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On the Equal Hull Problem for Nontrivial Semiconvex Hulls

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We define a nontrivial semiconvex hull $qr_\alpha(K)$ of a compact set $K \subset M^{N\times n}$ called the α-rank-one convex quadratic hull and establish the equalities of semiconvex hulls with respect to $qr_\alpha(K)$ by showing that $L_c(K) = qr_\alpha(K)$ if and only if $Q(K) = qr_\alpha(K)$, $0 < \alpha < 1$, where $Q(K)$ and $L_c(K)$ are the quasiconvex convex hull and the closed lamination convex hull of K respectively. We also show that $qr_\alpha(K)$ is a nontrivial semiconvex hull, that is, $qr_\alpha(K) \neq C(K)$ if $R(K) \neq C(K)$.

Keywords: Semiconvex hulls, equal hull properties, nontrivial semiconvex hull, rank-one convex quadratic functions

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Let $M^{N\times n}$ the linear space of real $N \times n$ matrices with the standard Euclidean norm of $\mathbb{R}^{Nn}$, and assume that N, $n \geq 2$. In the vectorial calculus of variations, we may define, for a compact set $K \subset M^{N\times n}$ the corresponding semiconvex hulls by using cosets. Let $R(K)$, $Q(K)$, $P(K)$, $qr(K)$ and $C(K)$ be the rank-one convex, the quasiconvex, the polyconvex, the quadratic rank-one convex and the convex hulls of K respectively. We may also define the so called closed lamination convex hull $L_c(K)$ of K (see below for definitions). We have

$$K \subset L_c(K) \subset R(K) \subset Q(K) \subset qr(K)\&P(K) \subset C(K). \tag{1}$$

A surprising connection among these semiconvex hulls is the following equalities of semiconvex hulls with respect to the trivial hull $C(K)$:

(a) $Q(K) = C(K) \iff R(K) = C(K)$ [15],
(b) when $n = M = 2$, $P(K) = C(K) \iff R(K) = C(K)$ [15] and in fact this result holds if and only if $\min\{N, n\} = 2$ [4].
(c) $qr(K) = C(K) \iff R(K) = C(K)$ [18] and $qr(K) = P(K)$ if and only if $\min\{N, n\} = 2$ [12, 4].

Item (c) above unifies results in (a) and (b) and the proof is more elementary. We may view the results a) - c) as equal hull properties with respect to the trivial hull $C(K)$.

Naturally, one would ask whether the equal hull properties holds with respect to other semiconvex hulls such as $qr(K)$ or $P(K)$, that is, whether for a compact set $K \subset M^{N\times n}$, $L_c(K) = qr(K)$ if and only if $Q(K) = qr(K)$, or $L_c(K) = P(K)$ if and only if $Q(K) = P(K)$ when $\min\{N, n\} = 2$.

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It can be shown by using an example due to Šverák [9] that the answers to both questions are negative at least for some compact sets K in $M^{6\times 2}$ (see Example 4 below). The question then is whether there are nontrivial semiconvex hulls of K such that the equal hull property holds. In this paper we define a family of semiconvex hulls called α-rank-one convex quadratic hull $qr_\alpha(K)$ lying between the convex hull $C(K)$ and the quadratic rank-one convex hull $q(K)$ by using a class of rank-one convex quadratic functions which are either convex or strictly rank-one convex. We are then able to establish the equal hull property with respect to $qr_\alpha(K)$. We have

Theorem 1. *Let $K \subset M^{N\times n}$ be compact, then for $0 < \alpha < 1$,*

(i) $L_c(K) = C(K)$ *if and only if* $qr_\alpha(K) = C(K)$.

(ii) $L_c(K) = qr_\alpha(K)$ *if and only if* $Q(K) = qr_\alpha(K)$.

Remark 2. Since we have

$$L_c(K) \subset R(K) \subset Q(K) \subset qr(K) \subset qr_\alpha(K) \subset C(K), \tag{2}$$

and (i) implies that $qr_\alpha(K)$ is a nontrivial semiconvex hull, we see that the equal hull property holds for a family of nontrivial semiconvex hulls.

Before we prove our main results, let us first introduce some notation and definitions.

Let $f : M^{N\times n} \to \mathbb{R}$ be a continuous function. The following are some conditions related to weak lower semicontinuity of the integral (c.f. [2, 7, 5, 1])

$$I(u) = \int_\Omega f(Du(x))dx$$

(i) f is rank-one convex if for each matrix $A \in M^{N\times n}$ and each rank-one matrix $B = a \otimes b \in M^{N\times n}$, the function $t \to f(A + tB)$ is convex.

(ii) f is quasiconvex at $A \in M^{N\times n}$ on Ω, if for any smooth function $\phi : \Omega \to \mathbb{R}^N$ compactly supported in Ω,

$$\int_\Omega f(A + D\phi(x))dx \geq \int_\Omega f(A)dx$$

holds. f is quasiconvex if it is quasiconvex at every $A \in M^{N\times n}$. The class of quasiconvex functions is independent of the choice of Ω.

(iii) f is polyconvex if $f(A) =$ convex function of minors of the matrix A.

(iv) f is a rank-one convex quadratic function if $f(A) = q(A) + l(A)$ with $q(\cdot)$ a rank-one convex quadratic form and $l(\cdot)$ an affine function.

It is well-known that (iii)$\Rightarrow$(ii)$\Rightarrow$(i), while (i)$\not\Rightarrow$(ii)$\not\Rightarrow$(iii) (cf. [2, 7, 5, 13]). However, if f is a quadratic function, (i) is equivalent to (ii).

Let $E \subset M^{N\times n}$ be a linear subspace without rank-one matrices, and $E^\perp$ being its orthogonal complement. Let

$$q_E(A) = |P_{E^\perp}(A)|^2 - \lambda_E |P_E(A)|^2, \tag{3}$$

where $P_{E^\perp}$ and P_E are orthogonal projections to $E^\perp$ and E respectively, where $\lambda_E > 0$ is defined by

$$\frac{1}{\lambda_E} = \sup\left\{\frac{|P_E(a\otimes b)|^2}{|P_{E^\perp}(a\otimes b)|^2}, \qquad a\in\mathbb{R}^N,\, b\in\mathbb{R}^n\, |a| = |b| = 1\right\} < \infty. \tag{4}$$

Then q_E is a rank-one convex quadratic form [3]. For convenience, we also define $\lambda_E = 0$ if E has rank-one matrices, and

$$q_{\mu,E} = |P_{E^\perp}(A)|^2 - \mu\lambda_E|P_E(A)|^2$$

for some $0\le\mu\le 1$. Clearly, $q_{\mu,E}$ is a rank-one convex quadratic form. Let $\mathcal{E}$ be the set of all non-zero linear subspaces of $M^{N\times n}$ and define for a fixed $0\le\alpha<1$,

$$QR_\alpha = \{\sigma q_{\mu,E} + l,\ \sigma\ge 0,\, 0\le\mu\le\alpha,\, E\in\mathcal{E},\ l \text{ affine}\}.$$

Although the definition of the family QR_α of rank-one convex quadratic functions looks complicated, in fact it is among the simplest collection of strictly (when $0<\alpha<1$) rank one convex quadratic functions that separate points.

Definition 3. We define the α-quadratic rank-one convex envelope $qr_\alpha(f)$ of $f : M^{N\times n}\to\mathbb{R}$ as

$$qr_\alpha(f)(A) = \sup\{q(A),\, q\le f,\, q\in QR_\alpha\}, \tag{5}$$

In the study of material microstructure, the following concepts of semiconvex hulls for a compact set $K\subset M^{N\times n}$ are naturally introduced.

A compact set $K\subset M^{N\times n}$ is called stable under lamination (or lamination convex) [8] if A, $B\in K$ are rank-one connected, that is, $\operatorname{rank}(A-B)=1$, then for all $\lambda\in(0,1)$, one has $(1-\lambda)A+\lambda B\in K$. For a given $K\subset M^{N\times n}$, the *lamination convex hull* $L(K)$ is defined as the smallest lamination convex set that contains K [8]. We also define the *closed lamination convex hull* $L_c(K)$ as the smallest closed lamination convex set that contains K [15].

By using the coset definition, we may define semiconvex hulls $S(K)$ of K as follows,

$$S(K) = \{X\in M^{N\times n},\ f(X)\le\sup_{Y\in K} f(Y), \text{ for every S-convex } f : M^{N\times n}\to\mathbb{R}\}.$$

If we replace S-convex by rank-one convex, quasiconvex, polyconvex, quadratic rank-one convex functions respectively, we obtain the rank-one convex hull $R(K)$, the quasiconvex hull $Q(K)$, the polyconvex hull $P(K)$ [14], the quadratic rank-one convex hull $qr(K)$ [18] and the convex hull $C(K)$. Clearly, if K is closed, (1) holds. If $L_c(K)$ is convex, obviously, all other 'semiconvex' hulls are identical to $C(K)$.

Later we will use some facts from the theory of gradient Young measures and homogeneous (gradient) Young measures supported on compact sets in $M^{N\times n}$ [10, 6]: (i) If $X_0\in Q(K)$, there is a homogeneous gradient Young measure ν supported in K such that the integral average $\bar\nu := \int_K \lambda d\nu = X_0$ (also see [16]). (ii) For a rank-one convex quadratic form q satisfying $q(a\otimes b)\ge c|a|^2|b|^2$ for $a\in\mathbb{R}^N$, $b\in\mathbb{R}^n$, with $c>0$ a constant, one has for the Young measure above, $\int_K q(\lambda)d\nu \ge q(X_0) + c\int_K|\lambda - X_0|^2 d\nu$ (also see [17]).

Before we introduce the α-quadratic rank-one convex hull $qr_\alpha(K)$, let us examine the following counter-example of the equal hull property with respect to $qr(K)$ and $P(K)$ due to Šverák [9].

Example 4. Let $u : \mathbb{R}^2 \to \mathbb{R}^6$ be defined as

$$u(x_1, x_2) = (\cos x_1,\, \sin x_1,\, \cos x_2,\, \sin x_2,\, \cos(x_1 + x_2),\, \sin(x_1 + x_2)),$$

and notice that u is a periodic smooth mapping. We define K as the image of the gradient of u,

$$K = \{Du(x) \in M^{6\times 2},\, x \in \mathbb{R}^2\}.$$

It was shown by Šverák that $L_c(K) = R(K) = K$ while $P(K) = Q(K) = K \cup \{0\}$. We have $P(K) = qr(K) \neq L_c(K)$ while $qr(K) = Q(K)$. Thus the equal hull property does not hold in $M^{6\times 2}$.

Definition 5. For a compact set $K \subset M^{N\times n}$, the α-quadratic rank-one convex hull $qr_\alpha(K)$ is defined by

$$qr_\alpha(K) = \{A \in M^{N\times n},\, q(A) \leq \sup_{B\in K} q(B),\, q \in QR_\alpha\}. \tag{6}$$

If $qr_\alpha(K) = K$, we call K an α-quadratic rank-one convex set. Clearly, $qr_\alpha(K)$ satisfies (2).

Remark 6. Our choice of QR_α of quadratic functions is because it gives us several advantages.

(i) QR_α separates points for any plane $E \subset M^{N\times n}$ without rank-one connection in the sense that for every $X \in E$, there is some $q \in QR_\alpha$ such that $q(Y) < 0$ for $Y \in E$, $Y \neq X$ and $q(X) = 0$. We may construct such a quadratic function as follows. Let E_0 be the subspace parallel to E given by $E_0 = \{Y - X, Y \in E\}$, then we define

$$q(A) = |P_{E_0^\perp}(A - X)|^2 - \alpha\lambda_{E_0}|P_{E_0}(A - X)|^2,$$

then $q \in QR_\alpha$ and for every $Y \in E$, $Y \neq X$, $q(Y) = -\alpha\lambda_{E_0}|P_{E_0}(Y - X)|^2 < 0 = q(X)$.

(ii) Every 'bounded' set of QR_α is sequentially compact. To be more precisely, let

$$q_k(A) = \sigma_k\left(|P_{E_k^\perp}(A)|^2 - \mu_k\lambda_{E_k}|P_{E_k}(A)|^2\right) + B_k \cdot A + c_k$$

be a sequence in QR_α such that σ_k, B_k and c_k are bounded. Then there is a subsequence q_{k_j} and some $q \in QR_\alpha$ such that for each fixed $A \in M^{N\times n}$, $q_{k_j}(A) \to q(A)$.

This can be easily proved because $\dim(E_k)$ is a sequence of bounded integers. Therefore, there is a subsequence such that $\dim(E_k) = m$. For each k we take an orthonormal basis $\{E_k^{(1)}, \ldots, E_k^{(m)}\} \subset E_k$ and that of $E_k^\perp$. Then up to a subsequence, the two basis of E_k and $E_k^\perp$ converge to basis of E and $E^\perp$ respectively (this is easy to verify). Note also that λ_{E_k} is bounded and equals zero if E_k has a rank-one matrix. We may also assume that $\sigma_k \to \sigma \geq 0$, $\mu_k \to \mu \leq \alpha$, $B_k \to B$, $c_k \to c$, and $\lambda_{E_k} \to \lambda_E$ up to a subsequence. We will check the last assertion later. Now, let

$$q(A) = \sigma\left(|P_{E^\perp}(A)|^2 - \mu\lambda_E|P_E(A)|^2\right) + B \cdot A + c,$$

then clearly $q_k(A) \to q(A)$ for each fixed $A \in M^{N\times n}$ and $q \in QR_\alpha$.

To show that $\lambda_{E_k} \to \lambda_E$, we have, by definition, $|P_{E_k^\perp}(a\otimes b)|^2 \geq \lambda_{E_k}|P_{E_k}(a\otimes b)|^2$, and for some $a_k \in \mathbb{R}^n$, $b_k \in \mathbb{R}^N$ with $|a_k| = |b_k| = 1$, $|P_{E_k^\perp}(a_k\otimes b_k)|^2 = \lambda_{E_k}|P_{E_k}(a_k\otimes b_k)|^2$. Passing to the limit $k \to \infty$, one obtains, $|P_{E^\perp}(a\otimes b)|^2 \geq \lambda|P_E(a\otimes b)|^2$, so that $\lambda \leq \lambda_E$. On the other hand, up to a subsequence, $a_k \to a_0$, $b_k \to b_0$ as $k \to \infty$, we also have $|P_{E^\perp}(a_0\otimes b_0)|^2 = \lambda|P_E(a_0\otimes b_0)|^2$. Thus $\lambda = \lambda_E$.

(iii) The semiconvex hull $qr_\alpha(K)$ can be represented as the intersection of the convex hull $C(K)$ and sub-level sets of a family of simple rank-one convex quadratic functions.

From the definition of $qr_\alpha(K)$, for any $X_0 \in qr_\alpha(K)$, $q(X_0) \leq \max\{q(X),\, X \in K\} := c_q$, for $q \in QR_\alpha$, if we let $q_c = q - c_q$, then $q_c \in QR_\alpha$ and $q_c(X_0) \leq 0$. Let $K_{q_c} = \{X \in M^{N\times n},\ q_c(X) \leq 0\}$, then

$$qr_\alpha(K) = \cap_{q\in QR_\alpha} K_{q_c}.$$

Since every $q \in QR_\alpha$ can be written as

$$q(X) = \sigma\left(|P_{E^\perp}(X)|^2 - \mu\lambda_E|P_E(X)|^2\right) + B\cdot A + c,$$

where $\sigma \geq 0$. If $\sigma = 0$, q is affine hence q_c is affine. If $\sigma > 0$, we may divide q_c by σ and consider quadratic functions in the form

$$q^{(*)}(X) = \left(|P_{E^\perp}(X)|^2 - \mu\lambda_E|P_E(X)|^2\right) + B^{(*)}\cdot X + c^{(*)}, \tag{7}$$

under the assumption that E does not have rank-one matrices and $0 < \mu \leq \alpha$ Clearly $q_c(X) \leq 0$ if and only of $q^{(*)}(X) \leq 0$. If $\mu = 0$ or E has rank-one matrices, $q^{(*)}$ is then convex. However, recall [11] that convex functions can be represented by 'sup' of affine functions, we define a subset $QR_\alpha(K)$ of QR_α by

$$\begin{aligned}&QR_\alpha(K) = \\ &\{q \in QR_\alpha,\ \ q \text{ satisfies (7)},\ \mu\lambda_E > 0,\ q(X) \leq 0,\ X \in K,\ \exists Y_q \in K,\ q(Y_q) = 0\}.\end{aligned} \tag{8}$$

It is easy to verify that

$$qr_\alpha(K) = C(K) \cap \left(\cap_{q\in QR_\alpha(K)} K_q\right), \quad \text{ where } \ K_q = \{X \in M^{N\times n},\ q(X) \leq 0\}.$$

We only prove Theorem 1 (ii) here and leave the proof for (i) at the end of this paper.

Proof of Theorem 1 (ii). We may assume that $K = L_c(K)$. In other words, K is a closed laminated convex set. If $qr_\alpha(K) \neq K$, we show that $qr_\alpha(K) \neq Q(K)$. Let m be the smallest affine dimension of $C(K)$ such that our statement fails. It is easy to show that this dimension m is greater than 1.

In fact, if the affine dimension of $C(K)$ is 1, K is contained in a straight line. If the line contains rank-one connections, then $L_c(K) = C(K)$ and our claim is true. If the line does not have rank-one connections, then it separates points as shown in Remark 6.(i). Thus in this case $qr_\alpha(K) = K = L_c(K)$. We examine two cases.

Case (I) There is some $X_0 \in \partial C(K) \cap qr_\alpha(K)$ while $X_0 \notin K$, where $\partial C(K)$ is the relative boundary of $C(K)$ in the plane containing $C(K)$ with dimension m.

Case (II) $qr_\alpha(K) \cap \partial C(K) \subset K$.

We need the following Lemma for Case (I).

Lemma 7. *Let E be a proper supporting plane [11] of $C(K)$ then $qr_\alpha(K) \cap E = qr_\alpha(K \cap E)$.*

We prove Lemma 7 after we finish the proof of Theorem 1.

Proof of Theorem 1, (ii), Case (I). Accepting Lemma 7 for the moment. We seek to prove that $Q(K) \neq qr_\alpha(K)$. Lemma 7 implies that there is a supporting plane E of $C(K)$ passing through X_0, hence $qr_\alpha(K) \cap E = qr(K \cap E)$. On the other hand, we have $Q(K \cap E) = Q(K) \cap E$ and $K \cap E$ is still a closed lamination convex set. Since $\dim(E \cap C(K)) < m$, we see that $Q(K \cap E) \neq qr_\alpha(K \cap E)$. Thus $Q(K) \cap E \neq qr_\alpha(K) \cap E$ so that $Q(K) \neq qr_\alpha(K)$. Therefore the proof of Case (I) is finished pending the proof of Lemma 7.

Proof of Theorem 1, (ii), Case (II). Without loss of generality, we may assume that $0 \in K$. Since

$$qr_\alpha(K) = C(K) \cap \left(\cap_{q \in QR_\alpha(K)} K_q\right) \subset \operatorname{span}[C(K)] := E_0,$$

we see that E_0 must have rank-one matrices. Otherwise $qr_\alpha(K) = Q(K) = L_c(K) = K$.

We further consider two different subcases:

Case (IIa) The relative boundary of $qr_\alpha(K)$ in E_0 is contained in K and there is a relative interior point X_0 of $qr_\alpha(K)$ such that $X_0 \notin Q(K)$.

Case (IIb) There is a relative boundary point X_0 of $qr_\alpha(K)$ such that $X_0 \notin Q(K)$.

Proof of Case (IIa). Let $A_0 \in E_0$ be a rank-one matrix, and let us consider the line $X_0 + tA_0$. It is easy to see that since $qr_\alpha(K)$ is compact and X_0 is a relative interior point, there are $t_1 > 0$, $t_2 < 0$ such that $A_1 = X_0 + t_1 A_0$, $A_2 = X_0 + t_2 A_0$ are both on the boundary $\partial qr_\alpha(K) \subset K$. Thus $X_0 \in L_c(K)$, which is a contradiction. □

Proof of Case (IIb). Since X_0 is a boundary point of $qr_\alpha(K) = C(K) \cap (\cap_{q \in QR_\alpha(K)} K_q)$ while $X_0 \notin \partial C(K)$, we see that there is a sequence (q_k) in $QR_\alpha(K)$ such that $q_k(X) \leq 0$, $-1/k \leq q_k(X_0) \leq 0$, $k = 1, 2, \dots$.

Clearly, we may write q_k as $q_k(X) = q_k^{(0)}(X) + B_k \cdot (X - X_0) + c_k$, where

$$q_k^{(0)}(X) = |P_{E_k^\perp}(X)|^2 - \mu_k \lambda_{E_k} |P_{E_k}(X)|^2, \qquad X \in M^{N \times n}.$$

We see from $-1/k \leq q_k(X_0) \leq 0$ that

$$-\frac{1}{k} \leq q_k^{(0)}(X_0) + c_k \leq 0.$$

Since $q_k^{(0)}(X_0)$ is bounded, c_k is then bounded. We also claim that $|B_k|$ is bounded. Otherwise, up to a subsequence, $|B_k| \to \infty$. Hence up to a subsequence $B_k/|B_k| \to B_0$ for some $B_0 \in M^{N\times n}$, $|B_0| = 1$. Now we dividing q_k by $|B_k|$ and pass to the limit $k \to \infty$ for each fixed X, we see that $q_k(X)/|B_k| \to l(X) = B_0 \cdot (X - X_0)$. We also have $l(X) \le 0$ for $X \in K$ and $l(X_0) = 0$. Thus $X_0 \in \partial C(X)$, which is a contradiction.

Now, since B_k and c_k are both bounded, we may pass to the limit $k \to \infty$ (up to a subsequence) so that $q_k(X) \to q(X)$ where

$$q(X) = |P_{E^\perp}(X)|^2 - \mu\lambda_E|P_E(X)|^2 + B\cdot(X - X_0) + c,$$

and we let

$$q^{(0)}(X) = |P_{E^\perp}(X)|^2 - \mu\lambda_E|P_E(X)|^2.$$

If E has rank-one matrices, hence $\lambda_E = 0$ or $\mu = 0$, q is then convex which again implies $X_0 \in \partial C(K)$ and leads to a contradiction. Thus we may claim that $0 < \mu \le \alpha$ and $\lambda_E > 0$.

Since $X_0 \in Q(K)$, there is a homogeneous gradient Young measure ν supported in K such that $\int_K \lambda d\nu = X_0$. Due to the fact that q is quasiconvex, we have

$$\int_K q(\lambda)d\nu \ge q(X_0).$$

On the other hand, $q(X) \le 0$ for $X \in K$, we have $\int_K q(\lambda)d\nu = q(X_0)$. Thus ν is supported on the level set $\{X \in K,\, q(X) = 0\}$. Since $\sigma \le \alpha$, we see that the quadratic form $q^{(0)}$ above satisfies

$$q^{(0)}(a\otimes b) \ge (1-\alpha)\lambda_E|P_{E^\perp}(a\otimes b)|^2 \ge (1-\alpha)\lambda_E\frac{\lambda_E}{1+\lambda_E}|a|^2|b|^2,$$

so that for the homogeneous Young measure ν above, we have

$$0 = \int_K (q(\lambda) - q(X_0))d\nu \ge (1-\alpha)\lambda_E\frac{\lambda_E}{1+\lambda_E}\int_K |\lambda - X_0|^2 d\nu \ge 0.$$

Thus $\nu = \delta_{X_0}$ is a Dirac mass, hence $X_0 \in K$, a contradiction. The proof for Case (IIb) is complete. □

Proof of Lemma 7. Let E_1 be the plane in $M^{N\times n}$ containing $C(K)$ with the same dimension as $C(K)$ [11]. Obviously, $qr_\alpha(K\cap E) \subset qr_\alpha(K)\cap E$. Let $X \in qr_\alpha(K)\cap E$. There is an affine function l defined on $M^{N\times n}$ such that $l < 0$ on the open half plane in E_1 containing $C(K)\setminus E$, $l = 0$ on E and $l > 0$ on the opposite half plane to $C(K)$ in E_1. We also define

$$E(\epsilon) = \{A \in E_1,\, \mathrm{dist}(A, E) \le \epsilon,\, l(A) \le 0\}$$

which is a set on the same side as $C(K)$ in E_1, where $\mathrm{dist}(A, E_1)$ is the Euclidean distance from A to E_1. For any fixed $q \in QR_\alpha$ we consider, for every integer $n > 0$ the quadratic function $q(\cdot) + nl(\cdot) \in QR_\alpha$. Since for any $A \in E$, $l(A) = 0$, we have, for each fixed $X \in qr_\alpha(K)\cap E$,

$$q(X) = q(X) + nl(X) \le \sup_{A\in K}[q(A) + nl(A)].$$

Since $q + nl$ is continuous and K compact, the maximum is attained at some $A_n \in K$, that is, $\sup_{A\in K}[q(A) + nl(A)] = q(A_n) + nl(A_n)$, so that $q(X) \le q(A_n) + nl(A_n)$. Since K is compact there is a subsequence $A_{n_k} \to A_0 \in K$ as $k \to \infty$. Notice that $l(A_n) \le 0$ for all n. If we let $k \to \infty$ we see that $\delta_k := \mathrm{dist}(A_{n_k}, E) \to 0$. Otherwise $q(X)$ cannot be finite. Now we have

$$q(X) \le q(A_{n_k}) + n_k l(A_{n_k}) \le \sup\{q(A),\, A \in K \cap E(\delta_k)\}. \tag{9}$$

Again the 'sup' in (9) can be reached by, say $B_k \in K \cap E(\delta_k)$, and up to a subsequence, we have $B_k \to B_0 \in K \cap E$ as $k \to \infty$.

Passing to the limit $k \to \infty$ on both side of the inequality $q(X) \le q(B_k)$ and noticing that $B_0 \in K \cap E$, we have $q(X) \le q(B_0) \le \sup_{A\in K\cap E} q(A)$, hence $X \in qr_\alpha(K \cap E)$, Lemma 7 is then proved by noticing also that $C(K) \cap E = C(K \cap E)$. □

Remark 8. Our argument breaks down if we allow all rank-one convex quadratic functions to be considered. The reason is that non-convex quadratic functions are no longer strictly quasiconvex. The problem is reduced to the study of homogeneous Young measures ν supported on a compact set $K \subset \{X \in M^{N\times n},\, q(X) = 0\}$ for some quadratic rank-one convex function q with $q(0) = 0$ under the conditions that $0 \notin L_c(K)$. Example 4 shows that it is possible that the average of Young measure $\bar\nu = 0$ hence the equal hull property fails.

However, it is interesting to find examples of non-convex rank-one convex quadratic functions and compact sets K contained in the level set $q = 0$ such that $qr(K) \ne R(K)$ while $qr(K) = Q(K)$.

Proof of Theorem 1 (i). For any nontrivial supporting plane of $C(K)$, we have, from Lemma 7 that $qr_\alpha(K) \cap E = qr(K \cap E)$. Notice also that $C(K) \cap E_1 = C(K \cap E_1)$.

Suppose $L_c(K) \ne C(K)$, while $qr_\alpha(K) = C(K)$. We may assume that K is a closed laminated convex set. Then among all these K's there is one for which the affine dimension $\dim C(K) \ge 1$ of $C(K)$ is the smallest. For such K we claim that the plane E spanned by $C(K)$ does not have rank-one connections. Otherwise it is easy to see [15] that there is a supporting plane E of $C(K)$ such that $E \cap K$ is not convex and is still a closed laminated convex set while

$$qr_\alpha(K \cap E) = qr_\alpha(K) \cap E = C(K) \cap E$$

is convex. This contradicts to the fact that the dimension $\dim C(K)$ is the smallest. Now since $C(K) \subset E$ and E does not have rank-one connection, there is some $X \in C(K) \ne K$. If we define q_X as in Remark 6(i), there is some $\delta > 0$, such that $q_X(X) = 0 > -\delta = \sup_{A\in K\subset E} q_X(A)$. Hence $X \notin qr_\alpha(K)$ and $qr_\alpha(K) \ne C(K)$, a contradiction. □

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