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Degenerate Perturbations of a Two-Phase Transition Model

Roberto Monti*
Dipartimento di Matematica, Università di Trento,
Via Sommarive 14, 38050, Povo (Trento), Italy
rmonti@science.unitn.it

Francesco Serra Cassano*
Dipartimento di Matematica, Università di Trento,
Via Sommarive 14, 38050, Povo (Trento), Italy
cassano@science.unitn.it

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We study the $\Gamma-$convergence as $\varepsilon \to 0^+$ of the family of degenerate functionals

$$Q_\varepsilon(u) = \varepsilon \int_\Omega \langle ADu, Du\rangle\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx$$

where $A(x)$ is a symmetric, *non negative* $n\times n$ matrix on Ω (i.e. $\langle A(x)\xi,\xi\rangle \geq 0$ for all $x\in\Omega$ and $\xi\in\mathbb{R}^n$) with regular entries and $W:\mathbb{R}\to[0,+\infty)$ is a double well potential having two isolated minimum points. Moreover, under suitable assumptions on the matrix A, we obtain a minimal interface criterion for the $\Gamma-$limit functional exploiting some tools of Analysis in Carnot-Carathéodory spaces. We extend some previous results obtained for the non degenerate perturbations Q_ε in the classical gradient theory of phase transitions.

Keywords: Phase transitions, Γ-convergence, Carnot-Carathéodory spaces, minimal interface criterion

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1. Introduction

In this paper we study the variational convergence for a family of anisotropic degenerate perturbations of a non convex functional which arises in the theory of two-phase transitions. Let us consider the family of functionals

$$Q_\varepsilon(u) = \varepsilon \int_\Omega q(x, Du)\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx, \quad \varepsilon > 0, \tag{1}$$

where Ω is a smooth, bounded open set of $\mathbb{R}^n$, $u:\Omega\to\mathbb{R}$, and $W:\mathbb{R}\to[0,+\infty)$ is a double-well potential that supports two phases of the model (i.e. W has two isolated global minimum points). For the sake of simplicity we assume here $W(u) = u^2(1-u)^2$ but W can be more general (see Section 3). The integral perturbation with integrand

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function $q : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a term that penalizes the formation of interfaces in the model and it may degenerate in the sense that q could vanish on big parts of $\Omega \times \mathbb{R}^n$.

Functionals of type (1) have arisen in a variety of applications as, for instance, in the study of stable configurations in the context of Van der Waals-Cahn-Hilliard theory of phase transitions (see [16], [33]). This model can be described by a fluid under isothermal conditions which is confined in a bounded container Ω and whose Gibbs free energy per unit volume is a prescribed non convex function W of the density function u. The space of admissible smooth densities is the class

$$\mathcal{A} = \Big\{ u : \Omega \to [0,1] : u \in C^1(\Omega),\ \int_\Omega u\,dx = V \Big\},$$

where $0 < V < |\Omega|$ is the given total mass of the fluid in Ω.

In the classic isotropic model to every density u one can associate the energy

$$\mathcal{E}_\varepsilon(u) = \varepsilon Q_\varepsilon(u)$$

where

$$q(x,\xi) = |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n, \tag{2}$$

and $\varepsilon > 0$ is a small parameter (see [33] for a physical motivation and also [1] for a simple nice introduction to the subject). The problem of determining the stable configurations is the study of the variational problem

$$\inf\{\mathcal{E}_\varepsilon(u) :\ u \in \mathcal{A}\},$$

and the mathematical problem is then to study the asymptotic behavior as $\varepsilon \downarrow 0$ of the solutions u_ε of these problems or equivalently, as the sets of the solutions agree, the ones of the rescaled problems

$$\inf\{Q_\varepsilon(u) :\ u \in \mathcal{A}\}. \tag{P_ε}$$

A relevant variational convergence which turned out to be very useful to this goal is the $\Gamma-$convergence introduced by De Giorgi (see [20] and [19] for an introduction to this topic). More precisely, the functional $Q_\varepsilon : \mathcal{A} \to [0, +\infty]$ can be extended, with a slight abuse of notation, to a functional $Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ defined $+\infty$ outside $\mathcal{A}$, and now the variational problem is the existence and characterization of $Q = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon\downarrow 0} Q_\varepsilon$.

In the isotropic scalar case, i.e. when q is as in (2), the existence and characterization of the Γ-limit functional was first conjectured by De Giorgi and Franzoni ([20]). Then, this variational problem was studied in some particular situations by Gurtin ([33]), who also proposed several conjectures (see also [34]). Following a Gurtin's conjecture and using previous $\Gamma-$convergence arguments contained in [42], Modica ([41]) proved that

$$Q(u) = \begin{cases} 2\alpha|\partial E|(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}(\Omega),\ |E \cap \Omega| = V \\ +\infty & \text{otherwise,} \end{cases} \tag{3}$$

where $|\partial E|(\Omega)$ is the *perimeter* of E in Ω, $\mathrm{BV}(\Omega)$ is the set of functions with bounded variation in Ω (see [6]) and

$$\alpha = \int_0^1 \sqrt{W(s)}\,ds, \tag{4}$$

(see also [49]). Let us recall that by a well-known De Giorgi's result

$$|\partial E|(\Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$$

where $\partial^* E \subset \partial E$ is the *reduced boundary* of E and $\mathcal{H}^{n-1}$ is the $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$ (see [6]).

Moreover, in [41] Modica also proved the existence of a sequence $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ of solutions of the relaxed problems (P_{ε_h}) strongly converging in $\mathrm{L}^1(\Omega)$ as $\varepsilon_h \downarrow 0$ to a function $u_0 = \chi_E$ solution of the geometric problem

$$\inf\{2\alpha\mathcal{H}^{n-1}(\partial^* E \cap \Omega) : \chi_E \in \mathrm{BV}(\Omega), |E \cap \Omega| = V\}. \tag{5}$$

In particular, this result yields a "selection criterion" singling out a solution u_0 among the infinite collection of the ones of the imperturbated classical physical problem

$$\min\left\{\int_\Omega W(u)\,dx : u \in \mathrm{L}^1(\Omega), \int_\Omega u\,dx = V\right\} \tag{6}$$

(see [33] for a discussion of the physical meaning of this problem).

These results were generalized by Bouchitté ([14]) and Owen-Sternberg ([47]) to anisotropic functionals Q_ε allowing the function q to be very general but always assuming at least a coercivity property which, in the case when q is a positive quadratic form, i.e.

$$q(x,\xi) = \langle A(x)\xi, \xi\rangle \quad x \in \Omega \;\text{ and }\; \xi \in \mathbb{R}^n, \tag{7}$$

with $A(x)$ symmetric $n \times n$ matrix, amounts to the existence of a constant $\lambda_0 > 0$ such that

$$\langle A(x)\xi, \xi\rangle \geq \lambda_0|\xi|^2 \quad \text{for all } x \in \Omega \;\text{ and }\; \xi \in \mathbb{R}^n. \tag{8}$$

Under this hypothesis Bouchitté proved in [14] that there exists a limit solution $u_0 = \chi_E$ which solves the following geometric problem

$$\inf\left\{2\alpha\int_{\Omega\cap\partial^* E}\langle A(x)\nu_E(x), \nu_E(x)\rangle^{1/2} d\mathcal{H}^{n-1} : \chi_E \in \mathrm{BV}(\Omega), |E\cap\Omega| = V\right\} \tag{9}$$

where ν_E denotes the *generalized outward normal* to E (see [6]) and α is the constant (4).

The isotropic vector valued-case, i.e. if $u : \Omega \to \mathbb{R}^p$ and $q : \Omega \times \mathbb{R}^{pn} \to [0, +\infty)$ is as in (2), was studied by Sternberg ([49]), by Kohn and Sternberg ([38]), by Baldo [9] and by Fonseca and Tartar ([22]). The anisotropic vector-valued case was also studied by Barroso and Fonseca ([10]) and recently by Ambrosio, Colli Franzone and Savaré when a degeneration in the potential W is admitted too ([5]). Moreover, other variations of the functionals Q_ε in (1) have been studied by Alberti and Bellettini ([2] and [3]), Alberti, Bouchitté and Seppecher ([4]) and Fonseca and Mantegazza ([21]). Finally, Baldi and Franchi ([8]) informed us of a $\Gamma-$convergence result for the family of functionals $(Q_\varepsilon)_\varepsilon$ in the special case when $q(x,\xi) = |\xi|^2\omega(x)^{1-2/n}$ and ω is a strong$-A_\infty$ weight on $\mathbb{R}^n$.

In this paper we obtain $\Gamma-$convergence results in the case when $q : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is a non negative quadratic form, i.e. q is as in (7) but the matrix $A(x)$ is only non negative definite on Ω; in particular (8) may fail. More precisely, suppose that there exists a $m \times n$ matrix $C(x) = [c_{ji}(x)]$ with Lipschitz continuous entries on $\mathbb{R}^n$ such that

$$A(x) = C(x)^T C(x) \quad \text{for all } x \in \Omega, \tag{10}$$

where $C(x)^T$ denotes the transposed matrix of $C(x)$, define the $A-$*variation* in Ω of a function $f \in \mathrm{L}^1(\Omega)$ as

$$|Df|_A(\Omega) = \sup\Big\{ \int_\Omega f \operatorname{div}(C^T\psi)\, dx : \psi = (\psi_1, ..., \psi_m) \text{ is such that } C^T\psi \in \mathrm{Lip}_0(\Omega;\mathbb{R}^n),\ |\psi| \le 1 \Big\}.$$

Note that $|Df|_A(\Omega)$ does not depend on the particular factorization (10) (see (27), Proposition 2.1 and Remark 2.3). Finally define

$$\mathrm{BV}_A(\Omega) = \{f \in \mathrm{L}^1(\Omega) : |Df|_A(\Omega) < +\infty\}.$$

In a natural way the $A-$*perimeter measure* in Ω of a measurable set $E \subset \mathbb{R}^n$ is

$$|\partial E|_A(\Omega) = |D\chi_E|_A(\Omega). \tag{11}$$

Now, let $Q : \mathrm{L}^1(\Omega) \to [0, +\infty]$ be the functional

$$Q(u) = \begin{cases} 2\alpha|\partial E|_A(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_A(\Omega),\ |E \cap \Omega| = V \\ +\infty & \text{otherwise,} \end{cases} \tag{12}$$

where α is the constant (4).

Then, under assumption (10) we prove that

$$Q = \Gamma(\mathrm{L}^1(\Omega)) - \lim_{\varepsilon\downarrow 0} Q_\varepsilon \tag{13}$$

for every bounded open set $\Omega \subset \mathbb{R}^n$ with boundary of class C^2 (see Theorem 3.5 and Remark 3.6). The proof relies on some preliminary results that have been established in [43].

The result (13) shows that the definition of the $A-$perimeter measure $|\partial E|_A$ is stable with respect to $\Gamma-$convergence. Indeed, only assumption (10), which is satisfied for instance by all matrices $A(x)$ with smooth entries (see Lemma 2.2), is needed in order to assure the $\Gamma-$convergence result.

Under the weak assumption (10) only, the result (13) does not provide a significative selection criterion to single out preferred solutions among the ones of the limit geometric problem

$$\inf\{2\alpha|\partial E|_A(\Omega) : E \subset \mathbb{R}^n,\ |E \cap \Omega| = V\}, \tag{14}$$

because a minimizing sequence $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ of the problems (P_{ε_h}) need not be relatively compact in $\mathrm{L}^1(\Omega)$ if A vanishes on big parts of Ω.

On the other hand, we are able to prove a selection criterion providing a control to this lack of coerciveness by means of a *Carnot-Carathéodory* (hereafter cc) *distance* d induced by the matrix A. This results also requires that the geometry of Ω be smooth in the metric space $(\mathbb{R}^n, d)$.

Namely, let $X(x) = (X_1(x), ..., X_m(x))$ be the family of Lipschitz continuous vector fields whose coefficients are the rows of the matrix $C(x)$ in (10), i.e.

$$X_j(x) = \sum_{i=1}^n c_{ji}(x)\partial_i, \quad x \in \mathbb{R}^n,\ j = 1, ..., m, \tag{15}$$

and call $X-$*subunit* a Lipschitz continuous curve $\gamma : [0,T] \to \mathbb{R}^n$ such that

$$\langle \dot\gamma(t), \xi\rangle^2 \leq \sum_{j=1}^{m} \langle X_j(\gamma(t)), \xi\rangle^2 \quad \text{for a.e. } t \in [0,T] \text{ and for all } \xi \in \mathbb{R}^n, \tag{16}$$

denoting $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$. The cc distance between the points $x, y \in \mathbb{R}^n$ is defined as

$$d(x,y) = \inf\{T \geq 0 : \begin{array}{l} \text{there exists an } X-\text{subunit curve } \gamma : [0,T] \to \mathbb{R}^n \\ \text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{array} \tag{17}$$

If the above set is empty put $d(x,y) = +\infty$. If d is finite on $\mathbb{R}^n$ it turns out to be a metric and the metric space $(\mathbb{R}^n, d)$ is called *cc space.*

Under the hypotheses

(H1) X is a family of *Hörmander* or *Grushin's type* vector fields (see respectively Example 5.1 and Example 5.2 in Section 5) and

(H2) Ω is a bounded open set of class C^2 and a *Boman domain* in $(\mathbb{R}^n, d)$ (see Definition 5.4)

we prove that the relaxed problem of (P_ε) has a solution u_ε in the anisotropic Sobolev space $\mathrm{H}^1_X(\Omega)$, the set of functions $f \in \mathrm{L}^2(\Omega)$ such that $X_j f \in \mathrm{L}^2(\Omega)$ $(j = 1, ..., m)$ in distributional sense (see (69) and Theorem 4.3). Moreover, a sequence of solutions $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^1(\Omega)$, and using the $\Gamma-$convergence result (13) we show that, up to a subsequence, it strongly converges in $\mathrm{L}^1(\Omega)$ to a solution $u_0 = \chi_E$ of problem (14) (see Theorem 5.8).

We stress that the degeneration makes things deeply different from the coercive case. Indeed, if the matrix $A(x)$ is not positive definite in Ω the domain of the functional Q defined in (12) may be bigger than the domain of the one in (3). Moreover, Rellich-Kondrachov compactness theorems for anisotropic Sobolev spaces are critical and depend on the cc geometry of the domain Ω.

Finally, a natural question is whether the geometric problem (14) can be translated in a minimum problem involving Hausdorff measures induced by the cc distance d. A representation of the perimeter measure $|\partial E|_A$ in terms of Hausdorff measures is in general not possible (see Section 5 Example 5.15 Remark 5.19), but in some special cases such a representation is available (see Section 5 Example 5.9).

We would like to notice that the use of cc metrics to control the lack of coerciveness of a quadratic form is well known in the literature, specially in applications in the setting of degenerate elliptic PDE's (see, for instance, [25], [26], [23], [24], [17] and references therein). In this paper we show that such metrics can be useful also in the study of some functionals of Calculus of Variations.

We give a short abstract of the paper. In Section 2 we introduce our notation and some preliminary technical results. In Section 3 we prove the $\Gamma-$convergence results for the involved perturbated functionals and in Section 4 we study the asymptotic behavior of their minimizers and minima. Finally, in Section 5 we give some examples where our main results apply.

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2. Definitions and preliminary results

Let $X = (X_1, ..., X_m)$ be a family of locally Lipschitz continuous vector fields of the form (15). Let us denote the matrix of their coefficients

$$C(x) = [c_{ji}(x)]_{j=1,\dots,m\, i=1,\dots,n}, \tag{18}$$

and let $d_X \equiv d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty]$ be the cc metric induced by X according to (17). The following $X-$connectivity assumption

(Xc) the metric d is finite and the identity map $\mathrm{Id} : (\mathbb{R}^n, d) \to (\mathbb{R}^n, |\cdot|)$ is a homeomorphism,

will be discussed in Section 5. In this section we shall introduce some functional spaces associated with vector fields and recall some properties of cc spaces.

We denote by X_j^* the operator formally adjoint to X_j in $\mathrm{L}^2(\mathbb{R}^n)$, that is the operator which for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} \varphi X_j \psi \, dx = \int_{\mathbb{R}^n} \psi X_j^* \varphi \, dx.$$

Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f \in C^1(\Omega)$ and $\varphi \in C^1(\Omega; \mathbb{R}^m)$, define the $X-$*gradient* and $X-$*divergence*

$$Xf := (X_1 f, ..., X_m f), \quad \mathrm{div}_X(\varphi) := -\sum_{j=1}^m X_j^* \varphi_j.$$

If $1 \le p \le \infty$ we can introduce the anisotropic Sobolev space

$$\mathrm{H}_X^{1,p}(\Omega) = \big\{ f \in \mathrm{L}^p(\Omega) : \text{there exists } X_j f \in \mathrm{L}^p(\Omega) \text{ for } j = 1, ..., m, \text{ in distributional sense} \big\}. \tag{19}$$

Is is well known that $\mathrm{H}_X^{1,p}(\Omega)$ endowed with the norm

$$\|u\|_{\mathrm{H}_X^{1,p}(\Omega)} = \|u\|_{\mathrm{L}^p(\Omega)} + \sum_{j=1}^m \|X_j u\|_{\mathrm{L}^p(\Omega)}$$

is a Banach space. We shall write $\mathrm{H}_X^1(\Omega) := \mathrm{H}_X^{1,2}(\Omega)$.

We introduce the functions with *bounded variation* with respect to the vector fields X. Let

$$F(\Omega; \mathbb{R}^m) := \{\varphi \in C_0^1(\Omega; \mathbb{R}^m) : |\varphi(x)| \le 1 \text{ for all } x \in \Omega\}, \tag{20}$$

and if $f \in \mathrm{L}^1(\Omega)$ define

$$\|Xf\|(\Omega) := \sup_{\varphi \in F(\Omega; \mathbb{R}^m)} \int_\Omega f \, \mathrm{div}_X(\varphi) \, dx < +\infty. \tag{21}$$

The space of the functions with bounded $X-$variation is

$$\mathrm{BV}_X(\Omega) := \big\{ f \in \mathrm{L}^1(\Omega) : \|Xf\|(\Omega) < +\infty \big\}. \tag{22}$$

A measurable set $E \subset \mathbb{R}^n$ is of *locally finite $X-$perimeter* (or an *$X-$Caccioppoli set*) if $\chi_E \in \mathrm{BV}_X(U)$ for any open set $U \Subset \mathbb{R}^n$, namely if

$$|\partial E|_X(U) := \|X\chi_E\|(U) < +\infty. \tag{23}$$

By means of Riesz representation Theorem one can prove that if $f \in \mathrm{BV}_X(\Omega)$ then $\|Xf\|$ is a Radon measure on Ω. Moreover, the total variation is lower semicontinuous with respect to the $\mathrm{L}^1(\Omega)$ convergence, i.e. if $f, f_k \in \mathrm{L}^1(\Omega)$, $k \in \mathbb{N}$, and $f_k \to f$ in $\mathrm{L}^1(\Omega)$ then

$$\liminf_{k\to\infty} \|Xf_k\|(\Omega) \geq \|Xf\|(\Omega). \tag{24}$$

Finally, the $X-$perimeter has the following representation. If $E \subset \mathbb{R}^n$ is an $X-$Caccioppoli set with C^1 boundary then

$$|\partial E|_X(\Omega) = \int_{\partial E\cap\Omega} |Cn| d\mathcal{H}^{n-1}, \tag{25}$$

where $n(x)$ is the Euclidean normal to ∂E at x and $C(x)$ is the matrix in (18).

We now recall the definition of the space of functions with bounded variation with respect to a symmetric, non negative matrix, space introduced in [28] (see also [12] for some general motivations in the case when the matrix is is positive definite).

Let $A(x)$ be a symmetric, non negative $n \times n$ matrix defined for $x \in \Omega$. Let $V_x \subset \mathbb{R}^n$ be the range of $A(x)$, i.e. $V_x = \{A(x)\xi : \xi \in \mathbb{R}^n\}$. Denote the linear map associated with $A(x)$ by $L_x : V_x \to V_x$, i.e. $L_x(\xi) = A(x)\xi$ for all $x \in \Omega$ and $\xi \in V_x$. The map L_x is invertible and it can be easily checked that

$$|v|_x := \langle v, L_x^{-1} v\rangle^{1/2}, \quad v \in V_x$$

is a norm on V_x. Let

$$F_A(\Omega) := \{\psi \in \mathrm{Lip}_0(\Omega;\mathbb{R}^n) : \psi(x) \in V_x \text{ and } |\psi(x)|_x \leq 1 \text{ for all } x \in \Omega\}, \tag{26}$$

and define

$$|Df|_A(\Omega) := \sup_{\psi\in F_A(\Omega)} \int_\Omega f \,\mathrm{div}(\psi)\, dx, \qquad |\partial E|_A(\Omega) := |D\chi_E|_A(\Omega) \tag{27}$$

and

$$\mathrm{BV}_A(\Omega) := \big\{ f \in \mathrm{L}^1(\Omega) : |Df|_A(\Omega) < +\infty \big\}. \tag{28}$$

An interesting relation between the spaces $\mathrm{BV}_X(\Omega)$ and $\mathrm{BV}_A(\Omega)$ is given by the following result (see [28, Proposition 2.1.7 and Remark 2.1.8]).

Proposition 2.1. *If $A(x) = C(x)^T C(x)$ for all $x \in \Omega$ for some $m \times n-$matrix C with locally Lipschitz continuous entries, then $\mathrm{BV}_X(\Omega) = \mathrm{BV}_A(\Omega)$, the total variations in (21) and (27) are equal, and moreover*

$$\|Xf\|(\Omega) = |Df|_A(\Omega) = \inf\Big\{ \liminf_{h\to\infty} \int_\Omega \langle ADf_h, Df_h\rangle^{1/2} dx : (f_h)_{h\in\mathbb{N}} \subset C^1(\Omega), \; f_h \to f \text{ in } \mathrm{L}^1(\Omega)\Big\}. \tag{29}$$

The factorization $A = C^T C$, the matrix C having Lipschitz continuous entries, is not always possible. The following lemma gives a sufficient condition (see for instance [48, Theorem 5.2.3]).

Lemma 2.2. *Let $A(x)$ be a symmetric, non negative $n \times n-$matrix with entries of class $C^2(\mathbb{R}^n)$ and assume there exists $\Lambda_0 > 0$ such that*

$$|\langle \frac{\partial^2 A}{\partial x_i^2}(x)\xi, \xi\rangle| \leq \Lambda_0 |\xi|^2 \quad \text{for all } x, \xi \in \mathbb{R}^n \text{ and } i = 1, ..., n. \tag{30}$$

Then there exists a symmetric $n \times n-$matrix $C(x)$ with Lipschitz continuous entries such that $A(x) = C(x)^T C(x)$ for all $x \in \mathbb{R}^n$.

Remark 2.3. If $A(x) = C(x)^T C(x)$ definition (27) can be equivalently given as

$$|Df|_A(\Omega) = \sup\Big\{ \int_\Omega f \operatorname{div}(C^T\psi)\, dx : \psi = (\psi_1, ..., \psi_m) \text{ is such that } C^T\psi \in \mathrm{Lip}_0(\Omega; \mathbb{R}^n),\ |\psi| \leq 1 \Big\}.$$

Moreover, if A is positive definite on Ω, i.e. there exists a constant $\lambda_0 > 0$ such that

$$\langle A(x)\xi, \xi\rangle \geq \lambda_0 |\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n,$$

then $\mathrm{BV}_A(\Omega) = \mathrm{BV}(\Omega)$ (see [28]). On the other hand, the inclusion $\mathrm{BV}(\Omega) \subset \mathrm{BV}_A(\Omega)$ always holds but it may be strict (see Remark 5.12).

We turn back to cc metrics and recall some results that will be needed. Consider a cc space $(\mathbb{R}^n, d)$. A function $f : (\mathbb{R}^n, d) \to \mathbb{R}$ is $L-$Lipschitz if

$$|f(x) - f(y)| \leq L d(x, y) \tag{31}$$

for all $x, y \in \mathbb{R}^n$. In this case we shall write $f \in \mathrm{Lip}(\mathbb{R}^n, d)$. The infimum of the constants L such that (31) holds will be denoted by $\mathrm{Lip}(f)$.

The following coarea formulas were proved in [28], [30], [43].

Theorem 2.4. *Let $X_1, ..., X_m \in \mathrm{Lip}(\mathbb{R}^n; \mathbb{R}^n)$. Then, if $f \in \mathrm{BV}_X(\Omega)$*

$$\|Xf\|(\Omega) = \int_{-\infty}^{+\infty} |\partial E_t|_X(\Omega)\, dt, \tag{32}$$

where $E_t = \{x \in \mathbb{R}^n : f(x) > t\}$.

Moreover, if $X = (X_1, ..., X_m)$ satisfies (Xc)*, then for every $f \in \mathrm{Lip}(\mathbb{R}^n, d)$ and $u \in \mathrm{L}^1(\mathbb{R}^n)$*

$$\int_{\mathbb{R}^n} u\, |Xf|\, dx = \int_{-\infty}^{+\infty} \Big(\int_{\{f=t\}} u\, d\mu_t \Big)\, dt, \tag{33}$$

where $\mu_t = |\partial E_t|_X$ is the perimeter measure of the level set E_t.

The following result shows that, in view of those applications which are local in nature, we can always assume the vector fields to be bounded and globally Lipschitz on $\mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $r \geq 0$ define the open Euclidean and cc ball respectively as

$$B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\} \quad \text{and} \quad B_C(x,r) = \{y \in \mathbb{R}^n : d(y,x) < r\}.$$

If $K \subset \mathbb{R}^n$ define its Euclidean and cc diameter respectively as

$$\operatorname{diam}(K) = \sup\{|x-y| : x,y \in K\} \quad \text{and} \quad \operatorname{diam}_C(K) = \sup\{d(x,y) : x,y \in K\}.$$

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that $\Omega \Subset \Omega_0 := B(x_0, r_0)$ with $x_0 \in \Omega$ and $r_0 > 0$. Let $X = (X_1, ..., X_m)$, $X_j := \sum_{i=1}^n c_{ji}\partial_i$, be a family of vector fields on Ω_0 such that*

(i) $X_j \in \mathrm{Lip}(\Omega_0; \mathbb{R}^n)$ *for $j = 1, ..., m$;*
(ii) *the cc metric d induced by X on Ω_0 is finite and the map* $\mathrm{Id} : (\Omega_0, d) \to (\Omega_0, |\cdot|)$ *is a homeomorphism.*

Then there exists a family $\widetilde{X} = (\widetilde{X}_1, ..., \widetilde{X}_{m+n})$ of vector fields on $\mathbb{R}^n$, $\widetilde{X}_j = \sum_{i=1}^n \widetilde{c}_{ji}\partial_i$, and there exists $L > 0$ such that

(1) $|\widetilde{X}_j(x)| := \Big(\sum_{i=1}^n \widetilde{c}_{ji}(x)^2\Big)^{1/2} \leq L$ *for all $x \in \mathbb{R}^n$ and $j = 1, ..., m+n$;*
(2) $|\widetilde{X}_j(x) - \widetilde{X}_j(y)| \leq L|x-y|$ *for all $x, y \in \mathbb{R}^n$ and $j = 1, ..., m+n$;*
(3) $\widetilde{X}(x) = (X_1(x), ..., X_m(x), 0, ..., 0)$ *for all $x \in \Omega$;*
(4) *hypothesis* ($\widetilde{\mathrm{X}}$c) *holds;*
(5) *let $M_0 := \sup_{x\in\Omega_0} |X(x)|$ and assume that*

$$\operatorname{diam}(\Omega) < \frac{r_0}{2} \quad \textit{and} \quad \operatorname{diam}_C(\Omega) < \frac{r_0}{2M_0}.$$

Then $d(x,y) = \widetilde{d}(x,y)$ for all $x, y \in \Omega$.

Proof. Fix $0 < s < t < 1$ and define $\Omega_1 := B(x_0, tr_0)$ and $\Omega_2 := B(x_0, sr_0)$. We can choose $s \in (0,1)$ such that $\Omega \Subset \Omega_2$. By the Lipschitz extension theorem we can assume $c_{ji} \in \mathrm{Lip}(\mathbb{R}^n)$ and denote by Λ a Lipschitz constant for $X_1, ..., X_m$. Define for $j = 1, ..., m$ and $i = 1, ..., n$

$$b_{ji}(x) := \max\{-M_0, \min\{M_0, c_{ji}(x)\}\}.$$

Clearly, $b_{ji} \in \mathrm{Lip}(\mathbb{R}^n)$, $|b_{ji}(x)| \leq M_0$ for all $x \in \mathbb{R}^n$, and $b_{ji}(x) = c_{ji}(x)$ for all $x \in \Omega_0$, $i = 1, ..., n$, $j = 1, ..., m$.

Let $\varphi \in C^\infty(\mathbb{R}^n)$ be a function such that $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ on Ω_2 and $\varphi \equiv 1$ on $\mathbb{R}^n \setminus \Omega_1$. Define

$$\widetilde{X}_j(x) = \sum_{i=1}^n b_{ji}(x)\partial_i \quad \text{if} \quad j = 1, ..., m \quad \text{and}$$

$$\widetilde{X}_j(x) = \varphi(x)\partial_{j-m} \quad \text{if} \quad j = m+1, ..., m+n.$$

Let $L_1 = \max\{1, M_0\}$ and $L_2 = \max\{\Lambda, \max_{x\in\mathbb{R}^n} |D\varphi(x)|\}$. If we choose $L = \max\{L_1, L_2\}$ then claims (1), (2) and (3) are verified.

It is easy to check that $\mathbb{R}^n$ is $\widetilde{X}-$connected, i.e. that for any couple of points $x, y \in \mathbb{R}^n$ there exists an $\widetilde{X}-$subunit curve connecting them. We prove that $(\mathbb{R}^n, \widetilde{d})$ and $(\mathbb{R}^n, |\cdot|)$ are homeomorphic. First of all notice that for all $x, y \in \mathbb{R}^n$

$$|x - y| \leq L_1 \widetilde{d}(x, y). \tag{34}$$

Indeed, if $\gamma : [0, T] \to \mathbb{R}^n$ is an $\widetilde{X}-$subunit curve such that $\gamma(0) = x$ and $\gamma(T) = y$

$$|x - y| = |\gamma(0) - \gamma(T)| = \Big| \int_0^T \dot{\gamma}(s)\, ds \Big| \leq \int_0^T |\dot{\gamma}(s)|\, ds \leq L_1 T,$$

as the subunit condition implies

$$|\dot{\gamma}(s)| = \Big| \sum_{j=1}^{m+n} h_j(s) \widetilde{X}_j(\gamma(s)) \Big| \leq L_1.$$

From (34) it follows that the map $\mathrm{Id} : (\mathbb{R}^n, \widetilde{d}) \to (\mathbb{R}^n, |\cdot|)$ is continuous. We prove that $\mathrm{Id}^{-1} : (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}^n, \widetilde{d})$ is continuous, too. We show that if $|x_h - x| \to 0$ then $\widetilde{d}(x_h, x) \to 0$. If $x \in \Omega_0$ we can assume $x_h \in \Omega_0$ for all $h \in \mathbb{N}$, and since $\widetilde{d}(x_h, x) \leq d(x_h, x)$, the claim follows from hypothesis (ii). If $x \in \mathbb{R}^n \setminus \Omega_0$ we can assume $x_h \in \mathbb{R}^n \setminus \Omega_1$ for all $h \in \mathbb{N}$. And since $\widetilde{d}(x_h, x) \leq |x_h - x|$ if h is large enough, the claim follows.

We prove (5). Since every $X-$subunit curve is also $\widetilde{X}-$subunit then $\widetilde{d}(x, y) \leq d(x, y)$ for all $x, y \in \mathbb{R}^n$. Fix $s \in (0, 1)$ in such a way that

$$\mathrm{diam}(\Omega) < \frac{sr_0}{2} \quad \text{and} \quad \mathrm{diam}_C(\Omega) < \frac{sr_0}{2M_0},$$

and choose $0 < \varepsilon < (sr_0/(2M_0) - \mathrm{diam}_C(\Omega))$. Let $x, y \in \Omega$. Every $\widetilde{X}-$subunit curve $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma(T) = y$ with $T \leq d(x, y) + \varepsilon$ is $X-$subunit (with the same coefficients). Indeed

$$TM_0 \leq (\mathrm{diam}_C(\Omega) + \varepsilon) M_0 < \frac{sr_0}{2},$$

and the argument in [37, Lemma 11.1] implies that $|\gamma(t) - x| < sr_0/2$ for all $t \in [0, T]$. Since $|x - x_0| \leq sr_0/2$ it follows that $|\gamma(t) - x_0| < sr_0$, that is $\gamma(t) \in \Omega_2$ for all $t \in [0, T]$. □

Remark 2.6. From (3) in Proposition 2.5 it follows that if $u \in \mathrm{H}^{1,p}_X(\Omega)$ then $|Xu| = |\widetilde{X}u|$ a.e., and hence $\mathrm{H}^{1,p}_X(\Omega) = \mathrm{H}^{1,p}_{\widetilde{X}}(\Omega)$, $p \geq 1$. Analogously, $\|Xu\|(\Omega) = \|\widetilde{X}u\|(\Omega)$ for all $u \in \mathrm{BV}_X(\Omega)$ and thus $\mathrm{BV}_X(\Omega) = \mathrm{BV}_{\widetilde{X}}(\Omega)$.

Remark 2.7. Assume that there exists $L > 0$ such that

$$|X_j(x)| = \Big(\sum_{i=1}^n c_{ji}(x)^2 \Big)^{1/2} \leq L \tag{35}$$

for all $x \in \mathbb{R}^n$ and $j = 1, ..., m$, and

$$|X_j(x) - X_j(y)| \leq L|x - y| \tag{36}$$

for all $x, y \in \mathbb{R}^n$ and $j = 1, ..., m$.

Let $\sigma > 0$ and consider the family of vector fields $X_{\sigma,\eta} = (X_1^\eta, ..., X_m^\eta, \sigma\partial_1, ..., \sigma\partial_n)$ where

$$X_j^\eta = \sum_{i=1}^n (c_{ji} * J_\eta)\partial_i, \quad j = 1, ..., m,$$

and $(J_\eta)_{\eta>0}$ is a family of mollifiers. We claim that

$$\sum_{j=1}^m \langle X_j(x), \xi\rangle^2 \leq \sigma^2|\xi|^2 + \sum_{j=1}^m \langle X_j^\eta(x), \xi\rangle^2 \tag{37}$$

for all $x \in \mathbb{R}^n$, for all $\xi \in \mathbb{R}^n$ and for all $0 < \eta \leq \sigma^2/(2mL^2)$, where $L > 0$ is a constant such that (35) and (36) hold. Indeed

$$\begin{aligned}
\sigma^2|\xi|^2 + \sum_{j=1}^m \langle X_j^\eta(x), \xi\rangle^2 &= \sigma^2|\xi|^2 + \sum_{j=1}^m \Big(\langle X_j^\eta(x) - X_j(x), \xi\rangle - \langle X_j(x), \xi\rangle\Big)^2 \\
&\geq \sigma^2|\xi|^2 + \sum_{j=1}^m \langle X_j(x), \xi\rangle^2 - 2|\xi|^2 \sum_{j=1}^m |X_j(x)||X_j^\eta(x) - X_j(x)| \\
&\geq (\sigma^2 - 2m\eta L^2)|\xi|^2 + \sum_{j=1}^m \langle X_j(x), \xi\rangle^2.
\end{aligned}$$

We used $|X_j^\eta(x) - X_j(x)| \leq L\eta$.

Now let $\eta_\sigma = \sigma^2/(4mL^2)$ and define

$$X_\sigma = X_{\sigma,\eta_\sigma}. \tag{38}$$

The coefficients of the vector fields X_σ are of class C^∞ and if d_σ is the cc metric induced by them then the cc space $(\mathbb{R}^n, d_\sigma)$ is actually a complete Riemannian manifold.

3. The results of Γ−convergence

This section deals with the Γ−convergence results. For a comprehensive introduction to Γ−convergence we refer to [19]. We introduce the involved functionals.

Let $W \in C^2(\mathbb{R})$ be a function with two "wells" of equal depth

$$W(0) = W(1) = 0, \quad W(s) > 0 \ \text{ if } \ s \neq 0, 1, \quad W''(0) > 0, \ W''(1) > 0. \tag{39}$$

Fix a bounded open set $\Omega \subset \mathbb{R}^n$ and for $\varepsilon > 0$ define the functionals $F_\varepsilon, F : \mathrm{L}^1(\Omega) \to [0, +\infty]$

$$F_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\varepsilon|Xu|^2 + \frac{1}{\varepsilon}W(u)\Big)\,dx & \text{if } u \in \mathrm{H}^1_X(\Omega) \\ +\infty & \text{if } u \in \mathrm{L}^1(\Omega) \setminus \mathrm{H}^1_X(\Omega), \end{cases}$$

and

$$F(u) = \begin{cases} 2\alpha|\partial E|_X(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_X(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

where $\alpha = \int_0^1 \sqrt{W(s)}\,ds$.

Let $0 < V < |\Omega|$, introduce the set of admissible functions

$$\mathcal{A}_V = \Big\{ u \in \mathrm{L}^1(\Omega) : \int_\Omega u\,dx = V,\ 0 \le u \le 1 \text{ a.e. in } \Omega \Big\}, \tag{40}$$

and let I_V be the *indicator function* of $\mathcal{A}_V$, i.e. the function which takes the value 0 on $\mathcal{A}_V$ and $+\infty$ outside. Finally, define

$$G_\varepsilon = F_\varepsilon + I_V \quad \text{and} \quad G = F + I_V. \tag{41}$$

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$ and let $G_h = G_{\varepsilon_h}$, $F_h = F_{\varepsilon_h}$.

Theorem 3.1. *Suppose that $X_1, ..., X_m \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n)$, let $W \in C^2(\mathbb{R})$ be as in* (39) *and let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then*

$$G = \Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{h\to\infty} G_h,$$

i.e. by definition

$$\forall u \in \mathrm{L}^1(\Omega) \text{ and } \forall (u_h) \subset \mathrm{L}^1(\Omega) \text{ if } u_h \to u \text{ in } \mathrm{L}^1(\Omega) \text{ then } G(u) \le \liminf_{h\to\infty} G_h(u_h), \tag{42}$$

$$\forall u \in \mathrm{L}^1(\Omega)\, \exists (u_h) \subset \mathrm{L}^1(\Omega) \text{ such that } u_h \to u \text{ in } \mathrm{L}^1(\Omega) \text{ and } G(u) \ge \limsup_{h\to\infty} G_h(u_h). \tag{43}$$

Remark 3.2. The $\Gamma-$convergence of the family $(F_\varepsilon)_{\varepsilon>0}$ to F (with $W(u) = u^2(1-u)^2$ and without volume constraint) was proved in [43] assuming the regularity of the vector fields $X_1, ..., X_m$ and of Ω ($c_{ji} \in C^\infty(\mathbb{R}^n)$ and Ω with C^∞ boundary), and finally assuming hypothesis (Xc) and an eikonal equation for the cc metric d. Even under all these stronger regularity assumptions Theorem 3.1 is not implied by the results in [43] since the indicator function I_V is not a continuous perturbation of F_ε in the $\mathrm{L}^1(\Omega)$ topology.

We begin with a refinement of the approximation theorem for BV_X functions which is necessary in order to bypass the following technical difficulty. In the Euclidean setting one of the main tools in the approximation of a set of finite perimeter in Ω by means of sets with regular boundary in $\mathbb{R}^n$ (not only in Ω) is the property of a function $u \in \mathrm{BV}(\Omega)\cap\mathrm{L}^\infty(\Omega)$ to be extendible to a function $\widetilde{u} \in \mathrm{BV}(\mathbb{R}^n) \cap \mathrm{L}^\infty(\mathbb{R}^n)$ with $\|D\widetilde{u}\|(\partial\Omega) = 0$, if Ω has Lipschitz boundary (see [41, Lemma 1] and [49, Lemma 1]). It is not known if such a property does hold for $\mathrm{BV}_X(\Omega)$ functions. Nevertheless, we can prove the following result.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary, and let $E \subset \Omega$ be a measurable set such that $|\partial E|_X(\Omega) < +\infty$ and $0 < |E| < |\Omega|$. Then there exists a sequence $(E_h)_{h\in\mathbb{N}}$ of open sets of $\mathbb{R}^n$ such that*

(i) E_h *is bounded and* ∂E_h *is of class* C^∞ *for all* $h \in \mathbb{N}$;
(ii) $E_h \to E$ *in* $\mathrm{L}^1(\Omega)$;
(iii) $|\partial E_h|_X(\Omega) \to |\partial E|_X(\Omega)$;
(iv) $\mathcal{H}^{n-1}(\partial E_h \cap \partial\Omega) = 0$ *for all* $h \in \mathbb{N}$;
(v) $|E_h \cap \Omega| = |E|$ *for all* $h \in \mathbb{N}$.

As a first step we prove the following Lemma.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let $E \subset \Omega$ be a measurable set such that $|\partial E|_X(\Omega) < +\infty$ and $0 < |E| < |\Omega|$. Then there exists a sequence $(E_h)_{h\in\mathbb{N}}$ of open sets in $\mathbb{R}^n$ such that*

(i) E_h *is bounded and* $\partial E_h \cap \Omega$ *is of class* C^∞ *for all* $h \in \mathbb{N}$*;*
(ii) $E_h \to E$ *in* $\mathrm{L}^1(\Omega)$*;*
(iii) $|\partial E_h|_X(\Omega) \to |\partial E|_X(\Omega)$*;*
(iv) $|E_h \cap \Omega| = |E|$ *for all* $h \in \mathbb{N}$*.*

Proof. We first show that it is not restrictive to assume $\mathrm{int}(E) \neq \emptyset$ and $\mathrm{int}(\Omega \setminus E) \neq \emptyset$. Recall the definition of *interior in measure* of a set $F \subset \mathbb{R}^n$

$$\mathrm{int}_{\mathrm{M}}(F) = \left\{ x \in \mathbb{R}^n : \text{ there exists } \lim_{r\downarrow 0} \frac{|F \cap B(x,r)|}{|B(x,r)|} = 1 \right\}.$$

Since $0 < |E| < |\Omega|$ from Lebesgue differentiation Theorem there exist $x_1 \in \mathrm{int}_{\mathrm{M}}(\Omega \setminus E)$ and $x_2 \in \mathrm{int}_{\mathrm{M}}(E)$. Let $r_0 := \min\{\mathrm{dist}(x_1, \partial\Omega), \mathrm{dist}(x_2, \partial\Omega), |x_1 - x_2|\}$, and if $0 \le r_1, r_2 < r_0$ define

$$\varphi(r_1, r_2) = |(E \cup B(x_1, r_1)) \setminus B(x_2, r_2))| - |E|.$$

If $0 < r < r_0$ then

$$\varphi(r, 0) = |E \cup B(x_1, r)| - |E| > 0,$$
$$\varphi(0, r) = |E \setminus B(x_2, r))| - |E| < 0.$$

Since φ is continuous, for all $0 < r < r_0$ there exists $\alpha_r \in (0, 1)$ such that $\varphi(\alpha_r r, (1 - \alpha_r)r) = 0$. Define

$$E_r = (E \cup B(x_1, \alpha_r r)) \setminus B(x_2, (1 - \alpha_r)r),$$

and notice that $\mathrm{int}(E_r) \neq \emptyset$, $\mathrm{int}(\Omega \setminus E_r) \neq \emptyset$, $|E_r \Delta E| \le 2\omega_n r^n$, $|E_r| = |E|$ and

$$\begin{aligned} |\partial E_r|_X(\Omega) &\le |\partial E|_X(\Omega) + |\partial B(x_1, \alpha_r r)|_X(\mathbb{R}^n) + |\partial B(x_2, (1 - \alpha_r)r)|_X(\mathbb{R}^n) \\ &\le |\partial E|_X(\Omega) + Cr^{n-1}. \end{aligned}$$

These inequalities and the lower semicontinuity of the perimeter with respect to the convergence $E_r \to E$ in $\mathrm{L}^1(\Omega)$ as $r \downarrow 0$ imply

$$|\partial E|_X(\Omega) \le \liminf_{r\downarrow 0} |\partial E_r|_X(\Omega) \le \limsup_{r\downarrow 0} |\partial E_r|_X(\Omega) \le |\partial E|_X(\Omega),$$

and thus equalities hold and $|\partial E_r|_X(\Omega) \to |\partial E|_X(\Omega)$.

We now turn to the proof of the lemma. There exist $x_1 \in E$, $x_2 \in \Omega \setminus E$ and $r_0 > 0$ such that

$$B_1 = B(x_1, r_0) \subset E, \quad B_2 = B(x_2, r_0) \subset \Omega \setminus E.$$

Using the same notation as in [28, Theorem 2.2.2] write $u = \chi_E$ and let $\Omega_i = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \frac{1}{k+i}\}$ for $i \in \mathbb{N}$. If k is sufficiently large we can assume that Ω_0 is such that $B_1 \cup B_2 \Subset \Omega_0 \Subset \Omega$. There exists a sequence $(u_h)_{h\in\mathbb{N}} \subset C^\infty(\Omega)$ such that

$$u_h \to u \text{ in } \mathrm{L}^1(\Omega) \quad \text{and} \quad \lim_{h\to\infty} \int_\Omega |Xu_h(x)|\, dx = |\partial E|_X(\Omega).$$

Such functions may assumed to be of the form

$$u_h = \sum_{i=0}^{\infty} (u\varphi_i) * J_{\varepsilon_i},$$

where $(J_\varepsilon)_{\varepsilon>0}$ is a family of mollifiers, ε_i depend on h and are small, and $(\varphi_i)_{i\in\mathbb{N}}$ is a partition of unity of Ω subordinate to the covering $\{\Omega_{i+1} \setminus \overline{\Omega}_{i-1} : i \in \mathbb{N}\}$ $(\Omega_{-1} = \emptyset)$. In particular $\varphi_0 \equiv 1$ on Ω_0 and $\varphi_i \equiv 0$ on Ω_0 if $i \geq 1$. Moreover, we can choose ε_i sufficiently small in order that $\mathrm{supp}((u\varphi_i) * J_{\varepsilon_i}) \subset \Omega \setminus \overline{\Omega}_0$ for all $i \geq 1$.

If $h \in \mathbb{N}$, $\varepsilon_0 < r_0/2$ and $x \in B(x_1, r_0/2) \cup B(x_2, r_0/2)$ then

$$u_h(x) = \sum_{i=0}^{\infty} ((u\varphi_i) * J_{\varepsilon_i})(x) = ((u\varphi_0) * J_{\varepsilon_0})(x) = u(x). \tag{44}$$

For suitable sequences $(h_k)_{k\in\mathbb{N}}$ and $(t_k)_{k\in\mathbb{N}}$ the sets $\widehat{E}_k = \{x \in \Omega : u_{h_k}(x) > t_k\}$ are regular and verify

$$\widehat{E}_k \to E \text{ in } \mathrm{L}^1(\Omega) \quad \text{and} \quad \lim_{k\to\infty} |\partial \widehat{E}_k|_X(\Omega) = |\partial E|_X(\Omega). \tag{45}$$

This can be proved exactly as in [43, Theorem 7.1].

The sets $\widehat{E}_k$ can be modified in order that the volume constraint be satisfied. Let $\lambda_k = |\widehat{E}_k| - |E|$ and define

$$E_k = \begin{cases} \widehat{E}_k \setminus B(x_1, r_k) & \text{if } \lambda_k > 0 \\ \widehat{E}_k & \text{if } \lambda_k = 0 \\ \widehat{E}_k \cup B(x_2, r_k) & \text{if } \lambda_k < 0, \end{cases}$$

where $r_k > 0$ is such that $|B(x_1, r_k)| = |B(x_2, r_k)| = |\lambda_k|$.

We show that $|E_k \cap \Omega| = |E|$. Notice that

$$|\lambda_k| \leq |(\widehat{E}_k \Delta E) \cap \Omega| \to 0 \quad \text{as } k \to \infty, \tag{46}$$

and therefore $\lim_{k\to\infty} r_k = 0$. For k sufficiently large we can assume $r_k < r_0/2$. Moreover, by (44) $B(x_1, r_0/2) \subset E_k$ and $B(x_2, r_0/2) \subset \Omega \setminus E_k$, whence

$$\begin{aligned} |E_k| &= |\widehat{E}_k| - |B(x_1, r_k)| = |E| \quad \text{if } \lambda_k > 0, \\ |E_k| &= |\widehat{E}_k| + |B(x_2, r_k)| = |E| \quad \text{if } \lambda_k < 0. \end{aligned}$$

This proves (iv). From (46) we also get (ii). Indeed

$$|(E_k \Delta E) \cap \Omega| \leq (\widehat{E}_k \Delta E_k) \cap \Omega| + |\widehat{E}_k \Delta E| \leq |\lambda_k| + |(\widehat{E}_k \Delta E) \cap \Omega| \to 0.$$

Finally notice that

$$|\partial E_k|_X(\Omega) = |\partial \widehat{E}_k|_X(\Omega) + \int_{\partial B(x_i, r_k)} |Cn| \, d\mathcal{H}^{n-1}$$

for $i = 1$ or $i = 2$, where n is the Euclidean normal to $\partial B(x_i, r_k)$ and C is the matrix $C(x) = [(c_{ji}(x))]$. From (45) and $r_k \to 0$ we get (iii). $\square$

Proof of Proposition 3.3. By Lemma 3.4 we can assume without loss of generality that $E \subset \Omega$ is an open set such that $\partial E \cap \Omega$ is of class C^∞. We shall divide the proof in two steps.

Step 1. Assume that $|\partial E|_X(\partial\Omega) = 0$. In this case

$$\begin{aligned}|\partial E|_X(\mathbb{R}^n) &= |\partial E|_X(\Omega) + |\partial E|_X(\partial\Omega) + |\partial E|_X(\mathbb{R}^n \setminus \overline{\Omega}) \\ &= |\partial E|_X(\Omega) < +\infty.\end{aligned}$$

Let $(J_\varepsilon)_{\varepsilon>0}$ be a family of mollifiers, write $u = \chi_E$ and define $u_\varepsilon = u * J_\varepsilon$. From [28, Theorem 2.2.2] it follows that $u_\varepsilon \to u$ in $\mathrm{L}^1(\mathbb{R}^n)$, $\lim_{\varepsilon\downarrow 0} |\{x \in \mathbb{R}^n : |u_\varepsilon(x) - u(x)| \geq \eta\}| = 0$ for any $\eta > 0$ and $\lim_{\varepsilon\downarrow 0} \int_{\mathbb{R}^n} |Xu_\varepsilon(x)|\, dx = |\partial E|_X(\mathbb{R}^n)$. Moreover, since $|\partial E|_X(\partial\Omega) = 0$ we also have

$$\lim_{\varepsilon\downarrow 0} \int_\Omega |Xu_\varepsilon|\, dx = |\partial E|_X(\Omega).$$

Thus we can proceed exactly as in [41, Lemma 1, proof] replacing the gradient ∇ with X, the Euclidean perimeter $|\partial E|$ with $|\partial E|_X$ and taking into account the coarea formula (33).

Step 2. Assume that $|\partial E|_X(\partial\Omega) > 0$. We show that for all $\varepsilon > 0$ there exists an open set $E_\varepsilon \subset \mathbb{R}^n$ such that $|\partial E_\varepsilon|_X(\Omega) < \infty$, $|E_\varepsilon \cap \Omega| = |E|$, $|\partial E_\varepsilon|_X(\partial\Omega) = 0$ and

$$\lim_{\varepsilon\downarrow 0} |(E_\varepsilon \Delta E) \cap \Omega| = 0 \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} |\partial E_\varepsilon|_X(\Omega) = |\partial E|_X(\Omega). \tag{47}$$

Because E is open, $\partial E \cap \Omega$ is C^∞ and $0 < |E| < |\Omega|$ there exist $x_1 \in E$, $x_2 \in \Omega \setminus E$ and $r_0 > 0$ such that $B_1 = B(x_1, r_0) \subset E$ and $B_2(x_2, r_0) \subset \Omega \setminus E$. We shall use the technique introduced in [43, Proposition 6.3, *Step 3*]. For $\varepsilon > 0$ fixed let $0 < t_0$, Ω_{t_0} and $\widehat{E}$ be as in [43, (6.8)]. If Ω is of class C^2 then $\partial\widehat{E} \cap \Omega$ is of class C^1. If t_0 and r_0 are small enough then

$$B_1 \subset \Omega_{t_0} \cap E \quad \text{and} \quad B_2 \subset \Omega_{t_0} \setminus E. \tag{48}$$

Let $\lambda_\varepsilon = |\widehat{E} \cap \Omega| - |E|$ ($\widehat{E}$ depends on ε) and define

$$E_\varepsilon = \begin{cases} \widehat{E} \setminus B(x_1, r_\varepsilon) & \text{if } \lambda_\varepsilon > 0 \\ \widehat{E} & \text{if } \lambda_\varepsilon = 0 \\ \widehat{E} \cup B(x_2, r_\varepsilon) & \text{if } \lambda_\varepsilon < 0 \end{cases}$$

where $r_\varepsilon > 0$ is chosen in such a way that $|B(x_1, r_\varepsilon)| = |B(x_2, r_\varepsilon)| = |\lambda_\varepsilon|$.

Since $B_1 \subset \widehat{E}$ and $B_2 \subset \Omega \setminus \widehat{E}$, arguing as in the proof of Lemma 3.4 we get (ii), (iii) and (iv).

We finally prove that $|\partial E_\varepsilon|_X(\partial\Omega) = 0$. Since $\partial E_\varepsilon \cap \partial\Omega = (\partial\widehat{E} \cup \partial B(x_i, r_\varepsilon)) \cap \partial\Omega$ for $i = 1$ or $i = 2$ with $\partial B(x_i, r_\varepsilon) \cap \partial\Omega = \emptyset$, from the definition of E_ε we get $|\partial E_\varepsilon|_X(\partial\Omega) = |\partial\widehat{E}|_X(\partial\Omega) = 0$, because of the definition of $\widehat{E}$ (see [43, (6.10)]). □

Proof of Theorem 3.1. We divide the proof in two steps.

Step 1. Assume that $X_1, \dots, X_m \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $j = 1, \dots, m$, and that the family $X = (X_1, \dots, X_m)$ satisfies hypothesis (Xc) and let d be the induced cc metric. We also assume the following eikonal equation:

(Ek) Let $K \subset \mathbb{R}^n$ be a closed set. If $d_K(x) := \inf_{y\in K} d(x,y)$ then $Xd_K(x) = (X_1 d_K(x), \dots, X_m d_K(x)) \in \mathbb{R}^m$ exists and $|Xd_K(x)| = 1$ for a.e. $x \in \mathbb{R}^n \setminus K$.

Under such hypotheses we shall prove the thesis.

The proof of the lower bound estimate (42) is verbatim contained in [43, Theorem 6.5, proof]. A few modifications will be needed in order to prove the upper bound estimate (43). By Proposition 3.3 and by [42, Lemma IV] we can reduce to prove (43) for $u = \chi_E$, $E \subset \mathbb{R}^n$ bounded open set with C^∞ boundary such that $|E \cap \Omega| = V$ and $\mathcal{H}^{n-1}(\partial\Omega \cap \partial E) = 0$.

Define $\varrho : \mathbb{R}^n \to [0, +\infty)$

$$\varrho(x) = \begin{cases} \min\limits_{y\in\partial E} d(x,y) & x \in E \\ -\min\limits_{y\in\partial E} d(x,y) & x \in \mathbb{R}^n \setminus E, \end{cases}$$

and write $\chi_0(t) = \chi_{(0,+\infty)}(t)$. Then $u(x) = \chi_0(\varrho(x))$ for all $x \in \mathbb{R}^n$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the maximal solution of the Cauchy problem

$$\begin{cases} \chi'(t) = \sqrt{W(\chi(t))} \\ \chi(0) = \frac{1}{2}. \end{cases}$$

It is easy to see that, as $W(0) = W(1) = 0$, χ is a strictly increasing C^2 function such that $\lim_{t\to+\infty} \chi(t) = 1$ and $\lim_{t\to-\infty} \chi(t) = 0$. Moreover, there exist $\bar{t} \in \mathbb{R}$, $c_1, c_2 > 0$ such that (see [49, (1.21)])

$$1 - \chi(t) \le c_1 e^{-c_2 t}, \quad \text{for all } t \ge \bar{t}. \tag{49}$$

We follow the proof contained in [43] (see also [12]). Fix $\varepsilon > 0$ and write $t_\varepsilon = \vartheta\varepsilon \log 1/\varepsilon$ where $\vartheta \ge 3$ is a constant that will be determined later. Define the function $\Lambda_\varepsilon : \mathbb{R} \to \mathbb{R}$ in the following way

$$\Lambda_\varepsilon(t) = \begin{cases} \chi(t) & \text{if } 0 \le t < \frac{t_\varepsilon}{\varepsilon} \\ p_\varepsilon(t) & \text{if } \frac{t_\varepsilon}{\varepsilon} \le t < \frac{2t_\varepsilon}{\varepsilon} \\ 1 & \text{if } t \ge \frac{2t_\varepsilon}{\varepsilon} \\ 1 - \Lambda_\varepsilon(-t) & \text{if } t < 0, \end{cases}$$

where $p_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the uniquely determined polynomial of degree 3 for which $\Lambda_\varepsilon \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{\pm t_\varepsilon/\varepsilon, \pm 2t_\varepsilon/\varepsilon\})$ (see [13] for the construction of p_ε) satisfying

$$\|p_\varepsilon - 1\|_{\mathrm{L}^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)} = O(\varepsilon^{2\vartheta-1}) \quad \text{and} \quad \|p'_\varepsilon\|_{\mathrm{L}^\infty(t_\varepsilon/\varepsilon, 2t_\varepsilon/\varepsilon)} = O(\varepsilon^{2\vartheta}). \tag{50}$$

Now define $\chi_\varepsilon(t) = \Lambda_\varepsilon(t/\varepsilon)$ for $t \in \mathbb{R}$ and $v_\varepsilon(x) = \chi_\varepsilon(\varrho(x))$. It is easy to see that $v_\varepsilon \in \mathrm{H}^{1,\infty}_X(\Omega)$ and $Xv_\varepsilon(x) = \chi'_\varepsilon(\varrho(x))X\varrho(x)$ a.e. Then, from [43, Theorem 6.5]

$$\lim_{\varepsilon\downarrow 0} \int_\Omega |v_\varepsilon - u| dx = 0, \tag{51}$$

$$\limsup_{\varepsilon\downarrow 0} F_\varepsilon(v_\varepsilon) \le F(u) = G(u). \tag{52}$$

The functions v_ε will be now perturbated so as to satisfy the integral constraint without disturbing inequality (52). Let us begin to show that if $\delta_\varepsilon = \int_\Omega v_\varepsilon \, dx - V$, then $\delta_\varepsilon = O(\varepsilon)$

(see also [49, Theorem 1]). Notice that

$$\begin{aligned}\delta_\varepsilon &= \int_\Omega (v_\varepsilon - u)\,dx \\ &= \int_{\{x\in\Omega:0<\varrho(x)<t_\varepsilon\}} (\chi(\varrho(x)/\varepsilon)-1)\,dx + \int_{\{x\in\Omega:t_\varepsilon\le\varrho(x)\le 2t_\varepsilon\}} (p_\varepsilon(\varrho(x)/\varepsilon)-1)\,dx \\ &\quad + \int_{\{x\in\Omega:-t_\varepsilon<\varrho(x)<0\}} (1-\chi(-\varrho(x)/\varepsilon))\,dx + \int_{\{x\in\Omega:-2t_\varepsilon\le\varrho(x)\le t_\varepsilon\}} (1-p_\varepsilon(-\varrho(x)/\varepsilon))\,dx.\end{aligned}$$

Because of (50), if $\vartheta \ge 1$ the second and fourth integrals are $O(\varepsilon)$.

We estimate the first one. By hypothesis (Ek) $|X\varrho| = 1$ a.e. on $\mathbb{R}^n$ and using the coarea formula (32) we get for $t \ge 0$

$$V^+(t) := |\{x\in\Omega : 0<\varrho(x)\le t\}| = \int_0^t |\partial E_s|_X(\Omega)\,ds,$$

where $E_s := \{x\in\mathbb{R}^n : \varrho(x) > s\}$. By the coarea formula (33) and integrating by parts

$$\begin{aligned}\int_{\{x\in\Omega:0<\varrho(x)<t_\varepsilon\}} (1-\chi(\varrho(x)/\varepsilon))\,dx &= \int_0^{t_\varepsilon} (1-\chi(s/\varepsilon))|\partial E_s|_X(\Omega)\,ds \\ &= V^+(t_\varepsilon)(1-\chi(\vartheta\log(1/\varepsilon))) + \frac{1}{\varepsilon}\int_0^{t_\varepsilon} \chi'(s/\varepsilon)V^+(s)ds.\end{aligned}$$

By [43, Theorem 5.1] (see also [7]) $V^+(t) = Lt + t\delta^+(t)$, where $L = |\partial E|_X(\Omega)$ and $\delta^+ : [0,+\infty) \to \mathbb{R}$ is a function such that

$$\lim_{\varepsilon\downarrow 0} \sup_{s\in[0,t_\varepsilon]} |\delta^+(s)| = 0.$$

By (49) it follows that $V^+(t_\varepsilon)(1-\chi(\vartheta\log(1/\varepsilon))) = O(\varepsilon)$ if $\vartheta c_2 \ge 1$. Moreover

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int_0^{t_\varepsilon} \chi'(s/\varepsilon)V^+(s)\,ds\right| &\le \frac{1}{\varepsilon}\int_0^{t_\varepsilon} \sqrt{W(\chi(s/\varepsilon))}V^+(s)\,ds \\ &\le (L + \sup_{s\in[0,t_\varepsilon]}|\delta^+(s)|)\frac{1}{\varepsilon}\int_0^{t_\varepsilon} s\sqrt{W(\chi(s/\varepsilon))}\,ds \\ &\le \varepsilon(L + \sup_{s\in[0,t_\varepsilon]}|\delta^+(s)|)\int_0^{+\infty} s\sqrt{W(\chi(s))}\,ds,\end{aligned}$$

and the integral in the last expression is bounded because of (49). In conclusion, if we choose $\vartheta \ge \max\{3, 1/c_2\}$ this ends the proof of $\delta_\varepsilon = O(\varepsilon)$.

Consider now the family of functions $u_\varepsilon = (1+\eta_\varepsilon)v_\varepsilon$ with $\eta_\varepsilon = -\delta_\varepsilon/\int_\Omega v_\varepsilon dx$. Of course, $u_\varepsilon \in \mathrm{H}^{1,\infty}_X(\Omega)$ and $u_\varepsilon \in \mathcal{A}_V$ since $1+\eta_\varepsilon > 0$ and $\int_\Omega u_\varepsilon dx = V$. If we show that

$$\limsup_{\varepsilon\downarrow 0} G_\varepsilon(u_\varepsilon) \le \limsup_{\varepsilon\downarrow 0} F_\varepsilon(v_\varepsilon), \tag{53}$$

statement (43) will be proved.

Notice that

$$\begin{aligned} G(u_\varepsilon) = & \int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} \Big(\varepsilon(1+\eta_\varepsilon)^2|Xv_\varepsilon|^2 + \frac{1}{\varepsilon}W(v_\varepsilon+\eta_\varepsilon v_\varepsilon)\Big)\,dx \\ & + \frac{1}{\varepsilon}W(1+\eta_\varepsilon)|\{x\in\Omega:\varrho(x)>2t_\varepsilon\}| \\ \le\ & \varepsilon\int_\Omega |Xv_\varepsilon|^2\,dx + \frac{\eta_\varepsilon(2+\eta_\varepsilon)}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} |\Lambda'_\varepsilon(\varrho/\varepsilon)|^2\,dx \\ & + \frac{1}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} W(v_\varepsilon+\eta_\varepsilon v_\varepsilon)\,dx + \frac{1}{\varepsilon}W(1+\eta_\varepsilon)|\{x\in\Omega:\varrho(x)>2t_\varepsilon\}|. \end{aligned}$$

By (39) and by Taylor's formula

$$\frac{1}{\varepsilon}W(1+\eta_\varepsilon)|\{x\in\Omega:\varrho(x)>2t_\varepsilon\}| \le \frac{|\Omega|}{2\varepsilon}W''(\xi_\varepsilon)\eta_\varepsilon^2$$

for some $\xi_\varepsilon\in(1-\eta_\varepsilon,1+\eta_\varepsilon)$ and hence this term is $O(\varepsilon)$. Moreover, since

$$\begin{aligned} \int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} |\Lambda'_\varepsilon(\varrho/\varepsilon)|^2\,dx \le & \sup|\chi'|^2|\{x\in\Omega:|\varrho(x)|\le t_\varepsilon\}| \\ & + \|p'_\varepsilon\|^2_{L^\infty(t_\varepsilon/\varepsilon,2t_\varepsilon/\varepsilon)}|\{x\in\Omega: t_\varepsilon<|\varrho(x)|\le 2t_\varepsilon\}|, \end{aligned}$$

by (50) we get

$$\lim_{\varepsilon\downarrow 0}\frac{\eta_\varepsilon(2+\eta_\varepsilon)}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|\le 2t_\varepsilon\}} |\Lambda'_\varepsilon(\varrho/\varepsilon)|^2\,dx = 0.$$

In order to prove (53) it suffices to show that

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|<2t_\varepsilon\}} (W(u_\varepsilon)-W(v_\varepsilon))\,dx = 0.$$

Indeed, by the Mean Value Theorem there exists $\tau>0$ such that

$$\frac{1}{\varepsilon}\int_{\{x\in\Omega:|\varrho(x)|<2t_\varepsilon\}} |W(u_\varepsilon)-W(v_\varepsilon)|\,dx \le \frac{|\eta_\varepsilon|}{\varepsilon}|\{x\in\Omega:|\varrho(x)|<2t_\varepsilon\}|\sup_{s\in[0,1+\tau]}|W'(s)|,$$

and the last quantity approaches to zero as $\varepsilon\downarrow 0$.

Step 2. We prove the thesis under the only assumption $X_1,...,X_m\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Thanks to Proposition 2.5 $X=(X_1,...,X_m)$ may be assumed to satisfy (35) and (36). For $\sigma>0$ let X_σ be the family of vector fields defined in (38), i.e.

$$X_\sigma = (X_1^{\eta_\sigma},...,X_m^{\eta_\sigma},\sigma\partial_1,...,\sigma\partial_n) \equiv (X_1^\sigma,...,X_{m+n}^\sigma).$$

Now, $X_j^\sigma\in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ for all $j=1,...,m+n$, these vector fields are bounded on $\mathbb{R}^n$ and by (37)

$$\sum_{j=1}^m \langle X_j(x),\xi\rangle^2 \le \sum_{j=1}^{m+n}\langle X_j^\sigma(x),\xi\rangle^2 \quad \text{for all } x,\xi\in\mathbb{R}^n. \tag{54}$$

The cc distance d_σ induced on $\mathbb{R}^n$ by X_σ is a Riemannian metric and since the vector fields are bounded $(\mathbb{R}^n, d_\sigma)$ is a complete metric space. Hypothesis (X_σc) holds, and by [43, Theorem 3.1] the family X_σ satisfies the Eikonal hypothesis (Ek).

Therefore the first step of the proof does apply to the functionals $G^\sigma_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$

$$G^\sigma_\varepsilon(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega |X_\sigma u|^2\, dx + \frac{1}{\varepsilon}\int_\Omega W(u)\, dx & \text{if } u \in \mathrm{H}^1_{X_\sigma}(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise.} \end{cases} \tag{55}$$

Precisely, for all $\sigma > 0$

$$\Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon \downarrow 0} G^\sigma_\varepsilon = G^\sigma, \tag{56}$$

where $G^\sigma : \mathrm{L}^1(\Omega) \to [0, +\infty]$ is the functional

$$G^\sigma(u) = \begin{cases} 2\alpha |\partial E|_{X_\sigma}(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_{X_\sigma}(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise.} \end{cases} \tag{57}$$

By the vector fields' form

$$\mathrm{H}^1_{X_\sigma}(\Omega) = \mathrm{H}^1(\Omega) \subset \mathrm{H}^1_X(\Omega), \quad \text{for all } \sigma > 0,$$

and then by (54)

$$G_\varepsilon(u) \le G^\sigma_\varepsilon(x), \quad \text{for all } u \in \mathrm{L}^1(\Omega) \text{ and for all } \varepsilon, \sigma > 0. \tag{58}$$

Let $G', G'' : \mathrm{L}^1(\Omega) \to [0, +\infty]$ be respectively the lower and upper Γ−limits of $(G_\varepsilon)_{\varepsilon > 0}$ (see [19, Chapter 4]), i.e. if $u \in \mathrm{L}^1(\Omega)$

$$G'(u) = \Gamma(\mathrm{L}^1(\Omega))\text{-}\liminf_{\varepsilon \downarrow 0} G_\varepsilon(u),$$
$$G''(u) = \Gamma(\mathrm{L}^1(\Omega))\text{-}\limsup_{\varepsilon \downarrow 0} G_\varepsilon(u).$$

Then, from [19, Proposition 6.7], (58) and (56)

$$G'(u) \le G''(u) \le G^\sigma(u) \quad \text{for all } u \in \mathrm{L}^1(\Omega) \text{ and for all } \sigma > 0. \tag{59}$$

We claim that

$$G(u) \le G'(u) \quad \text{for all } u \in \mathrm{L}^1(\Omega). \tag{60}$$

Indeed, by [19, Proposition 8.1] we have to prove that for every $u \in \mathrm{L}^1(\Omega)$, for every sequence $(u_h)_{h \in \mathbb{N}} \subset \mathrm{L}^1(\Omega)$ strongly converging to u in $\mathrm{L}^1(\Omega)$ and for every sequence $(\varepsilon_h)_{h \in \mathbb{N}}$ of real numbers such that $\varepsilon_h \downarrow 0$

$$G(u) \le \liminf_{h \to \infty} G_{\varepsilon_h}(u_h),$$

and this can be done exactly as in [43, Theorem 6.5, proof] where only the coarea formula (32) is involved.

Define

$$\mathcal{D} = \{\chi_E : E \subset \mathbb{R}^n \text{ bounded open set}, \partial E \in C^\infty, |E \cap \Omega| = V, \mathcal{H}^{n-1}(\partial E \cap \partial\Omega) = 0\},$$

and notice that $\mathcal{D} \subset \mathrm{BV}_{X_\sigma}(\Omega)$ for all $\sigma > 0$. If $u = \chi_E \in \mathcal{D}$ then from (25)

$$G^\sigma(u) = 2\alpha |\partial E|_{X_\sigma}(\Omega) = 2\alpha \int_{\partial E \cap \Omega} |C^\sigma n| \, d\mathcal{H}^{n-1}, \tag{61}$$

where $C^\sigma(x)$ is the $(m+n) \times n$ matrix of the coefficients of the vector fields X_j^σ's as in (18), and n is the Euclidean normal to ∂E.

In particular, from (61) we get for all $u = \chi_E \in \mathcal{D}$

$$\lim_{\sigma \downarrow 0} G^\sigma(u) = 2\alpha \int_{\partial E \cap \Omega} |Cn| \, d\mathcal{H}^{n-1} = G(u), \tag{62}$$

being $C(x)$ the matrix of the coefficients of the vector fields X_j's. On the other hand, from (60), (59) and (62)

$$G(u) \le G'(u) \le G''(u) \le G(u) \quad \text{for all } u \in \mathcal{D},$$

whence

$$G(u) = \Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon \downarrow 0} G_\varepsilon(u) \quad \text{for all } u \in \mathcal{D}. \tag{63}$$

Applying (60), (63), Proposition 3.3 and the Reduction Lemma [42, Lemma IV] we finally find

$$G = \Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon \downarrow 0} G_\varepsilon.$$

□

The last result in this section deals with the $\Gamma-$convergence of functionals defined with degenerate quadratic forms. Let $A(x)$ be a symmetric, non negative matrix and consider the functionals $Q, Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ defined as

$$Q_\varepsilon(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega \langle ADu, Du \rangle \, dx + \frac{1}{\varepsilon} \int_\Omega W(u) \, dx & \text{if } u \in C^1(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases} \tag{64}$$

and

$$Q(u) = \begin{cases} 2\alpha |\partial E|_A(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_A(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases} \tag{65}$$

where V, $\mathcal{A}_V$, W and α are as in Theorem 3.1.

Theorem 3.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^2 boundary and let $A(x)$ be a symmetric, non negative $n \times n-$matrix , i.e. $\langle A(x)\xi, \xi \rangle \ge 0$ for all $x, \xi \in \mathbb{R}^n$. Suppose that A has C^2 entries and satisfies* (30). *Moreover, assume that there exist $C \ge 1$, $u_0 > 0$ and $p \ge 1$ such that*

$$C^{-1}|u|^p \le W(u) \le C|u|^p \quad \textit{for all } |u| \ge u_0. \tag{66}$$

Then

$$Q = \Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon \downarrow 0} Q_\varepsilon. \tag{67}$$

Remark 3.6. When the matrix A is *positive definite* on Ω, i.e. there exists $\lambda_0 > 0$ such that $\langle A(x)\xi, \xi\rangle \geq \lambda_0|\xi|^2$ for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$ Theorem 3.5 is well known under the only hypothesis of continuity of the matrix entries (see [14] and [12]).

In the degenerate setting we are dealing with, requiring the matrix A to be of class C^2 is necessary in order to assure the factorization $A = C^T C$ as in Lemma 2.2. Actually, the assumptions on A in Theorem 3.5 can be weakened requiring only $A(x) = C(x)^T C(x)$ for all $x \in \Omega$ and for some $m \times n$ matrix $C(x)$ with Lipschitz continuous entries. Without such a factorization we do not know if Theorem 3.5 still holds.

Proof of Theorem 3.5. By Lemma 2.2 there exists a $n \times n$ matrix $C(x)$ with Lipschitz continuous entries such that $A(x) = C(x)^T C(x)$ for all $x \in \mathbb{R}^n$. Let $X_1, ..., X_n$ be the family of vector fields whose coefficients are the rows of the matrix $C(x)$ (see (18)). By Proposition 2.1 we can write the functionals Q_ε and Q as follows

$$Q_\varepsilon(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega |Xu|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx & \text{if } u \in C^1(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$Q(u) = \begin{cases} 2\alpha|\partial E|_X(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_X(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise.} \end{cases}$$

By a general $\Gamma-$convergence result (see [19, Proposition 6.11]) (67) holds if and only if

$$Q = \Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon\downarrow 0} \mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon, \tag{68}$$

where $\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ is the *relaxed functional* of Q_ε with respect to the topology of $\mathrm{L}^1(\Omega)$.

Recalling Theorem 3.1 we only have to prove that for every $\varepsilon > 0$

$$\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) = G_\varepsilon(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega |Xu|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx & \text{if } u \in \mathrm{H}^1_X(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise.} \end{cases} \tag{69}$$

The inequality $\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) \geq G_\varepsilon(u)$ follows at once by a well known characterization of the relaxed functional (see, for instance, [19, Proposition 3.6]) and by the lower semicontinuity of G_ε with respect to the topology of $\mathrm{L}^1(\Omega)$. We claim that

$$\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) \leq G_\varepsilon(u) \quad \text{for all } u \in \mathrm{L}^1(\Omega). \tag{70}$$

If $G_\varepsilon(u) = +\infty$ there is nothing to prove. Let $u \in \mathrm{H}^1_X(\Omega) \cap \mathcal{A}_V$ be such that $G_\varepsilon(u) < +\infty$. The growth condition (66) implies $u \in \mathrm{L}^p(\Omega)$. Since $u \in \mathrm{H}^1_X(\Omega)$ by [28, Theorem 1.2.3] there exists a sequence $(v_h)_{h\in\mathbb{N}} \subset C^1(\Omega) \cap \mathrm{H}^1_X(\Omega)$ such that $v_h \to u$ in $\mathrm{H}^1_X(\Omega)$. Moreover, as $u \in \mathrm{L}^p(\Omega)$ and the technique of approximation by convolution is involved, it is not restrictive to assume that $v_h \to u$ in $\mathrm{L}^p(\Omega)$. Let $c_h = \int_\Omega u\,dx / \int_\Omega v_h\,dx$ and define $u_h = c_h v_h$. Then $u_h \in \mathrm{H}^1_X(\Omega) \cap \mathcal{A}_V$, $u_h \to u$ in $\mathrm{H}^1_X(\Omega)$ and

$$u_h \to u \quad \text{in } \mathrm{L}^p(\Omega). \tag{71}$$

By (66), (71) and Carathéodory continuity Theorem (see [19, Example 1.22])

$$\lim_{h\to\infty}\int_\Omega W(u_h)\,dx = \int_\Omega W(u)\,dx.$$

Eventually

$$\begin{aligned}\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) &\le \liminf_{h\to\infty}\Big(\varepsilon\int_\Omega |Xu_h|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u_h)\,dx\Big)\\ &\le \varepsilon\int_\Omega |Xu|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u)\,dx = G_\varepsilon(u).\end{aligned}$$

This proves (70). As a consequence, (69) and (68) do hold. □

4. Convergence of minima and minimizers

In this section we study existence and asymptotic behavior of minima and minimizers of the functionals G_ε and Q_ε defined in (41) and (64). To this purpose we recall the following fundamental variational property of $\Gamma-$convergence (see [19, Corollary 7.20]).

Theorem 4.1. *Let (M,ϱ) be a metric space and let $F, F_h : M \to [0,+\infty]$ be such that $F = \Gamma(M)\text{-}\lim_{h\to\infty} F_h$. Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$, and let $(u_h)_{h\in\mathbb{N}} \subset M$ be a relatively compact sequence of ε_h-minimizers, i.e. $F_h(u_h) \le \inf_M F_h + \varepsilon_h$ for all $h \in \mathbb{N}$. Then*

(i) $\min_{u\in M} F(u) = \lim_{h\to\infty}\inf_{u\in M} F_h(u)$;

(ii) *every cluster point $u \in M$ of $(u_h)_{h\in\mathbb{N}}$ is a minimum of F, i.e. $F(u) = \min_{v\in M} F(v)$.*

In order to apply Theorem 4.1 a fundamental tool will be the compact embedding of $\mathrm{H}^{1,p}_X(\Omega)$ in $\mathrm{L}^p(\Omega)$ which will be discussed more in detail in Section 5. An open set $\Omega \subset \mathbb{R}^n$ will be said to support the $\mathrm{H}^{1,p}_X(\Omega)-$compact embedding, $1 \le p \le +\infty$, if

$(\mathcal{C})_p$ the embedding $\mathrm{H}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ is compact.

In the Euclidean case the compact embedding is known to imply a Poincaré inequality (see, for instance, [32]). Following the same proof an analogous result for vector fields can be obtained.

Proposition 4.2. *Let $X = (X_1, ..., X_m)$ be a family of Lipschitz vector fields on $\mathbb{R}^n$ satisfying* (Xc). *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set. If $(\mathcal{C})_p$ holds for $1 \le p < +\infty$ then there exists $C > 0$ such that*

$$\int_\Omega |u-u_\Omega|^p\,dx \le C\int_\Omega |Xu|^p\,dx \tag{72}$$

for all $u \in \mathrm{H}^{1,p}_X(\Omega)$, where $u_\Omega := \fint_\Omega u\,dx$.

Let G_ε be as in (41). The first result of this section is the existence of minima for the functionals G_ε and the compactness of the family of such minima.

Theorem 4.3. *Let $X = (X_1, ..., X_m)$ be a family of Lipschitz vector fields on $\mathbb{R}^n$ satisfying* (Xc)*, let $\Omega \subset \mathbb{R}^n$ be a connected, bounded open set such that the compact embedding $(\mathcal{C})_2$ holds, and finally let $W : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (66) *for some $p > 2$. Then for all $\varepsilon > 0$ there exists $u_\varepsilon \in \mathcal{A}_V$ such that*

$$G_\varepsilon(u_\varepsilon) = \min_{u \in \mathrm{L}^1(\Omega)} G_\varepsilon(u). \tag{73}$$

If, in addition, Ω supports the compact embedding $(\mathcal{C})_1$, then the family $\{u_\varepsilon : \varepsilon > 0\}$ is relatively compact in $\mathrm{L}^1(\Omega)$.

Let G be the functional defined in (41). Choosing $M = \mathrm{L}^1(\Omega)$, $F_h = G_{\varepsilon_h}$ and $F = G$ in Theorem 4.1 and taking into account Theorem 3.1 and Theorem 4.3 we get the following Corollary.

Corollary 4.4. *Let X, Ω and W be as in Theorem 4.3. Moreover, assume that Ω is of class C^2 and W satisfies* (39)*. Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$. Then:*

(i) *there exists* $\displaystyle\min_{u\in \mathrm{L}^1(\Omega)} G(u) = \lim_{h\to\infty} \min_{u\in \mathrm{L}^1(\Omega)} G_{\varepsilon_h}(u)$;

(ii) *if $(u_h)_{h\in\mathbb{N}}$ is a sequence of minimizers of $(G_{\varepsilon_h})_{h\in\mathbb{N}}$ ($G_{\varepsilon_h}(u_h) = \min_{u\in \mathrm{L}^1(\Omega)} G_{\varepsilon_h}(u)$) then there exist a subsequence $(u_{h_j})_{j\in\mathbb{N}}$ and a function $u_0 = \chi_E \in \mathrm{BV}_X(\Omega)$ such that $u_{h_j} \to u_0$ in $\mathrm{L}^1(\Omega)$ and $G(u_0) = \min_{u\in \mathrm{L}^1(\Omega)} G(u)$.*

Proof of Theorem 4.1. The proof can be essentially carried out as in [41] and we shall only sketch the main steps.

The existence of $u_\varepsilon \in \mathcal{A}_V$ such that (73) holds can be proved by the direct method of Calculus of Variations. To this aim we have to check that $G_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ is lower semicontinuous and coercive (see, for instance, [19, Theorem 1.15]). The lower semicontinuity and the coerciveness follow as in the classic case by the compact embedding $(\mathcal{C})_2$, by the Poincaré inequality (72) and by Fatou Lemma.

Let us prove that the family of minima $\{u_\varepsilon : \varepsilon > 0\}$ is relatively compact in $\mathrm{L}^1(\Omega)$. Define $\varphi \in C^1(\mathbb{R})$ by $\varphi(t) = \int_0^t \sqrt{W(s)}\, ds$, and let $v_\varepsilon(x) := \varphi(u_\varepsilon(x)) \in \mathrm{H}^1_X(\Omega)$. By (66) and arguing as in [41, Proposition 3, proof] we get the existence of two positive constants c_3, c_4 such that

$$\int_\Omega v_\varepsilon\, dx \le c_3|\Omega| + c_4 G_\varepsilon(u_\varepsilon) \qquad \text{for all } \varepsilon \in (0,1),$$

and moreover

$$\int_\Omega |Xv_\varepsilon|\, dx = \int_\Omega \varphi'(u_\varepsilon)|Xu_\varepsilon|\, dx \le \frac{1}{2}\int_\Omega \Big(\varepsilon|Xu_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\, dx = \frac{1}{2}G_\varepsilon(u_\varepsilon).$$

If we show that $G_\varepsilon(u_\varepsilon) \le C < +\infty$ for all $\varepsilon > 0$ and for some $C > 0$, then the set $\{v_\varepsilon : \varepsilon > 0\}$ is bounded in $\mathrm{H}^{1,1}_X(\Omega)$ and hence relatively compact in $\mathrm{L}^1(\Omega)$ by the compact embedding $(\mathcal{C})_1$. The function

$$w_\varepsilon(x) = \begin{cases} 1 & \text{if } x_1 \le \delta_\varepsilon - \varepsilon \\ \frac{1}{2} + \frac{1}{2\varepsilon}(x_1 - \delta_\varepsilon) & \text{if } \delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon \\ 0 & \text{if } x_1 \ge \delta_\varepsilon + \varepsilon \end{cases}$$

belongs to $H^1_X(\Omega)$ for all $\varepsilon > 0$ and for all $\delta_\varepsilon \in \mathbb{R}$. Since $0 < V < |\Omega|$, $\delta_\varepsilon \in \mathbb{R}$ can be chosen in such a way that $w_\varepsilon \in \mathcal{A}_V$. If $x \in (\delta_\varepsilon - \varepsilon, \delta_\varepsilon + \varepsilon) \times \mathbb{R}^{n-1} \cap \Omega$ then

$$|Xw_\varepsilon(x)|^2 = \sum_{j=1}^m (X_j w_\varepsilon(x))^2 = \frac{1}{4\varepsilon^2}\sum_{j=1}^m (c_{j1}(x))^2 \leq C/\varepsilon^2.$$

Moreover $W(w_\varepsilon) \leq \sup_{t\in[0,1]} W(t)$ and thus

$$\begin{aligned} G_\varepsilon(w_\varepsilon) &= \int_{\Omega\cap\{\delta_\varepsilon-\varepsilon<x_1<\delta_\varepsilon+\varepsilon\}} \Big(\varepsilon|Xw_\varepsilon|^2 + \frac{1}{\varepsilon}W(w_\varepsilon)\Big)dx \\ &\leq \frac{C}{\varepsilon}|\Omega\cap\{\delta_\varepsilon - \varepsilon < x_1 < \delta_\varepsilon + \varepsilon\}| \leq C < +\infty. \end{aligned}$$

This proves that $G_\varepsilon(u_\varepsilon) \leq C < +\infty$ for all $\varepsilon > 0$.

Since the set $\{v_\varepsilon \in L^1(\Omega) : \varepsilon > 0\}$ is relatively compact there exist $v \in L^1(\Omega)$ and $\varepsilon_h \downarrow 0$ such that $v_{\varepsilon_h} \to v$ in $L^1(\Omega)$. The function φ is strictly increasing and thus there exists $\psi = \varphi^{-1} \in C^1(\mathbb{R})$. Define $u(x) := \psi(v(x))$ and notice that $u_{\varepsilon_h} = \psi(v_{\varepsilon_h})$. Arguing as in [41] we finally get $u_{\varepsilon_h} \to u$ in $L^1(\Omega)$. □

Let V and $\mathcal{A}_V$ be as in (40) and let Q_ε be the functionals defined in (64). The second result of this section deals with the compactness of Q_ε's minimizers.

Theorem 4.5. *Let Ω be a connected, bounded open set, let $A(x)$ be a symmetric matrix of functions on $\mathbb{R}^n$ and let $Y = (Y_1, ..., Y_r)$ be a family of Lipschitz continuous vector fields on $\mathbb{R}^n$ satisfying the connectivity hypothesis* (Yc). *Assume that:*

(i) *$A(x)$ has entries of class $C^2(\mathbb{R}^n)$ and satisfies* (30)*;*
(ii) *$\langle A(x)\xi, \xi\rangle \geq \sum_{j=1}^r \langle Y_j(x), \xi\rangle^2$ for all $x, \xi \in \mathbb{R}^n$;*
(iii) *the compact embeddings $(\mathcal{C})_1$ and $(\mathcal{C})_2$ hold with $X \equiv Y$ relatively to Ω;*
(iv) *the function W in the functional Q_ε satisfies* (39) *and* (66).

Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$. Then every sequence $(u_h)_{h\in\mathbb{N}}$ of ε_h−minimizers of Q_ε (i.e. $Q_{\varepsilon_h}(u_h) \leq \inf_{u\in\mathcal{A}_V} Q_{\varepsilon_h}(u) + \varepsilon_h$) is relatively compact in $L^1(\Omega)$.

Remark 4.6. The requirement on A to be of class C^2 is necessary in order to assure the factorization $A = C^T C$ as in Lemma 2.2. Actually, assumption (i) in Theorem 4.5 can be weakened requiring only $A(x) = C(x)^T C(x)$ for all $x \in \Omega$ for some $m \times n$ matrix $C(x)$ with Lipschitz continuous entries (see also Remark 3.6).

Let Q be the functional defined in (65). Choosing $M = L^1(\Omega)$, $F_h = Q_{\varepsilon_h}$ and $F = Q$ from Theorem 4.1 and Theorem 4.5 we get the following Corollary.

Corollary 4.7. *Let Ω, A and Y be as in Theorem 4.5. Assume that Ω has C^2 boundary and that W satisfies* (39) *and* (66)*. Let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$. Then:*

(i) *there exists* $\displaystyle\min_{u\in L^1(\Omega)} Q(u) = \lim_{h\to\infty} \inf_{u\in L^1(\Omega)} Q_{\varepsilon_h}(u)$;
(ii) *if $(u_h)_{h\in\mathbb{N}}$ is a sequence of ε_h−minimizers of $(Q_{\varepsilon_h})_{h\in\mathbb{N}}$ then there exist a subsequence $(u_{h_j})_{j\in\mathbb{N}}$ and a function $u_0 = \chi_E \in BV_A(\Omega)$ such that $u_{h_j} \to u_0$ in $L^1(\Omega)$ and $Q(u_0) = \min_{u\in L^1(\Omega)} Q(u)$.*

Proof of Theorem 4.5. By assumption (i) Lemma 2.2 can be applied and arguing as in the proof of Theorem 3.5 we conclude that

$$Q_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\varepsilon|Xu|^2 + \frac{1}{\varepsilon}W(u)\Big)\,dx & \text{if } u \in C^1(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

for a suitable family $X = (X_1, ..., X_n)$ of Lipschitz continuous vector fields. Moreover, for every $\varepsilon > 0$ and for all $u \in \mathrm{L}^1(\Omega)$

$$\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon(u) = G_\varepsilon(u),$$

being $\mathrm{sc}^-(\mathrm{L}^1(\Omega))Q_\varepsilon$ the relaxed functional of Q_ε with respect to the $\mathrm{L}^1(\Omega)$ topology and G_ε the functional defined in (41).

On the other hand, by assumptions (ii) X can be assumed to satisfy (Xc), and by (iii) $(\mathcal{C})_1$ and $(\mathcal{C})_2$ can be assumed to hold relatively to X and Ω. Theorem 4.3 can be applied. As pointed out in the first part of the proof of Theorem 4.3 G_ε is coercive with respect to the $\mathrm{L}^1(\Omega)$ topology and from a well-known result of relaxation theory (see, for instance, [19, Theorem 3.8]) there exists

$$\min_{u\in \mathrm{L}^1(\Omega)} G_{\varepsilon_h}(u) = \inf_{u\in \mathrm{L}^1(\Omega)} Q_{\varepsilon_h}(u).$$

The thesis follows. □

5. Examples and applications

In this section we give some important examples of families of vector fields to which our results of Sections 3 and 4 apply. Moreover, we study in detail a couple of examples that often play a paradigmatic role in the theory of cc spaces.

Example 5.1 (Hörmander vector fields). Let $X = (X, ..., X_m)$ with $X_j \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ and denote by $\mathcal{L}(X_1, ..., X_m)$ the Lie algebra generated by such vector fields by iterated commutators. If the Chow-Hörmander condition

$$\operatorname{rank} \mathcal{L}(X_1, ..., X_m)(x) = n \quad \text{for every } x \in \mathbb{R}^n, \tag{74}$$

holds, then X satisfies (Xc). Vector fields of this type were introduced in [36] and a deep study of the induced cc metric can be found in [46].

Example 5.2 (Grushin's type vector fields). Let $X = (X_1, ..., X_n)$ and $X_j = \lambda_j(x)\partial_j$, $j = 1, ..., n$ with $\lambda_j \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$, $\lambda_j \geq 0$. Assume that:

(i) $\lambda_1 \equiv 1$ and $\lambda_j(x) = \lambda_j(x_1, ..., x_{j-1})$, $j = 2, ..., n$;
(ii) $\lambda_j \in C^1(\mathbb{R}^n \setminus \Pi_j)$ where $\Pi_j = \{x \in \mathbb{R}^n : x_1 \cdot ... \cdot x_{j-1} = 0\}$;
(iii) $\lambda_j(x) = \lambda_j(x_1, ..., |x_k|, ..., x_{j-1})$ for all $k = 1, ..., j-1$ and $j = 2, ..., n$;
(iv) there exist positive constants α_{jk} such that $0 \leq x_k\partial_k\lambda_j(x) \leq \alpha_{jk}\lambda_j(x)$ for all $x \in \mathbb{R}^n \setminus \Pi_j$.

The vector fields X satisfy (Xc). Vector fields of this type were introduced in [25], [26], [23]. In the special case $\lambda_1 = ... = \lambda_r = 1$, $\lambda_{r+1} = ... = \lambda_n \equiv \lambda$ they were studied in [24] even under less restrictive assumptions on the regularity of λ. This class can be considered as a "weak-extension" to the non smooth case of Hörmander vector fields given in Example 5.1.

We introduce some basic notions on regular domains in metric spaces (see [18] for an account of recent results in this argument and see also [51]). The following definition gives a generalization of the well known "interior cone property" of domains in Euclidean spaces to domains in a general metric space (see, for instance, [37, §9] and [17]).

Definition 5.3. Let (M, d) be a metric space. A bounded open set $\Omega \subset M$ is a *John domain* if there exist $x_0 \in \Omega$ and $C > 0$ such that for every $x \in \Omega$ there exists a continuous rectifiable curve parameterized by arclength $\gamma : [0, T] \to \Omega$, $T \geq 0$, such that $\gamma(0) = x$, $\gamma(T) = x_0$ and $\text{dist}(\gamma(t), \partial\Omega) \geq Ct$.

If B is a ball in the metric space (M, d) and $\lambda \geq 0$ with λB we denote the ball with same center as B and radius $\lambda-$times that of B. The following definition extends the "Boman chain condition" to metric spaces (see [30, Definition 1.4] and see also [37, §9]).

Definition 5.4. Let (M, d) be a metric space and μ a positive Borel measure on M. A bounded open set $\Omega \subset M$ is a *Boman domain* if there exists a covering $\{B : B \in \mathcal{F}\}$ of Ω with balls, and there exist $N \geq 1$, $\lambda > 1$ and $\nu \geq 1$ such that:

(i) $\lambda B \subset \Omega$ for all $B \in \mathcal{F}$;
(ii) every point of Ω belongs to at most N balls λB with $B \in \mathcal{F}$;
(iii) there exists a central ball $B_0 \in \mathcal{F}$ such that for any $B \in \mathcal{F}$ there exists a chain of balls $B_1, ..., B_k$ such that $B_k = B$, $B_i \cap B_{i+1} \neq \emptyset$, $\mu(B_i \cap B_{i+1}) \geq 1/N \max\{\mu(B_i), \mu(B_{i+1})\}$ and $B \subset \nu B_i$ for all $i = 0, 1, ..., k$.

Definitions 5.3 and 5.4 turn out to identify the same class of domains in homogeneous metric spaces with geodesics (see [15], [30, Theorem 1.30] and [37, Proposition 9.6]).

Theorem 5.5. *Let (M, d) be a metric space endowed with a positive Borel measure μ. Assume that:*

(i) every couple of points can be connected by a geodesic;
(ii) there exists a constant $\delta > 0$ such that $0 < \mu(B(x, 2r)) \leq \delta\mu(B(x, r)) < +\infty$ for all $x \in M$ and $r \geq 0$.

Then, the class of John domains equals that of Boman domains.

In the examples we shall consider condition (i) is true, and condition (ii) is also true choosing μ to be the Lebesgue measure. Boman domains are of special interest because of the following Compactness Theorem which is proved in [30]. The metric space is a cc space $(\mathbb{R}^n, d)$ endowed with Lebesgue measure.

Theorem 5.6. *Let $(\mathbb{R}^n, d)$ be the cc space induced by a family $X = (X_1, ..., X_m)$ of Hörmander or Grushin's type vector fields (see Examples 5.1 and 5.2). If $\Omega \subset \mathbb{R}^n$ is a Boman domain then for all $1 \leq p < +\infty$ the embedding $\mathrm{H}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ is compact.*

Remark 5.7. Theorem 5.6 may fail if Ω is not a Boman domain in $(\mathbb{R}^n, d)$. Indeed, it is known that even for open sets $\Omega \subset \mathbb{R}^n$ with boundary of class C^∞ the Poincaré inequality (72) is no longer true (see, for instance, [35] and Remark 5.18).

From Theorems 5.6 and 4.5, Remark 4.6 and Corollary 4.7 we get the following result. Let Q_ε, Q be as in (64) and (65) and let W be a function which satisfies (39) and (66).

Theorem 5.8. *Let Ω be a connected, bounded open set of class C^2, let $A(x)$ be a matrix of functions on $\mathbb{R}^n$ and let $Y = (Y_1, ..., Y_r)$ be a family of Hörmander or Grushin's type vector fields on $\mathbb{R}^n$. Assume that:*

(i) *$A(x) = C^T(x)C(x)$ for all $x \in \Omega$ where $C(x)$ is a $m \times n$ matrix with Lipschitz continuous entries on $\mathbb{R}^n$;*

(ii) *$\langle A(x)\xi, \xi\rangle \geq \sum_{j=1}^r \langle Y_j(x), \xi\rangle^2$ for all $x, \xi \in \mathbb{R}^n$;*

(iii) *Ω is a Boman domain in $(\mathbb{R}^n, d)$, where d is the cc metric induced by the family of vector fields Y.*

If $(u_h)_{h\in\mathbb{N}}$ is a sequence of ε_h-minimizers of Q_{ε_h} ($Q_{\varepsilon_h}(u_h) \leq \inf_{u\in\mathcal{A}_V} Q_{\varepsilon_h}(u) + \varepsilon_h$ with $\varepsilon_h \downarrow 0$) then there exists a subsequence $(u_{h_j})_{j\in\mathbb{N}}$ and a function $u_0 = \chi_E \in \mathrm{BV}_A(\Omega)$ such that $u_{h_j} \to u_0$ in $\mathrm{L}^1(\Omega)$ and $Q(u_0) = \min_{u\in\mathrm{L}^1(\Omega)} Q(u)$.

We shall now study in detail two examples of vector fields which are respectively of Hörmander and Grushin's type. In particular, we shall see that in these cases a suitable Euclidean regularity of the domain Ω also provides its intrinsic regularity with respect to the induced cc metric (see Theorems 5.10 and 5.17).

The first example is the *Heisenberg group*, a Lie group whose origins can be found in quantum mechanics (see [50, §11]). The quadratic form associated with the Heisenberg vector fields is degenerate at every point of the manifold.

The second example is the so called *Grushin's space* where the degeneration of the quadratic form is concentrated on a small set but the coefficients of the vector fields may not be regular of class C^k for $k \geq 1$. Moreover, there is no Lie structure compatible with the cc metric of the Grushin space (see also Remark 5.19).

Example 5.9 (Heisenberg group). In $\mathbb{R}^{2n+1}$ we shall write the coordinates $(x, y, t) \in \mathbb{R}^{2n+1}$ with $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. The Heisenberg vector fields are

$$X_j = \partial_{x_j} + 2y_j\partial_t, \quad Y_j = \partial_{y_j} - 2x_j\partial_t \quad j = 1, ..., n, \tag{75}$$

which satisfy the commutation relations

$$\begin{array}{lll} [X_i, X_j] = 0 \quad \text{and} \quad [Y_i, Y_j] = 0, & & \text{for all } i, j = 1, ..., n, \\ [X_i, Y_j] = 0, & & \text{for all } i, j = 1, ..., n, \ i \neq j \\ [X_i, Y_i] = -4\partial_t & & \text{for all } i = 1, ..., n. \end{array} \tag{76}$$

The vector fields $X_1, ..., X_n, Y_1, ..., Y_n$ are a system of generators of the left invariant Lie algebra of $\mathbb{R}^{2n+1}$ when endowed with the Lie group product

$$(x, y, t) \cdot (\xi, \eta, \tau) = (x + \xi, y + \eta, t + \tau + 2(\langle y, \xi\rangle - \langle x, \eta\rangle)). \tag{77}$$

The group $(\mathbb{R}^{2n+1}, \cdot)$ is usually called the Heisenberg group and denoted by $\mathbb{H}^n$. This group is homogeneous in the sense that it admits a one parameter family of automorphisms $\delta_\lambda : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$, $\lambda > 0$, given by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. Lebesgue measure is the Haar measure of the Heisenberg group. Moreover, for any measurable set $E \subset \mathbb{R}^{2n+1}$ and $\lambda > 0$ we have $|\delta_\lambda(E)| = \lambda^N|E|$ with $N := 2n+2$. Here $|\cdot|$ stands for the $2n+1$-dimensional Lebesgue measure on $\mathbb{R}^{2n+1}$. The integer N is called the homogeneous dimension of $\mathbb{H}^n$. The vector fields (75) satisfy Hörmander condition (74) and therefore induce on $\mathbb{R}^{2n+1}$ a

Carnot-Carathéodory metric d verifying (Xc). $(\mathbb{R}^{2n+1}, d)$ is a metric space with Hausdorff dimension equal to N (see [40]).

The Heisenberg gradient is $\nabla_{\mathbb{H}} = (X_1, ..., X_n, Y_1, ..., Y_n)$, and if $f \in C^1(\mathbb{R}^{2n+1})$ we can write

$$|\nabla_{\mathbb{H}} f|^2 = \sum_{j=1}^{n} (X_j f)^2 + (Y_j f)^2 = \langle B\nabla f, \nabla f\rangle,$$

where B is the $(2n+1)-$square matrix with variable entries

$$B(x,y,t) = \begin{pmatrix} I_n & 0 & 2y^T \\ 0 & I_n & -2x^T \\ 2y & -2x & 4(|x|^2+|y|^2) \end{pmatrix}, \tag{78}$$

and I_n is the identity $(n \times n)-$matrix. Notice that $\det(B(x,y,t)) = 0$ for all $(x,y,t) \in \mathbb{R}^{2n+1}$: the degeneration of the quadratic form B is distributed at every point of the space.

Let now $\Omega \subset \mathbb{R}^{2n+1}$ be an open set. According to definition (19) denote by $\mathrm{H}^1_{\mathbb{H}}(\Omega) = \mathrm{H}^1_X(\Omega)$ the Sobolev space associated with the family of vector fields $X = \nabla_{\mathbb{H}}$.

Examples of Boman domains in the Heisenberg group are provided in [30] and [17]. The following Theorem proved in [44] and Theorem 5.5 give a general sufficient condition for a domain to be Boman.

Theorem 5.10. *Let $\Omega \subset \mathbb{R}^{2n+1}$ be a bounded open set with boundary of class C^2. Then Ω is a John domain.*

Remark 5.11. There are open sets which are John domains without being of class C^2. Examples are Carnot-Carathódory balls which are always John domains in general cc spaces (see [30]).

Let $E \subset \mathbb{H}^n$ be a measurable set. According to definition (23) denote by $|\partial E|_{\mathbb{H}}(\Omega) := |\partial E|_X(\Omega)$ and by $\mathrm{BV}_{\mathbb{H}}(\Omega) = \mathrm{BV}_X(\Omega)$ respectively the Heisenberg perimeter of E and the space of the functions with bounded variation associated with the family of vector fields $X = \nabla_{\mathbb{H}}$.

Remark 5.12. The space $\mathrm{BV}(\Omega)$, i.e. the space of the functions with bounded variation in Ω, is strictly contained in $\mathrm{BV}_{\mathbb{H}}(\Omega)$ (see [29, Example 1]).

The measure of a surface in the Heisenberg group can also be computed by means of suitable Hausdorff measures. Define the $(N-1)-$dimensional spherical measure of a set $A \subset \mathbb{R}^{2n+1}$ as

$$\mathcal{S}^{N-1}_d(A) = \lim_{\delta\downarrow 0} \inf\Big\{\gamma(N-1)\sum_{i=1}^{+\infty}(\mathrm{diam}(B_i))^{N-1} : A \subset \bigcup_{i=1}^{+\infty} B_i,\ \mathrm{diam}(B_i) \le \delta,\ B_i \subset \mathbb{R}^{2n+1}\Big\},$$

where $\gamma(N-1)$ is a geometric constant, B_i are closed balls in $(\mathbb{R}^{2n+1}, d)$ and $\mathrm{diam}(B_i)$ is the diameter of B_i with respect to d. We already noticed that the metric space $(\mathbb{R}^{2n+1}, d)$ has Hausdorff dimension $N = 2n+2$ and thus $N-1$ is the correct "surface dimension". The link between perimeter and spherical Hausdorff measure is given in the following theorem first proved in [29] when the Heisenberg group is equipped with a metric equivalent to the cc metric d. Later, in [39] the same result was obtained for the cc metric d.

Theorem 5.13. *Let $E \subset \mathbb{R}^{2n+1}$ be a measurable set and let $\Omega \subset \mathbb{R}^{2n+1}$ be an open set. Assume that $|\partial E|_{\mathbb{H}}(\omega) < +\infty$ for every open set $\omega \Subset \Omega$. Then there exists a Borel set $\partial^*_{\mathbb{H}} E \subset \partial E$ (called the $\mathbb{H}-$reduced boundary of E in Ω) such that $|\partial E|_{\mathbb{H}}(A) = \mathcal{S}_d^{N-1}(\partial^*_{\mathbb{H}} E \cap A)$ for every Borel set $A \subset \Omega$. Moreover, if E is an open set with boundary of class C^1 then $|\partial E|_{\mathbb{H}}(\Omega) = \mathcal{S}_d^{N-1}(\partial E \cap \Omega)$.*

We finally come to the applications of the results obtained in Section 3 and Section 4. Let $W \in C^2(\mathbb{R})$ be a function which satisfies (39) and (66) and let V and A_V be as in (40). For $\varepsilon > 0$ consider the functionals $G_\varepsilon, Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$

$$G_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\varepsilon|\nabla_{\mathbb{H}} u|^2 + \frac{1}{\varepsilon} W(u)\Big)\, dxdydt & \text{if } u \in \mathrm{H}^1_{\mathbb{H}}(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$Q_\varepsilon(u) = \begin{cases} \displaystyle\int_\Omega \Big(\varepsilon\langle ADu, Du\rangle + \frac{1}{\varepsilon} W(u)\Big)\, dxdydt & \text{if } u \in C^1(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

where $A = A(x, y, t)$ is a symmetric, non negative definite $(n \times n)-$matrix of C^2 functions on $\mathbb{R}^n$ verifying (30) and for some positive constant $C > 0$

$$\langle A(x, y, t)\xi, \xi\rangle \geq C\langle B(x, y, t)\xi, \xi\rangle \quad \text{for all } (x, y, t), \xi \in \mathbb{R}^{2n+1},$$

and B is the matrix (78).

Analogously, $G, Q : \mathrm{L}^1(\Omega) \to [0, +\infty]$ are the functionals defined by

$$G(u) = \begin{cases} 2\alpha|\partial E|_{\mathbb{H}}(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_{\mathbb{H}}(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

and

$$Q(u) = \begin{cases} 2\alpha|\partial E|_A(\Omega) & \text{if } u = \chi_E \in \mathrm{BV}_A(\Omega) \cap \mathcal{A}_V \\ +\infty & \text{otherwise,} \end{cases}$$

where $|\partial E|_A$ is the perimeter measure defined in (11) and $\alpha = \int_0^1 \sqrt{W(s)}\, ds$.

Then from Theorems 3.1, 3.5, 4.3, 4.5, 5.6, 5.10 and from Corollaries 4.4 and 4.7 we get at once the following result.

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^{2n+1}$ be a connected, bounded open set of class C^2. Then:*

- *(i)* $\Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon\downarrow 0} G_\varepsilon = G$;
- *(ii)* $\Gamma(\mathrm{L}^1(\Omega))\text{-}\lim_{\varepsilon\downarrow 0} Q_\varepsilon = Q$;
- *(iii) for all $\varepsilon > 0$ there exists $u_\varepsilon \in \mathrm{H}^1_{\mathbb{H}}(\Omega) \cap \mathcal{A}_V$ such that $G_\varepsilon(u_\varepsilon) = \min\limits_{u \in \mathrm{L}^1(\Omega)} G_\varepsilon(u)$.*

Moreover, let $(\varepsilon_h)_{h\in\mathbb{N}}$ be a sequence of real numbers such that $\varepsilon_h \downarrow 0$. Then:

- *(iv) the sequence $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ is relatively compact in $\mathrm{L}^1(\Omega)$;*

(v) *any sequence* $(v_{\varepsilon_h})_{h\in\mathbb{N}}$ *of* ε_h-*minimizers of* $(Q_{\varepsilon_h})_{h\in\mathbb{N}}$ *is relatively compact in* $\mathrm{L}^1(\Omega)$;

(vi) *there exist a subsequence* $(u_{\varepsilon_{h_j}})_{j\in\mathbb{N}}$ *of* $(u_{\varepsilon_h})_{h\in\mathbb{N}}$ *and a function* $u_0 = \chi_E \in \mathrm{BV}_{\mathbb{H}}(\Omega)$ *such that* $u_{\varepsilon_{h_j}} \to u_0$ *in* $\mathrm{L}^1(\Omega)$ *and* $G(u_0) = \min_{u\in\mathrm{L}^1(\Omega)} G(u)$;

(vii) *there exist a subsequence* $(v_{\varepsilon_{h_j}})_{j\in\mathbb{N}}$ *of* $(v_{\varepsilon_h})_{h\in\mathbb{N}}$ *and a function* $v_0 = \chi_E \in \mathrm{BV}_A(\Omega)$ *such that* $v_{\varepsilon_{h_j}} \to v_0$ *in* $\mathrm{L}^1(\Omega)$ *and* $Q(v_0) = \min_{v\in\mathrm{L}^1(\Omega)} Q(v)$.

Example 5.15 (Grushin vector fields). In this example we shall study a particular case of the vector fields introduced in Example 5.2.

In $\mathbb{R}^n$ we shall write the coordinates (x, y) with $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. Consider the vector fields

$$X_1 = \partial_{x_1}, ..., X_{n-1} = \partial_{x_{n-1}}, X_n = |x|^\beta \partial_y, \tag{79}$$

where $\beta \geq 1$ is a fixed real parameter. If β is not an even integer the Hörmander condition (74) can not be applied because the vector fields are not smooth. Nevertheless, according to Example 5.2 the vector fields induce on $\mathbb{R}^n$ a well defined cc metric d that was first introduced in [25]. The estimates proved in this paper show that the distance d satisfies hypothesis (Xc).

The Grushin gradient is $X = (X_1, ..., X_n)$. If $f \in C^1(\mathbb{R}^n)$ we can write

$$\begin{aligned} |Xf(x,y)|^2 &= \sum_{i=1}^n |X_i f(x,y)|^2 = |\nabla_x f(x,y)|^2 + |x|^{2\beta} |\partial_y f(x,y)|^2 \\ &= \langle B(x)\nabla f(x,y), \nabla f(x,y)\rangle, \end{aligned}$$

where B is the $(n \times n)-$matrix

$$B(x) = \begin{pmatrix} I_{n-1} & 0 \\ 0 & |x|^{2\beta} \end{pmatrix}. \tag{80}$$

Clearly, $\det(B(x)) = |x|^{2\beta}$ is zero when $x = 0$.

If $\Omega \subset \mathbb{R}^n$ is an open set the Sobolev space $\mathrm{H}^{1,p}_X(\Omega)$ is defined as in (19). Recall the Definition 5.4 of Boman domain and the Definition 5.3 of John domain. Here the metric space is $(\mathbb{R}^n, d)$ where d is the cc metric induced by the vector fields (79), and we put on $\mathbb{R}^n$ the Lebesgue measure. According to Theorem 5.5 and Theorem 5.6 if $\Omega \subset \mathbb{R}^n$ is a John domain then the embedding $\mathrm{H}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$ is compact. An answer to the problem of finding Boman domains in $(\mathbb{R}^n, d)$ is given in [45]. We introduce a definition.

Definition 5.16. Let $\Omega \subset \mathbb{R}^n$ be a connected open set with Lipschitz boundary such that $\partial\Omega$ is of class C^1 in a neighborhood of every point $(0, y) \in \partial\Omega$.

A point $(0, y) \in \partial\Omega$ will be said *flat* if there exist a neighborhood $\mathcal{V}$ of $(0, y)$ and a neighborhood $\mathcal{U}$ of the origin in $\mathbb{R}^{n-1}$ such that $\partial\Omega \cap \mathcal{V} = \{(x, \varphi(x)) : x \in \mathcal{U}\}$ for some $\varphi \in C^1(\mathcal{U}; \mathbb{R})$ with $\nabla\varphi(0) = 0$. A flat point $(0, y) \in \partial\Omega$ will be said $\beta-$*admissible* if there exists a constant $C > 0$ such that

$$|\nabla\varphi(x)| \leq C|x|^\beta \quad \text{for all } x \in \mathcal{U}. \tag{81}$$

Finally, Ω will be said $\beta-$*admissible* if flat points in $\partial\Omega$ are $\beta-$admissible or if Ω has no flat points.

For example, the cube $I = \{(x,y) \in \mathbb{R}^n : |y|, |x_i| < 1, \quad \text{for } i = 1, ..., n-1\}$ is $\beta-$admissible for all $\beta > 0$. Condition (81) states that in a neighborhood of the singular line $\{(x,y) \in \mathbb{R}^n : x = 0\}$ the boundary $\partial\Omega$ is suitably flat in connection with the power of degeneration of the quadratic form $B(x)$.

The following theorem, which is a special case of the results proved in [45], and Theorem 5.5 show that $\beta-$admissible domains support the compact embedding $\mathrm{H}^{1,p}_X(\Omega) \hookrightarrow \mathrm{L}^p(\Omega)$.

Theorem 5.17. *If $\Omega \subset \mathbb{R}^n$ is a $\beta-$admissible domain then it is a John domain.*

Remark 5.18. If Ω is a $\beta-$admissible domain then by Theorems 5.17, 5.5, 5.6 and by Proposition 4.2 it supports the Poincaré inequality (72) for all $1 \le p < +\infty$. Fix $n = 2$, $\beta = 3$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 < y < 1\}$. Then Ω is of class C^∞ in a neighborhood of the origin but it is not $\beta-$admissible. Taking $u(x,y) = y^{-3/4}$ it can be easily checked that

$$\int_\Omega |Xu|^2\, dxdy = \int_\Omega |x|^{2\beta} |\partial_y u|^2\, dxdy < +\infty \qquad \text{but} \qquad \int_\Omega |u|^2\, dxdy = +\infty,$$

and the Poincaré inequality (72) with $p = 2$ does not hold (see [35]).

The space $\mathrm{BV}_X(\Omega)$ of the function with bounded $X-$variation is defined as in (21) and (22). As usual, $|\partial E|_X(\Omega)$ denotes the $X-$perimeter of a measurable set E. If $E \subset \mathbb{R}^n$ has Lipschitz boundary then by (25)

$$|\partial E|_X(\Omega) = \int_{\partial E \cap \Omega} |Xn|\, d\mathcal{H}^{n-1},$$

where

$$|Xn(x,y)| := \Big(|n_x(x,y)|^2 + |x|^{2\beta} |n_y(x,y)|^2\Big)^{1/2}$$

and $n = (n_x, n_y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is the unit normal to ∂E which is defined $\mathcal{H}^{n-1}-$almost everywhere on ∂E.

Remark 5.19. Differently from the Heisenberg group (see Theorem 5.13) the metric space $(\mathbb{R}^n, d)$ has not a metric dimension constant at every point and a representation of the perimeter in terms of a unique intrinsic Hausdorff measures is not available. Indeed, let $n = 2$ and $X = (X_1, X_2)$ with $X_1 = \partial_x$ and $X_2 = x\partial_y$. Then it is easy to see that the Hausdorff dimension of $(\mathbb{R}^2, d)$ is $N = 2$. The set $E = \{(x,y) \in \mathbb{R}^2 : 0 < x, y < 1\}$ has Lipschitz boundary and $|\partial E|_X(\mathbb{R}^2) < +\infty$. On the other hand, it is easy to see that the Hausdorff dimension of $\partial E \cap \{(x,y) \in \mathbb{R}^n : x = 0\}$ is 2 whereas the one of $\partial E \cap \{(x,y) \in \mathbb{R}^n : x > 0\}$ is 1 (see also [43]).

We finally come to the applications of the main results of Section 3 and Section 4 to this example. Let W, V, $\mathcal{A}_V$ and α be as in Example 5.9. The functionals $G, G_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ are as in (41) but considering the Grushin vector fields $X = (\partial_{x_1}, ..., \partial_{x_{n-1}}, |x|^\beta \partial_y)$. Let $A(x,y)$ be a matrix as in Example 5.9 such that

$$\langle A(x,y)\xi, \xi\rangle \ge C\langle B(x)\xi, \xi\rangle \quad \text{for some } C > 0 \text{ and for all } (x,y), \xi \in \mathbb{R}^n,$$

where $B(x)$ is the matrix (80). Let $Q, Q_\varepsilon : \mathrm{L}^1(\Omega) \to [0, +\infty]$ be defined as above, as well.

Theorem 5.20. *Let $\Omega \subset \mathbb{R}^n$ be a connected bounded open set of class C^2 and assume that it is $\beta-$admissible. Then all statements* (i) − (vii) *of Theorem 5.14 hold replacing* $\mathrm{H}^1_{\mathbb{H}}(\Omega)$ *with* $\mathrm{H}^1_X(\Omega)$ *and* $\mathrm{BV}_{\mathbb{H}}(\Omega)$ *with* $\mathrm{BV}_X(\Omega)$.

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