \beta, \alpha+\beta \in D$. By continuity we deduce (13) for all $x \in S$ and $\alpha, \beta, \alpha+\beta \in [0,1]$. □

Recall that a *convex body* is a compact convex subset of Euclidean space having nonempty interior. A convex body is said to be *smooth* if each boundary point lies in only one supporting hyperplane; this is equivalent to requiring that the boundary is a surface of class C^1.

It should be noted that in the following Theorem, standard theory of ordinary differential equations does not suffice to construct a flow, having the semigroup property, for the vector field $(1-\xi(\cdot))v(\cdot)$ which is continuous but may not be Lipschitz.

Generally, in the case of a smooth convex body and an elliptical cone, an affine transformation must be performed to ensure the hypotheses of Theorem 4 hold; should this be necessary, the resulting flow will not have the same trajectories as the flow constructed by Hofer and Toland [2, 4].

Theorem 4. *Let $K \subset \mathbb{R}^d$ be a smooth convex body, let $C \subset \mathbb{R}^d$ be an elliptical open cone, and let S be the set of points of K that are maximal with respect to the ordering induced by C. Let ξ be a linear functional on $\mathbb{R}^d$ and suppose assumptions $\mathcal{A}1,2,3,4$ of the proof of Theorem 3 are satisfied. Let $h : S \times [0,1] \to S$ be the homotopy derived from the construction of Theorem 3, so $h(x,1) = a$ say for all $x \in S$, and define $p(x,t) = h(x,(1-\xi(x))(1-e^{-t}))$ for $x \in S$ and $t \geq 0$. For $x \in S \setminus \{a\}$ let $v(x)$ be the tangent vector to ∂K at x having least length subject to $\xi(v(x)) = 1$. Then $p(x,\cdot) \in C^1([0,\infty),\mathbb{R}^d)$ for each $x \in S$, p has the semigroup property on $S \times [0,\infty)$, and $\partial_t p(x,t) = (1-\xi(p(x,t)))v(p(x,t))$ for all $x \in S$ and $t \geq 0$, with the convention $0v(a) = 0$.*

Proof of Theorem 4. Let $x_0 \in S$ and $0 < \tau < 1$ with $h(x_0,\tau) \neq a$, with the object of studying $\partial_t h(x_0,\tau)$. We adopt the notation of the proof of Theorem 3. We identify ξ with the d-th coordinate function, and let π be the orthogonal projection map on $\Pi = \xi^{-1}(0)$. We can then express some neighbourhood of S relative to the surface of K as $\{(z,\phi(z)) | z \in U\}$ where $U = \pi(K^o)$ is a convex open subset of Π and $-\phi$ is a convex function of class C^1. Notice that $\nabla\phi(z) \neq 0$ for all $z \in U \setminus \{\pi a\}$, and that πS is a compact subset of U. For $z \in U$ we write $\widehat{z} = (z,\phi(z)) \in \partial K$.

If $z \in \pi S \setminus \{\pi a\}$ and $0 < t < \xi(a) - \phi(z)$, then setting $g := g(\widehat{z}, t)$ it follows that $\pi g \in \pi S \setminus \{\pi a\}$ is the minimiser of $||w - z||^2$ over w satisfying $\phi(w) = \phi(z) + t$, and therefore $\pi g - z = \lambda \nabla\phi(\pi g)$ for some $\lambda > 0$; thus $\phi(\pi g) = \phi(\pi g - \lambda\nabla\phi(\pi g)) + t$. We now fix $u = \pi h(x_0, \tau) \in U \setminus \{\pi a\}$, and show that the equation

$$\phi(w) = \phi(w - \lambda v) + t \tag{14}$$

defines λ as a C^1 function of (w, v, t) near to $(u, \nabla\phi(u), 0)$. Observe that $\partial_\lambda(\phi(u - \lambda\nabla\phi(u))) = ||\nabla\phi(u)||^2 > 0$ at $\lambda = 0$. Hence by the Implicit Function Theorem there are neighbourhoods $U_0 \subset U \setminus \{\pi a\}$ of u, $V_0 \subset \mathbb{R}^d$ of $\nabla\phi(u)$, and positive numbers η_0, δ, such that for $w \in U_0$, $v \in V_0$, and $|t| < \eta_0$, equation (14) has a unique solution $\lambda(w, v, t)$ in the interval $(-\delta, \delta)$, and that $\lambda(\cdot, \cdot, \cdot)$ is continuously differentiable.

Let $\varepsilon > 0$, and choose a neighbourhood $U_1 \subset U_0$ of u and and a number $0 < \eta_1 < \eta_0$ such that

$$y \in U_1,\ |t| < \eta_1 \ \Rightarrow \left\{ \begin{array}{l} \pi g(\widehat{y}, t) \in U_0,\ \nabla\phi(y) \in V_0,\ ||\nabla\phi(y) - \nabla\phi(u)|| < \varepsilon/2, \\ ||\nabla\phi(\pi g(\widehat{y}, t))||^{-1} ||\pi g(\widehat{y}, t) - y|| < \delta, \end{array} \right.$$

so that in particular

$$y \in U_1,\ |t| < \eta_1 \ \Rightarrow\ \pi g(\widehat{y}, t) - y = \lambda(\pi g(\widehat{y}, t), \nabla\phi(\pi g(\widehat{y}, t)), t)\nabla\phi(\pi g(\widehat{y}, t)).$$

We claim that U_1 and η_1 may also be assumed small enough that

$$y, z \in U_1,\ |t| < \eta_1 \ \Rightarrow\ |\lambda(z, \nabla\phi(z), t) - t\partial_t\lambda(y, \nabla\phi(y), 0)| \le \varepsilon t. \tag{15}$$

To see this, suppose $y, z \in U_1$ and $|t| < \eta_1$. Then $\lambda(z, \nabla\phi(z), 0) = 0$ so

$$\begin{aligned} & |\lambda(z, \nabla\phi(z), t) - t\partial_t\lambda(y, \nabla\phi(y), 0)| \\ = \ & |\lambda(z, \nabla\phi(z), t) - \lambda(z, \nabla\phi(z), 0) - t\partial_t\lambda(y, \nabla\phi(y), 0)| \\ = \ & |t\partial_t\lambda(z, \nabla\phi(z), \theta) - t\partial_t\lambda(y, \nabla\phi(y), 0)| \end{aligned}$$

for some θ, $|\theta| \le |t|$, by the Mean Value Theorem. Then by continuity of $\nabla\phi$ and $\partial_t\lambda$ we have (15) provided U_1 and η_1 are sufficiently small.

By property $(\mathcal{P})$ we can choose a neighbourhood $u \in U_2 \subset U_1$ and a number $0 < \eta_2 < \eta_1$ such that if $z \in U_2 \cap \pi F_m$, $t \in [0, \eta_2] \cap D_m$, and $m \ge 1$ then $\pi g_m(\widehat{z}, t) \in U_1$. We may further suppose η_2 is small enough that $h(x_0, \tau - \eta_2) \in U_2$, and that $\xi(h(x_0, \tau - \eta_2)) \in D$. Consider s, t with $\tau - \eta_2 \le s < t \le \tau$ such that $t - \tau + \eta_2,\ s - \tau + \eta_2 \in D$, so $t - s \in D$. Set $x := h(x_0, s)$; then $v := \pi x \in U_2 \cap F_m$ and $t - s \in D_m$ provided $m \ge 1$ is sufficiently large, an assumption we make henceforth. Thus $t - s = k2^{-m}$ for some k, $0 \le k \le 2^m$. Then for $1 \le j \le k$ we have $g_m(\widehat{v}, j2^{-m}) = g(g_m(\widehat{v}, (j-1)2^{-m}), 2^{-m})$ and $g_m(\widehat{v}, (j-1)2^{-m}) \in U_1$, hence by (15)

$$\begin{aligned} & ||\pi g_m(\widehat{v}, j2^{-m}) - \pi g_m(\widehat{v}, (j-1)2^{-m}) - 2^{-m}\partial_t\lambda(u, \nabla\phi(u), 0)\nabla\phi(u)|| \\ = \ & |\lambda(\pi g_m(\widehat{v}, j2^{-m}), \nabla\phi(\pi g_m(\widehat{v}, j2^{-m})), 2^{-m})\nabla\phi(\pi g_m(\widehat{v}, j2^{-m})) \\ & \quad -2^{-m}\partial_t\lambda(u, \nabla\phi(u), 0)\nabla\phi(u)| \\ \le \ & 2M\varepsilon 2^{-m}, \end{aligned}$$

where M is an upper bound for $||\nabla\phi(\cdot)||$ and $|\partial_t\lambda(\cdot, \nabla\phi(\cdot), 0)|$ on U_0. Summing over j and using the triangle inequality we obtain

$$||\pi g_m(\widehat{v}, k2^{-m}) - \pi g_m(\widehat{v}, 0) - k2^{-m}\partial_t\lambda(u, \nabla\phi(u), 0)\nabla\phi(u)|| \le 2M\varepsilon 2^{-m}k,$$

that is,

$$||\pi g_m(x, t - s) - v - (t - s)\partial_t \lambda(u, \nabla\phi(u), 0)\nabla\phi(u)|| \leq 2M\varepsilon(t - s).$$

Letting $m \to \infty$ yields

$$||\pi h(x, t - s) - v - (t - s)\partial_t \lambda(u, \nabla\phi(u), 0)\nabla\phi(u)|| \leq 2M\varepsilon(t - s),$$

that is,

$$||\pi h(x_0, t) - \pi h(x_0, s) - (t - s)\partial_t \lambda(u, \nabla\phi(u), 0)\nabla\phi(u)|| \leq 2M\varepsilon(t - s)$$

By continuity we deduce

$$||\pi h(x_0, t) - h(x_0, s) - (t - s)\partial_t \lambda(u, \nabla\phi(u), 0)\nabla\phi(u)|| \leq 2M\varepsilon(t - s)$$

for all real s, t with $\tau - \eta_2 \leq s < t \leq \tau$. Setting $t = \tau$ then letting $\varepsilon \downarrow 0$ and $s \uparrow \tau$ yields the existence and value of the left derivative

$$\partial_t^- \pi h(x_0, \tau) = \partial_t \lambda(u, \nabla\phi(u), 0)\nabla\phi(u).$$

Trivially

$$\partial_t \xi(h(x_0, \tau)) = 1.$$

Thus $\partial_t^- h(x_0, \tau)$ exists, and necessarily lies in the tangent plane to K at $h(x_0, \tau) = (u, \phi(u))$; since its component parallel to Π is parallel to $\nabla\phi(u)$, it follows that $\partial_t^- h(x_0, \tau)$ is orthogonal to all the tangent vectors parallel to Π, hence $\partial_t^- h(x_0, \tau) = v(h(x_0, \tau))$.

Since the above calculations show that $\partial_t^- h(x_0, t)$ is a continuous function of t, it easily follows that the two-sided derivative $\partial_t h(x_0, t)$ exists at values $t \in (0, 1)$ where $h(x_0, t) \neq a$. Moreover $h(x_0, \cdot)$ is right-continuous at 0 and $\partial_t h(x_0, \cdot)$ has a finite right limit at 0, hence we can also deduce the existence of the right derivative $\partial_t^+ h(x_0, 0)$.

Convexity of ϕ together with $\nabla\phi(0) = 0$ ensures $||\nabla\phi(z)||^{-1}\phi(z) \to 0$ as $z \to 0$, hence $(1 - \xi(x))v(x) \to 0$ as $x \to a$, suggesting the convention $0v(a) = 0$.

It follows from the definition that $1 - \xi(p(x, t)) = (1 - \xi(x))e^{-t}$ for $x \in S$ and $t \geq 0$. A direct calculation shows that p has the semigroup property, and that for $x \in S \setminus \{a\}$ and $t \geq 0$,

$$\begin{aligned} \partial_t p(x, t) &= \partial_t h(x, (1 - \xi(x))(1 - e^{-t})) \\ &= (1 - \xi(x))e^{-t} v(h(x, (1 - \xi(x))(1 - e^{-t}))) \\ &= (1 - \xi(p(x, t)))v(p(x, t)), \end{aligned}$$

and a is a fixed point of p. □

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