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Existence in Optimal Control with State Equation in Divergence Form Via Variational Principles

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We explore several relevant issues concerned with some typical two-dimensional optimal design problems by using a purely variational format. Specifically, we investigate situations of existence of optimal solutions and relaxation in terms of quasiconvexifications and Young measures.

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1. Introduction

Optimal design and structural optimization are two main areas of applied mathematics that have stirred fascinating mathematical developments in the last three decades. See for instance the monograph [7], the volume [8], or the notes [26]. Strictly speaking, we are talking about optimal control problems governed by boundary value problems. The distinguished feature of some of these problems is that the control acts on the principal part of the elliptic state equation which usually represents an equilibrium law for the particular system at hand. For this reason the structure of the underlying state equation or system incorporates all variables within a divergence.

Specifically, the general problem we would like to address is described as follows. Let Ω be a domain in $\mathbf{R}^2$, K a subset of $\mathbf{R}^n$, and F, G, V three Carathéodory functions

$$F : \Omega \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{M}^{2\times m} \to \mathbf{R},$$
$$G : \Omega \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{M}^{2\times m} \to \mathbf{M}^{2\times l},$$
$$V : \Omega \times \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{M}^{2\times m} \to \mathbf{R}^d,$$

with F bounded from below, and n, m, l, d positive integers. We would like to consider the optimization problem which we will identify by (P):

Find an optimal pair $(E(x), u(x))$, belonging to the admissible set of pairs verifying

$$E(x) \in \mathcal{U} = \{E \in L^\infty(\Omega) : E(x) \in K, \text{ a.e. } x \in \Omega\},$$

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$$\begin{cases} -\operatorname{div}\,[G(x,E(x),u(x),\nabla u(x))] = 0 & \text{in } \Omega, \\ u = u_0 \quad \text{on } \partial\Omega, \end{cases}$$

$$\int_\Omega V(x,E(x),u(x),\nabla u(x))\,dx \leq \gamma,$$

which minimizes the cost

$$I(E,u) = \int_\Omega F(x,E(x),u(x),\nabla u(x))\,dx.$$

The boundary data u_0 and the vector $\gamma \in \mathbf{R}^d$ are part of the data set. Other type of boundary conditions can be allowed. With respect to the state equation (system) we will place ourselves in the situation where weak solutions in some Sobolev space $W^{1,q}(\Omega)$, $q \geq 2$, exist for admissible controls $E \in \mathcal{U}$. A typical situation occurs when G has polynomial growth of order $q-1$ in ∇u, and G itself is the derivative of another convex, coercive density with respect to the gradient variable ([10]). Specifically, we explicitly suppose that

$$c\left(|A|^{q-1} - 1\right) \leq |G(x,E,u,A)|\,, \quad q \geq 1, c > 0,$$

and that the state equation admits solutions in $W^{1,q}(\Omega)$. We will also assume that the number of unknowns m is greater (usually equal) than the number of equations l. Target spaces for functions in Sobolev or Lebesgue spaces will not be indicated since this will not create any confusion.

Two typical situations correspond to optimal design in conductivity where $n = m = l = d = 1$,

$$G(x,E,u,A) = EA + \nabla p(x), \quad V(x,E,u,A) = E,$$

and p is a fixed, auxiliary function which is the solution of the problem

$$\Delta p = P \quad \text{in } \Omega, \quad p \in H_0^1(\Omega),$$

where

$$E(x) = \chi(x)E_1 + (1-\chi(x))E_2,$$

and E_i are the conductivity coefficients of two different materials. And structural optimization in the elasticity case which is formally similar to the conductivity problem but this time $n = 16$, $m = l = 2$, $d = 1$,

$$G(x,E,u,A) = E\,\frac{1}{2}(A + A^T) + \nabla p(x), \quad V(x,E,u,A) = E,$$

with the auxiliary vector p coming from the same Poisson vector problem

$$\Delta p = P \quad \text{in } \Omega, \quad p \in H_0^1(\Omega).$$

Here the equilibrium law is the system of linear elasticity

$$-\operatorname{div}\,(E(x)\epsilon(u)(x)) = P(x) \quad \text{in } \Omega,$$
$$E(x)\epsilon(u)(x)\cdot\eta = g(x) \quad \text{on } \Gamma_1, \qquad u(x) = u_0(x) \quad \text{on } \Gamma_0,$$

where $\epsilon(u)$ is the symmetrized gradient, g is a density surface load on a part of $\partial\Omega$, while the control E

$$E(x) = \chi(x)E_1 + (1 - \chi(x))E_2,$$

represents a spatial mixture of two linear elastic, homogeneous materials whose elastic constants are recorded in the fourth-order tensors E_i, $i = 1, 2$. We also assume in this situation a volume constraint so that V is the same as in the conductivity example. η is the outer normal to $\partial\Omega$.

These two situations are special in that the structure of the underlying equivalent variational principle is non-convex while in this paper we would like to focus on situations where existence of optimal solutions can be shown. From this point of view, these two examples are merely as a motivation for our analysis. On the other hand they are so important that a separate analysis is being carried out to understand them. See [6], [21] and comments below.

Problems of this nature have been considered for instance in [12], [13], [25], although the engineering literature is much more abundant when cost functionals do not depend on derivatives of states (see [7]).

The main motivation of our work is to analyze a general problem as the one described above by exploiting an alternative approach to this type of problems that permits a general dependence of the objective functional with respect to derivatives of the control $A = \nabla u$, and also a non-linear state equation. From this perspective, it is a natural continuation of [5]. We will however restrict our attention in the present paper mainly to an equilibrium law which is linear in the (gradient of the) state as indicated above. As pointed out, the main tool for the analysis is a reformulation of the whole optimization problem into a purely variational format where the non-local effect of the equilibrium law is substituted by the introduction of a new independent vector field in the form of a stream function or a potential. This transformation has already been used in some general, abstract fashion in a number of papers ([20]), and also in different contexts in ([15], [22]). Some of these situations correspond to non-existence, and relaxation in terms of gradient Young measures was the main point. In the present case, however, we would like to treat this sort of optimal design problems in a more systematic way, analyzing and tailoring situations where existence of optimal solutions can be shown. Existence is however much more involved than in dimension one ([5]). We then discuss in general terms the issue of how the fact that the integrand for the equivalent variational problem takes on the value $+\infty$ in a non-continuous fashion affects relaxation. This is relevant from the perspective of trying to approximate optimal microstructures by using relaxation in terms of gradient Young measures or laminates ([2]).

The structure of the paper is as follows. We will derive the variational reformulation in the next section, and use this new format to deepen our understanding of the reasons for non-existence in many of these optimal design problems by seeking situations where existence of optimal solutions is possible (Section 3). Since these theorems cannot be applied to the typical, important situation described earlier, we will briefly comment in Section 4 about relaxation of the equivalent variational principle in terms of convex hulls of functions and gradient Young measures.

2. Reformulation

Keep in mind all the notation introduced in Section 1 to describe problem (P). For the time being, we will neglect the integral constraints associated to the function V since the discussion that follows is not related to such restrictions. We will afterwards formulate and justify our conclusions incorporating such integral constraints.

Our basic starting element for the variational reformulation of the problem is the following well-known fact.

Theorem 2.1. ([11]) *Let Ω be an open subset of $\mathbf{R}^2$, regular and simply connected, and $v \in L^s(\Omega)$, $s \geq 2$. There exists a function $w \in W^{1,s}(\Omega)$ such that*

$$v = T\nabla w, \quad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

if and only if $div(v) = 0$ in a weak sense.

Assume now that the pair (E, u) is admissible for (P) and

$$I(E, u) < +\infty.$$

If

$$c\left(|G(x, E, u, B)|^s - 1\right) \leq F(x, E, u, B), \quad s \geq 2, c > 0$$

then

$$v(x) = G(x, E(x), u(x), \nabla u(x)) \in L^s(\Omega), \quad \mathrm{div}\,(v) = 0.$$

The above bound relating F and G is in a sense a minimal technical hypothesis to apply Theorem 2.1. By this theorem, there exists $w \in W^{1,s}(\Omega)$ such that

$$G(x, E(x), u(x), \nabla u(x)) = T\nabla w(x).$$

We define for $(x, u, A) \in \Omega \times \mathbf{R}^m \times (\mathbf{M}^{2\times m} \times \mathbf{M}^{2\times l})$

$$K(x, u, A) = \left\{E \in K : G(x, E, u, A^{(1)}) = TA^{(2)}\right\} \subset K,$$

where $A^{(1)} \in \mathbf{M}^{2\times m}$ and $A^{(2)} \in \mathbf{M}^{2\times l}$. We also put

$$\tilde{F}(x, u, A) = \min_{E\in K(x,u,A)} F(x, E, u, A^{(1)}).$$

It is important to rely on appropriate hypotheses to guarantee that the previous minimum is indeed attained somewhere in $K(x, u, A)$. Either K is assumed from the beginning to be compact, or else F must be coercive over K in the sense

$$\lim_{E\to\infty, E\in K} F(x, E, u, B) = +\infty.$$

Since in most of the situations of interest K should be bounded, we will restrict ourselves to this case. When $K(x, u, A)$ is empty, $\tilde{F}$ is set to be $+\infty$.

Consider the variational problem $(\tilde{P})$ associated with $\tilde{F}$

$$\text{Minimize} \quad \tilde{I}(U) = \int_\Omega \tilde{F}(x, U^{(1)}(x), \nabla U(x))\, dx$$

subject to

$$U = (U^{(1)}, U^{(2)}), \quad U \in W^{1,s}(\Omega), U^{(1)} - u_0 \in W_0^{1,q}(\Omega).$$

Notice that $q \leq s(q-1)$ and $s \leq s(q-1)$ if both q and s are greater than 2.

Proposition 2.2. *Assume the following*

1. Ω *is a regular, bounded, simply-connected domain;*
2. $K \subset \mathbf{R}^n$ *is a compact set;*
3. *there exists a constant* $c > 0$ *such that*

$$c\left(|B|^{q-1} - 1\right) \leq |G(x, E, u, B)|,$$

for all (x, E, u, B)*;*

4. *for some* $s \geq 2$

$$c\left(|G(x, E, u, B)|^s - 1\right) \leq F(x, E, u, B), \quad \textit{for all } (x, E, u, B).$$

Then (P) *and* $(\tilde{P})$ *are equivalent in the sense:*

1. *if* m *and* $\tilde{m}$ *stand for the infima for both problems, respectively, then* $m = \tilde{m}$*;*
2. *if the pair* (E, u) *is optimal for* (P)*, then* U *is optimal for* $(\tilde{P})$ *where*

$$U^{(1)} = u, \quad G(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x)) = T\nabla U^{(2)}(x);$$

3. *conversely, if* U *is optimal for* $(\tilde{P})$ *then the pair* (E, u) *where*

$$u = U^{(1)}, \quad \tilde{F}(x, u(x), \nabla U(x)) = F(x, E(x), u(x), \nabla u(x))$$

is optimal for (P)*.*

Proof. The proof has almost been indicated. In fact, the transformation from (P) to $(\tilde{P})$ has been dictated so that this equivalence is preserved.

Let (E, u) be admissible for (P). Then we can write, bearing in mind the preceding discussion

$$G(x, E(x), u(x), \nabla u(x)) = T\nabla w(x) \quad \text{a.e. } x \in \Omega.$$

Put $U = (u, w)$. By definition,

$$\tilde{F}(x, u(x), \nabla U(x)) \leq F(x, E(x), u(x), \nabla u(x))$$

pointwise, U is admissible for $(\tilde{P})$ and

$$\tilde{I}(U) \leq I(E, u).$$

Conversely, if U is admissible for $(\tilde{P})$ and $\tilde{I}(U) < +\infty$ then

$$\tilde{F}(x, U^{(1)}(x), \nabla U(x)) < +\infty \quad \text{a.e. } x \in \Omega.$$

Again by construction of $\tilde{F}$, this implies the existence of a measurable $E(x) \in K$ ([3]) such that

$$\tilde{F}(x, U^{(1)}(x), \nabla U(x)) = F(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x)),$$

$$G(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x)) = T\nabla U^{(2)}(x).$$

This implies that

$$\operatorname{div}\left[G(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x))\right] = 0 \quad \text{in } \Omega,$$
$$U^{(1)} = u_0 \quad \text{on } \partial\Omega,$$

so that if $u = U^{(1)}$, the pair (E, u) is admissible for (P) and

$$I(E, u) = \tilde{I}(U).$$

The conclusion of the proposition follows. □

Our approach to the analysis of (P) consists in examining $(\tilde{P})$. Concerning this variational problem, two comments are relevant. First, the new density $\tilde{F}$ may not be a Carathéodory function with respect to A, since in particular it may take on the value $+\infty$ abruptly. Indeed, in some of the typical situations the set where $\tilde{F}$ is finite is rather small as we will see. Secondly, we are facing a vector variational problem as the independent field for $(\tilde{P})$ is $U : \Omega \to (\mathbf{R}^m \times \mathbf{R}^l)$ and $1 \le l \le m$.

3. Existence results

In order to pursue a systematic approach to the sort of optimal design problems expressed in (P), we are going to deal in this section with some explicit situations where existence of optimal solutions can in fact be achieved. Specifically, what we seek is a set of hypotheses on the different ingredients determining the optimal design problem so as to permit a rigorous proof of existence of optimal solutions for $(\tilde{P})$ in the context of vector variational problems. We will first treat the case of absence of integral constraints, and then incorporate this additional restriction.

We will start by a general existence theorem based on the quasiconvexity ([9], [19]) and coercivity of the integrand $\tilde{F}$ so that the direct method will furnish optimal solutions; then we will look for fully explicit examples where this quasiconvexity can be shown: we will further state more restrictive hypotheses so that the integrand $\tilde{F}$ is in fact polyconvex.

For the precise statement of results we need the following additional notation

$$\Delta = \{(x, u, A) : K(x, u, A) \neq \emptyset\},$$
$$\Delta(x) = \{(u, A) : (x, u, A) \in \Delta\},$$
$$\Delta(x, u) = \{A : (x, u, A) \in \Delta\}.$$

Notice that Δ is the set where $\tilde{F}$ is finite. We also need, to be precise, to invoke the notion of quasiconvexity of functions taking on the value $+\infty$. This is a delicate issue. For our purposes and in order to avoid further technicalities, it is enough to adopt the following definition. Here $\mathbf{R}^*$ is $\mathbf{R} \cup \{+\infty\}$ and $\mathbf{M}$ is the space of matrices of certain dimensions.

Definition 3.1. ([18]) A function $\varphi : \mathbf{M} \to \mathbf{R}^*$ is called closed- $W^{1,p}$ quasiconvex if

$$\varphi\left(\int_{\mathbf{M}} A \, d\nu(A)\right) \le \int_{\mathbf{M}} \varphi(A) \, d\nu(A)$$

for every $W^{1,p}$-gradient Young measure.

A set of matrices Λ is called closed-$W^{1,p}$ quasiconvex if its characteristic function

$$\chi_\Lambda(A) = \begin{cases} 1, & A \in \Lambda, \\ +\infty, & A \notin \Lambda, \end{cases}$$

is closed-$W^{1,p}$ quasiconvex.

This quasiconvexity condition for sets amounts to having the center of mass of all $W^{1,p}$-gradient Young measures supported in that set, contained in the same set. In particular, convex sets of matrices are closed-$W^{1,p}$ quasiconvex for any p. The quasiconvexity of sets is also a very delicate issue ([4], [22], [23], [24]).

Theorem 3.2. *Assume, in addition to the hypotheses in Proposition* 2.2*, that*

1. $\Delta(x)$ *is closed for a.e.* $x \in \Omega$*;*
2. $\Delta(x, u)$ *is closed* $W^{1,s}$*-quasiconvex for a.e.* $x \in \Omega$ *and all* $u \in \mathbf{R}^m$*;*
3. *the integrand*
$$\tilde{F} : \Delta \to \mathbf{R}$$
is a Carathéodory function and
$$\tilde{F}(x, u, \cdot) : \Delta(x, u) \to \mathbf{R},$$
is closed-$W^{1,s}$ *quasiconvex for every pair* (x, u)*.*

Then the optimal design problem admits at least one optimal solution.

Proof. The proof follows along the lines of the direct method of the Calculus of Variations which basically consists in showing weak lower semicontinuity in terms of Young measures ([19]) and coercivity. Technicalities involve dealing with the peculiarity that $\tilde{F}$ takes on the value $+\infty$ in a non-continuous way.

We first show weak lower semicontinuity. Suppose $U_j \rightharpoonup U$ in $W^{1,s}(\Omega)$, where

$$\tilde{I}(U_j) < +\infty, \quad \liminf_{j\to\infty} \tilde{I}(U_j) < +\infty,$$

so that

$$(U_j^{(1)}(x), \nabla U_j(x)) \in \Delta(x) \quad \text{a.e. } x \in \Omega.$$

Consider the Young measure $\mu = \{\mu_x\}_{x\in\Omega}$ associated to the sequence

$$\left\{(U_j^{(1)}, \nabla U_j)\right\}.$$

By the compactness embedding of Sobolev spaces and the fact that

$$U_j^{(1)} = u_0 \quad \text{on } \partial\Omega$$

for all j, we see that $U_j^{(1)} \to U^{(1)}$ strong (in $L^{1,s(q-1)}(\Omega)$). This implies ([19]) that

$$\mu_x = \delta_{U^{(1)}(x)} \otimes \nu_x \quad \text{a.e. } x \in \Omega,$$

where $\nu = \{\nu_x\}_{x\in\Omega}$ is the Young measure associated with $\{\nabla U_j\}$. Since $\Delta(x)$ is closed,

$$\text{supp}\,(\nu_x) \subset \Delta(x, U^{(1)}(x)) \quad \text{a.e. } x \in \Omega.$$

Because $\Delta(x, U^{(1)}(x))$ is quasiconvex, its first moment

$$\nabla U(x) = \int_{\mathbf{M}^{2\times m}\times\mathbf{M}^{2\times l}} A\, d\nu_x(A)$$

should belong to $\Delta(x, U^{(1)}(x))$ as well. Further, due to the hypotheses in 3. of the statement, we have ([19])

$$\begin{aligned}\liminf_{j\to\infty} \int_\Omega \tilde{F}(x, U_j^{(1)}(x), \nabla U_j(x))\, dx &\geq \int_\Omega \int_{\mathbf{M}^{2\times m}\times\mathbf{M}^{2\times l}} \tilde{F}(x, U^{(1)}(x), A)\, d\nu_x(A)\, dx \\ &\geq \int_\Omega \tilde{F}(x, U^{(1)}(x), \nabla U(x))\, dx,\end{aligned}$$

as desired.

Coercivity in the appropriate spaces is a direct consequence of the definition of $\tilde{F}$ and the bounds on F and G assumed on Proposition 2.2. □

Much more explicit results could involve the polyconvexity of the function $\tilde{F}$. As an example we present one such situation where for simplicity we have taken $n = m = l = 1$.

Theorem 3.3. *Assume that*

$$G(x, E, u, B) = g(x, u, B) + h(x, u, E)$$

where

$$g : \Omega \times \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}^2,$$
$$h : \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^2,$$

are Carathéodory functions, affine on B and E, respectively, and otherwise chosen so that the state problem is well-posed. Take $s = q = 2$. Suppose in addition to the hypotheses of Proposition 2.2, that K is convex. If

$$f(x, u, A, t) : \Omega \times \mathbf{R} \times \mathbf{M}^{2\times 2} \times \mathbf{R} \to \mathbf{R}$$

is a Carathéodory function, convex (in the usual sense) in the variables (A, t) for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$, and

$$f(x, u, A, t) \geq c\left(|A|^2 - 1\right), \quad c > 0,$$

then the corresponding optimal control problem with objective integrand given by

$$F(x, E, u, B) = f\left(x, u, (B, -T(g(x, u, B) + h(x, u, E))), B \cdot (g(x, u, B) + h(x, u, E))\right),$$

will admit optimal solutions.

Proof. It is elementary to check

$$K(x,u,A)=\left\{E\in K: h(x,u,E)=TA^{(2)}-g(x,u,A^{(1)})\right\},$$
$$\tilde{F}(x,u,A)=\begin{cases} f(x,u,A,\det A), & \text{if } K(x,u,A) \text{ is not empty,}\\ +\infty, & \text{else.}\end{cases}$$

Since we also have the appropriate coercivity for f, $\tilde{F}$ is closed-H^1 quasiconvex since it is polyconvex. On the other hand, it is also straightforward to realize that the convexity of K together with the affinity of g and h, imply the convexity of each $\Delta(x,u)$. Theorem 3.2 applies. □

One typical situation where Theorem 3.3 can be applied is the following

$$\text{Minimize } I(E)=\int_\Omega \varphi\left(|\nabla u(x)|^2+E(x)\nabla u(x)\right)\,dx$$

where

$$E\in L^2(\Omega), E(x)\in K\subset \mathbf{R}^2, \text{ convex and compact,}$$
$$u\in H_0^1(\Omega):\ \operatorname{div}(-\nabla u+E)=0 \quad \text{in } \Omega,$$

and φ is convex and

$$\varphi(t)\geq |t|-c, \quad c, \text{ a constant.}$$

For instance $\varphi(t)=\sqrt{1+t^2}$, $\varphi(t)=t^2$, etc.

Due to the form of the state equation (linear in E and ∇u, and without any interaction between both terms), this results can be shown by using the Div-Curl lemma ([17], [24]) together with Meyer's regularity theorem ([14]). More general situations can, however, be easily described by tailoring the objective integrand to be quasiconvex but not polyconvex.

Proposition 3.4. *Under the same hypotheses of the previus theorem, let*

$$f:\mathbf{M}^{2\times 2}\times\mathbf{R}\to\mathbf{R}$$

be defined by putting

$$f(A,t)=|A|^2\left(|A|^2-2\gamma t\right).$$

There exists $\epsilon>0$ such that for $\gamma\in(1,1+\epsilon]$, the function $F(A)=f(A,\det A)$ is not polyconvex (which implies that f is not convex), yet the optimal control problem of Theorem 3.3 admits optimal solutions.

The proof is the same as that of Theorem 3.3, having in mind the results in [1] about the polyconvexity and quasiconvexity of that type of functions.

Even some non-linear state equations are elligible for this type of techniques. The proof of the next result follows along the lines of Theorem 3.3. A similar result can thus be proved allowing g to be nonlinear in B but K needs to be all of space.

Proposition 3.5. *Let*

$$G(x,E,u,B)=g(x,u,B)+h(x,u)E$$

where

$$g:\Omega\times\mathbf{R}\times\mathbf{R}^2\to\mathbf{R}^2,$$
$$h:\Omega\times\mathbf{R}\to\mathbf{R},$$

are Carathéodory functions. Suppose that

$$h\in L^\infty(\Omega\times\mathbf{R}),\quad h\geq c>0,$$
$$c\left(|G(x,E,y,B)|^s-1\right)\leq\left(|u|^p+|B|^p\right),\quad s\geq 2, c>0,$$
$$c\left(|B|^{p-1}-1\right)\leq G(x,E,u,B),$$

and that the equation of state is well-posed. Admissible controls are taken from all of $L^s(\Omega;\mathbf{R}^2)$*. If*

$$f(x,u,A,t):\Omega\times\mathbf{R}\times\mathbf{M}^{2\times 2}\times\mathbf{R}\to\mathbf{R}$$

is a Carathéodory function, convex in the variables (A,t) *for a.e.* $x\in\Omega$ *and all* $u\in\mathbf{R}$*, and*

$$f(x,u,A,t)\geq c\left(|A|^p-1\right),\quad c>0,$$

then the corresponding optimal control problem with objective integrand given by

$$F(x,E,u,B)=f\left(x,u,(B,-T(g(x,u,B)+h(x,u,E))),B\cdot(g(x,u,B)+h(x,u,E))\right),$$

will admit optimal solutions.

If we allow for additional integral constraints defined by an integrand V then more hypotheses are needed. One would have to ask for the quasiconvexity (polyconvexity) of the function $\tilde{V}$ just as we did with $\tilde{F}$. Furthermore there must be some compatibility between these two densities and the function G (see [5]). We put

$$\tilde{V}(x,u,A)=\min_{E\in K(x,u,A)}V(x,E,u,A^{(1)}).$$

The same observations we made for $\tilde{F}$ apply to $\tilde{V}$.

Theorem 3.6. *Suppose that the same hypotheses of Theorem* 3.2 *hold, and the same conditions we have for* $\tilde{F}$ *are true for* $\tilde{V}$*. Assume in addition that*

$$\mathit{argmin}_{E\in K(x,u,A)}F(x,E,u,A^{(1)})\cap\mathit{argmin}_{E\in K(x,u,A)}V(x,E,u,A^{(1)})$$

is not the empty set for any (x,u,A)*. Then the optimal design problem* (P) *at the beginning of Section* 2*, has at least one optimal solution.*

Proof. The proof is the same as that of Theorem 3.2. We only need to realize that the equivalent variational problem would be

$$\text{Minimize}\quad\tilde{I}(U)=\int_\Omega\tilde{F}(x,U^{(1)}(x),\nabla U(x))\,dx$$

subject to

$$U=(U^{(1)},U^{(2)}),\quad U\in W^{1,s}(\Omega),U^{(1)}-u_0\in W_0^{1,q}(\Omega),$$
$$\int_\Omega\tilde{V}(x,U^{(1)}(x),\nabla U(x))\,dx\leq\gamma.$$

The further hypothesis about argmin is necessary in order to ensure the possibility of having $E(x) \in K$ such that

$$\begin{aligned} \tilde{F}(x, U^{(1)}(x), \nabla U(x)) &= F(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x)), \\ \tilde{V}(x, U^{(1)}(x), \nabla U(x)) &= V(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x)), \\ G(x, E(x), U^{(1)}(x), \nabla U^{(1)}(x)) &= T\nabla U^{(2)}(x). \end{aligned}$$

Then the two optimization problems are equivalent. □

More explicit statements can be made enforcing the polyconvexity of $\tilde{V}$ as in Theorem 3.3.

4. Some comments on non-existence and relaxation

It occurs in practice that the optimal control problem in which one may be interested lacks optimal solutions ([16]). This is the typical situation for optimal design problems in conductivity and elasticity. In such cases the integrand for the equivalent variational principle cannot be quasiconvex in any sense. Optimality usually involves in these cases persistent oscillations in finer and finer spatial scales. Relaxation is the concept that one has to turn to in order to make an attempt to understand this sort of problems.

For the sake of the reader, we will make a disgression and remind some fundamental concepts in non-convex variational principles before proceeding to examine our situation.

Let $\varphi : \mathbf{M} \to \mathbf{R}$ be a continuous, non-quasiconvex function where $\mathbf{M}$ is the space of matrices of certain dimensions. It is well-known that the variational principle

$$\text{Minimize} \quad \int_\Omega \varphi(\nabla u(x))\, dx, \quad u \in \mathcal{A},$$

may lack optimal solutions. $\mathcal{A}$ is assumed to be a set of functions in some Sobolev space, closed under weak convergence. In this case relaxation must be examined. We would like to consider relaxation at two levels: one involves a change of integrand to the quasiconvex hull of φ; the other, a change in the set of admissible objects to the set of appropriate gradient Young measures. The quasiconvexification of φ is defined by

$$Q\varphi(A) = \inf\left\{ \frac{1}{|D|} \int_D \varphi(A + \nabla w(y))\, dy : w \in W_0^{1,\infty}(D) \right\},$$

where D is any regular domain. The set of homogeneous gradient Young measures generated by sequences of test deformations is

$$\{\nu : \nu \text{ is generated by a uniformly bounded sequence of gradients } \{\nabla w_j\}, w_j \in W_0^{1,\infty}(D)\}.$$

Both definitions do not depend on the domain D. The following is a typical relaxation theorem ([19]).

Theorem 4.1. *Let $\varphi : \mathbf{M} \to \mathbf{R}$ be a continuous function such that*

$$c\left(|A|^p - 1\right) \le \varphi(A) \le C\left(|A|^p - 1\right), \quad 0 < c < C, 1 < p.$$

Let $\mathcal{A}$ be a subset of $W^{1,p}(\Omega)$, closed under weak convergence, and $\overline{\mathcal{A}}$ the set of Young measures associated with sequences of gradients in $\mathcal{A}$, and let

$$m = \inf\left\{ \int_\Omega \varphi(\nabla u(x))\, dx : u \in \mathcal{A} \right\},$$

$$\tilde{m} = \inf\left\{ \int_\Omega Q\varphi(\nabla u(x))\, dx : u \in \mathcal{A} \right\},$$

$$\overline{m} = \inf\left\{ \int_\Omega \int_{\mathbf{M}} \varphi(A)\, d\nu_x(A)\, dx : \{\nu_x\} \in \overline{\mathcal{A}}, \right\},$$

Then

1. $m = \tilde{m} = \overline{m}$*;*
2. $\tilde{m}$ *and* $\overline{m}$ *are attained.*

There is a precise way of going from minimizers for $\tilde{m}$ to minimizers for $\overline{m}$, and viceversa, but it is not relevant to our discussion here.

When the integrand φ is continuous and only satisfies a lower bound

$$c\,(|A|^p - 1) \le \varphi(A), \quad 0 < c, 1 < p,$$

then all we can ensure is

$$m \ge \tilde{m} = \overline{m},$$

and a gap between m and $\tilde{m}$ may exist. When the integrand verifies a lower bound as above, but it is not continuous (in particular if it takes the value $+\infty$ abruptly) then

$$m \ge \tilde{m} \ge \overline{m},$$

and there might exist a gap between $\tilde{m}$ and $\overline{m}$.

Since the integrand $\tilde{F}$ coming from the reformulation of our optimal design problem lies in this last situation, we may in principle have that relaxation in terms of gradient Young measures could lead to a smaller value of the infimum. More importantly, a gradient Young measure minimizer will have its support contained in the set where $\tilde{F}$ is finite, and yet it may be impossible to generate this Young measure with a sequence of gradients taking on their values entirely on the finite set for the integrand $\tilde{F}$. What we know is that those gradients will be close (in measure) to the appropriate set. In particular, even though the set where the sequence of gradients does not lie in the finite set for $\tilde{F}$ were negligible as we proceed through the sequence, each one of those would not be admissible from the point of view of the initial optimal design problem. All of these technical issues must be addressed in each particular situation. The final answer and the techniques to deal with such difficulties may strongly depend on the particular problem. The case of conductivity has been treated in [6].

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