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Summability of Solutions of Elliptic and Parabolic Quasilinear Equations with Measures as Data

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In this paper, we give a result of summability of solution of elliptic and parabolic boundary and initial-boundary value problems associated with an operator of Leray-Lions type when the right-hand side and the initial condition are two bounded Radon measures. This work generalizes the results of papers by Boccardo-Gallouët [5] and Li Feng-Quan & Li Guang-Wei [6].

Keywords: Summability, Quasilinear elliptic and parabolic equations, Measures as data

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1. Introduction

Let Ω a bounded open subset of $\mathbb{R}^N$, $N \geq 2$. For $T > 0$, we denote Q the cylinder $\Omega \times]0, T[$. Also, we denote by $M(\Omega)$ ($M(Q)$ respectively) the space of bounded Radon measures in Ω (Q respectively). We consider an $\mathbb{R}^N$ valued function $\hat{a}$ defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N$ or in $Q \times \mathbb{R} \times \mathbb{R}^N$ which is of Carathéodory and satisfying, a.e. (x, t) in Q and $\forall s \in \mathbb{R}$ and $\forall \xi, \xi' \in \mathbb{R}^N$, the following:

$$\hat{a}(x,s,\xi)\cdot\xi \geq \alpha_0|\xi|^p \quad (\text{or} \quad \hat{a}(x,t,s,\xi)\cdot\xi \geq \alpha_0|\xi|^p) \tag{1}$$

$$|\hat{a}(x,s,\xi)| \leq \alpha\{|s|^{p-1} + |\xi|^{p-1} + a_0(x)\} \tag{2}$$
$$(\text{or } |\hat{a}(x,t,s,\xi)| \leq \alpha\{|s|^{p-1} + |\xi|^{p-1} + a_0(x,t)\})$$

$$[\hat{a}(x,s,\xi) - \hat{a}(x,s,\xi')]\cdot[\xi - \xi'] > 0 \tag{3}$$
$$(\text{or} \quad [\hat{a}(x,s,t,\xi) - \hat{a}(x,s,t,\xi')]\cdot[\xi - \xi'] > 0), \ \xi \neq \xi'.$$

Where p is a real number such that $2 - 1/N < p < N$, in the elliptic case and such that $2 - N/(N+1) < p$, in the parabolic one. Here, as usual, α_0 and α are positive real number. The function a_0 is positive and belonging to $L^{p'}(\Omega)$ or to $L^{p'}(Q)$, $p' = p/(p-1)$.

We are interested in the problems:

$$(E) \qquad \begin{cases} -\operatorname{div}(\hat{a}(x,u,\nabla u)) = \mu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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and

$$(P) \qquad \begin{cases} u' - \operatorname{div}(\hat{a}(x,t,u,\nabla u)) &= \mu \quad \text{in} \quad Q \\ u &= 0 \quad \text{on} \quad \partial\Omega\times]0,T[\\ u(\cdot,0) &= u_0 \quad \text{in} \quad \Omega, \end{cases}$$

where $\mu \in M(\Omega)$, in the first problem, and $\mu \in M(Q)$ and $u_0 \in M(\Omega)$, in the second one.

The Problem (E), see [3], [4] or [10], with the above restrictions on p has a solution u in the space $W_0^{1,q}(\Omega)$, $\forall q \in [1, q_0[$, where the critical exponent q_0 is given by $q_0 = N(p-1)/(N-1)$.

Also, the Problem (P) has a solution in the space $L^q(0,T;W_0^{1,q}(\Omega))$, $\forall q \in [1,\tilde{q}_0[$ with $\tilde{q}_0 = p - N/(N+1)$, see [11].

Recently the above regularity results were improved. In fact, in [5], the authors proved that $\nabla\Phi_0(u)$, with $\Phi_0(s) = s/\ln^\beta(2+|s|)$, $\forall \beta > 1/(p-1)$, belongs to the limiting space $W_0^{1,q_0}(\Omega)$. This result was extended to the parabolic case in [6], where it is proved that $\Phi_0(u)$ belongs to $L^{\tilde{q}_0}(0,T;(W_0^{1,\tilde{q}_0}(\Omega))$.

The aim of this paper is to prove for elliptic case that $\Phi(u) \in W_0^{1,q_0}(\Omega)$ and, for the parabolic one, that $\Phi(u) \in L^{\tilde{q}_0}(0,T;W_0^{1,\tilde{q}_0}(\Omega))$, where Φ is in a large class of functions, denoted in the sequel by $\mathcal{F}_p$, containing Φ_0 and functions increasing more rapidly than Φ_0 near infinity, this is in the sense that $\lim_{t\to\infty} \Phi_0(t)/\Phi_A(t) = 0$, see Section 2.

The rest of this paper will contains three sections. In the first of them, we presents the class $\mathcal{F}_p$, its relevant properties and give some important examples of function belonging to it. The others two will be devoted to the elliptic and parabolic Problems (E) and (P).

In this paper, the term "solution" means *weak solution* or solution in the sense of distributions.

Before giving our results, it seems of interest to make some remarks.

Remark 1.1. If the right-hand side member μ is in the Orlicz space $L\ln L(\Omega)$, then the Problem (E) has a solution in the limiting space $W_0^{1,q_0}(\Omega)$, see [4].

Remark 1.2. The Problem (E), considered in the framework of Orlicz spaces, with the following coerciveness condition:

$$\hat{a}(x,s,\xi)\cdot\xi \geq \alpha_0 B(|\xi|/\delta),\ \alpha_0 > 0,\ \delta > 0,$$

where B is an N-function such that $\int_d^\infty \frac{t^{p-1}}{B(t)}\,dt < +\infty$ $(d > 0)$, has a solution in the limiting space $W_0^{1,q_0}(\Omega)$. Note that the previous condition, which is more restrictive than (1), is not satisfied in the case of the p-Laplacian, see [2].

Remark 1.3. For $p \in]1, 2-1/N]$, the framework of Sobolev spaces is too narrow to contain the solution of (E). To overcome this difficulty, it is necessary to extend the framework and the notion of solution, see [1], [12].

2. The class $\mathcal{F}_p$

Definition 2.1. Let $p > 1$ a real number. The class $\mathcal{F}_p$ consists of odd functions of the form

$$\Phi_A(t) = \Big[\int_0^t \frac{d\tau}{\{A(\tau)\}^{1/p}}\Big]^{p'},\ t \geq 0, \quad p' = \frac{p}{p-1}, \tag{4}$$

where A is an even function, continuous in $\mathbb{R}$, positive on $]0,+\infty[$, and satisfies the properties:

(a) $\int_d^{+\infty} \frac{dt}{A(t)} < +\infty$, for some $d > 0$.

(b) $\lim_{t\to+\infty} \frac{A(t)}{t} = +\infty$.

(c) A has the following property of monotony: there exist two numbers $\beta > 0$ and $t_0 \geq 0$ such that $A(t_2) \geq \beta A(t_1)$, $\forall t_1, t_2$ with $t_2 > t_1 \geq t_0$.

(d) If $A(0) = 0$, it is also assumed that there exists a constant $C > 0$ such that $A(t) \geq C|t|$, $\forall t \in \mathbb{R}$ and $\Phi'_A(0)$ is finite.

Proposition 2.2. *Let $\Phi_A \in \mathcal{F}_p$. Then*

1. *Φ_A is solution of the differential equation*

$$A(t)[\Phi'_A(t)]^p = (p')^p|\Phi_A(t)|, \quad \forall t \in \mathbb{R}. \tag{5}$$

2. *There exists a constant $K > 0$ such that $|\Phi_A(t)| \leq K|t|$, $\forall t \in \mathbb{R}$.*

3. *Φ_A is globally Lipschitzian on $\mathbb{R}$.*

Proof. Deriving the relation (4), we get

$$\Phi'_A(t) = \frac{p'}{[A(t)]^{1/p}}\Big[\int_0^t \frac{d\tau}{\{A(\tau)\}^{1/p}}\Big]^{p'-1} = p'\Big[\frac{\Phi_A(t)}{A(t)}\Big]^{1/p}, \ t > 0$$

which implies the Equation (5).

For $A(0) \neq 0$, using Hölder inequality, we have $\Phi_A(t) \leq t[\int_0^t \frac{d\tau}{A(\tau)}]^{p'/p}$, $t \geq 0$. Thus, by symmetry, $|\Phi_A(t)| \leq K|t|$, $\forall t \in \mathbb{R}$, with $K \doteq (\int_0^\infty \frac{d\tau}{A(\tau)})^{p'/p}$.

If $A(0) = 0$, we use the Condition (d) to get $|\Phi_A(t)| \leq [\int_0^t \frac{d\tau}{(C\tau)^{1/p}}]^{p'} = \frac{(p')^{p'}}{C^{p'/p}} t$ for $t \geq 0$, which gives the same estimate.

To prove that the function Φ_A is globally Lipschitian, it suffices to write

$$|\Phi'_A(t)| = p'\Big|\frac{\Phi_A(t)}{A(t)}\Big|^{1/p} \leq p'K^{1/p}\Big|\frac{t}{A(t)}\Big|^{1/p} \leq C', \ \forall t \in \mathbb{R},$$

where K and C' are positive constants.

2.1. Examples of functions in $\mathcal{F}_p$

It is easy to see that $\Phi_A(t) = \ln^2(t + \sqrt{t^2+1})$ $(t \geq 0)$, the function determined by $A(t) = t^2+1$, is in $\mathcal{F}_2$. Also, it is easy to check that the function Φ_A determined by $A(t) = t$ for $0 \leq t \leq 1$ and $A(t) = t^\alpha$ for $t \geq 1$, with $1 < \alpha < p$, is in $\mathcal{F}_p$. Here Φ_A is equivalent at infinity to the function $Ct^{(p-\alpha)/(p-1)}$. The previous functions do not give any improvement for the smoothness of solution given in [3].

Let us now present some important examples which improve the smoothness of the solution. But, due to the difficulty to calculate the integrals involved, instead of giving the function A

and then determine Φ_A, we mention merely the functions Φ_A, where the index A is replaced by an numerical one.

The case studied by Boccardo-Gallouët corresponds to the function Φ_0 defined, for $t \geq 0$, by $\Phi_0(t) = t/[\ln^\alpha(\kappa_0 + t)]^{1/(p-1)}$, with $\kappa_0 = 2$ and $\alpha > 1$ a real number.

This last example is the weakest one of the family $\{\Phi_m\}_{m\in\mathbb{N}}$ where Φ_m is the function defined for $t \geq 0$ by $\Phi_m(t) = t/D_m(t)$ with

$$D_m(t) = \Big[\ln(\kappa_m + t)\ln(\ln(\kappa_m + t))\cdots\underbrace{\ln(\ln\cdots\ln}_{m\,\text{times}}(\kappa_m + t))\underbrace{\ln^\alpha(\ln\cdots\ln}_{m+1\,\text{times}}(\kappa_m + t))\Big]^{\frac{1}{p-1}},$$

where $\alpha > 1$ and κ_m is chosen greater enough to have $\underbrace{\ln(\ln\cdots\ln}_{m+1\,\text{times}}(\kappa_m)) > 0$. By direct calculation, one can see that the function Φ_m is in $\mathcal{F}_p$, $\forall m \in \mathbb{N}$. The function A_m associated to Φ_m is equivalent at the infinity to the function

$$Ct\ln(\kappa_m + t)\ln(\ln(\kappa_m + t))\cdots\underbrace{\ln(\ln\cdots\ln}_{m\,\text{times}}(\kappa_m + t))\underbrace{\ln^\alpha(\ln\cdots\ln}_{m+1\,\text{times}}(\kappa_m + t)),\ C = \text{const}.$$

Notice that $\lim_{t\to\infty} \Phi_i(t)/\Phi_j(t) = 0$, $\forall i < j$.

3. Summability of solution of Problem (E)

Theorem 3.1. *If Conditions* (1–3) *are satisfied, for $2 - 1/N < p < N$, the Problem* (E) *has a solution u such that*

$$\Phi_A(u) \in L^{q_0^\star}(\Omega) \quad \text{and} \quad \nabla\Phi_A(u) \in L^{q_0}(\Omega),\ \forall\Phi_A \in \mathcal{F}_p,$$

where $q_0 = N(p-1)/(N-1)$ and $q_0^\star = Nq_0/(N - q_0)$, the Sobolev exponent of q_0.

Proof. The proof needs four steps.

Step 1: Approximation. We replace the given Problem (E) by the family of approximate ones:

$$(E_k) \qquad \begin{cases} -\operatorname{div}(\hat{a}(x, u_k, \nabla u_k)) = \mu_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

where $\{\mu_k\}_{k=1}^\infty$ is a sequence of function in $L^\infty(\Omega)$ converging to μ in the weak-$\star$ topology of $M(\Omega)$ and there exists a positive constant C_0 such that $\|\mu_k\|_{L^1(\Omega)} \leq C_0$, $\forall k \geq 1$. The classical theory of monotone operators, see [7], shows that the problem (E_k) admits a solution in the space $W_0^{1,p}(\Omega)$.

Step 2: Uniform estimates on $\{\nabla u_k\}_{k=1}^\infty$.

Lemma 3.2. *Let A a function satisfying Conditions* (a)–(d) *of Definition 2.1. Then, there exists a constant $C_1 > 0$ such that*

$$\int_{\{|u_k|\geq n_0\}} \frac{|\nabla u_k|^p}{A(u_k)}\,dx \leq C_1,\ \forall k \geq 1, \tag{6}$$

where $\{|u_k| \geq n_0\} \doteq \{x \in \Omega \mid |u_k(x)| \geq n_0\}$ and $n_0 = [t_0] + 1$, $[t_0]$ being the integral part of t_0.

Proof of Lemma 3.2. For $n \in \mathbb{N}$, let φ_n be the real function defined on $\mathbb{R}$ by $\varphi_n(s) = \min\{(|s|-n)^+, 1\}\text{sign}\, s$, the superscript "+" stands for positive part, and put $B_n = \{n \leq |u_k| < n+1\}\cdot$ Taking $\varphi_n(u_k)$ as a test function in the variational formulation of Problem (E_k), using Condition (1), and noticing that $|\varphi_n| \leq 1$ and its derivative equals one on the set $[-n-1, -n] \cup [n, n+1]$ and zero elsewhere, we get:

$$\alpha_0 \int_{B_n} |\nabla u_k|^p \, dx \leq \int_\Omega |\mu_k| \, dx \leq C_0.$$

Putting now $n_0 = [t_0] + 1$ and using the monotony Condition (c), we can write

$$\begin{aligned} \int_{\{|u_k| \geq n_0\}} \frac{|\nabla u_k|^p}{A(u_k)} \, dx &= \sum_{n \geq n_0} \int_{B_n} \frac{|\nabla u_k|^p}{A(u_k)} \, dx \\ &\leq \sum_{n \geq n_0} \int_{B_n} \frac{|\nabla u_k|^p}{\beta A(n)} \, dx \leq \frac{C_0}{\alpha_0 \beta} \sum_{n \geq n_0} \frac{1}{A(n)} \doteq C_1. \end{aligned}$$

The convergence of this last series is a consequence of Conditions (a) and (c).

In the remainder of this paper, we denote by C_i, $i = 2, \ldots$, various positive constants.

Step 3: Uniform estimates on $\{\Phi_A(u_k)\}_{k=1}^\infty$ and $\{\nabla \Phi_A(u_k)\}_{k=1}^\infty$. Using Sobolev inequality, we can write

$$\begin{aligned} C_\Omega^{q_0} \Big(\int_\Omega |\Phi_A(u_k)|^{q_0^\star} \, dx \Big)^{q_0/q_0^\star} &\leq \int_\Omega |\nabla \Phi_A(u_k)|^{q_0} \, dx \\ &= \int_\Omega |\Phi_A'(u_k)|^{q_0} |\nabla u_k|^{q_0} \, dx = I_k^1 + I_k^2, \end{aligned}$$

where

$$I_k^1 = \int_{\{|u_k| < n_0\}} |\Phi_A'(u_k)|^{q_0} |\nabla u_k|^{q_0} \, dx \quad \text{and} \quad I_k^2 = \int_{\{|u_k| \geq n_0\}} |\Phi_A'(u_k)|^{q_0} |\nabla u_k|^{q_0} \, dx.$$

Here C_Ω^{-1} stands for a Sobolev constant. Ω being bounded, the estimate of the term $\int_{B_n} |\nabla u_k|^p \, dx$ seen in the proof of Lemma 3.2, the fact that $q_0 < p$, and the boundedness of the derivative of Φ_A', permit us to show that the sequence $\{I_k^1\}_{k=1}^\infty$ is bounded by a positive constant C_2.

To estimate the term I_k^2, we use Hölder inequality, Lemma 3.2, and Equation (5) to write:

$$\begin{aligned} I_k^2 &= \int_{\{|u_k| \geq n_0\}} \frac{|\nabla u_k|^{q_0}}{A(u_k)^{q_0/p}} |\Phi_A'(u_k)|^{q_0} A(u_k)^{q_0/p} \, dx \\ &\leq \Big[\int_{\{|u_k| \geq n_0\}} \frac{|\nabla u_k|^p}{A(u_k)} \, dx \Big]^{\frac{q_0}{p}} \Big[\int_{\{|u_k| \geq n_0\}} |\Phi_A'(u_k)|^{\frac{pq_0}{p-q_0}} A(u_k)^{\frac{q_0}{p-q_0}} \, dx \Big]^{1-\frac{q_0}{p}} \\ &\leq C_3 \Big[\int_{\{|u_k| \geq n_0\}} |\Phi_A(u_k)|^{\frac{q_0}{p-q_0}} \, dx \Big]^{1-\frac{q_0}{p}} \leq C_3 \Big[\int_\Omega |\Phi_A(u_k)|^{\frac{q_0}{p-q_0}} \, dx \Big]^{1-\frac{q_0}{p}}. \end{aligned}$$

Now, using the above estimates, we get

$$\Big(\int_\Omega |\Phi_A(u_k)|^{q_0^\star} \, dx \Big)^{q_0/q_0^\star} \leq C_4 + C_5 \Big[\int_\Omega |\Phi_A(u_k)|^{q_0^\star} \, dx \Big]^{1-\frac{q_0}{p}}, \ \forall k \in \mathbb{N}^\star.$$

Finally, because $\frac{q_0}{q_0^\star} = p - q_0 > \frac{p-q_0}{p} = 1 - \frac{q_0}{p}$, we deduce that

$$\int_\Omega |\Phi_A(u_k)|^{q_0^\star}\,dx \leq C_6 \quad \text{and} \quad \int_\Omega |\nabla\Phi_A(u_k)|^{q_0}\,dx \leq C_7,\ \forall k \in \mathbb{N}^\star. \tag{7}$$

Step 4: Passage to the limit. To this end, we use a pointwise compactness result (see [4] or [10]) on the sequence $\{u_k\}_{k=1}^\infty$ of approximate solutions and their gradients to construct a function u in $W_0^{1,q}(\Omega)$, $\forall q \in [1, q_0[$, solution of Problem (E). Using this convergence, we can make k goes to infinity in the estimate (7) to conclude that $\Phi_A(u) \in L^{q_0^\star}(\Omega)$ and $\nabla\Phi_A(u) \in L^{q_0}(\Omega)$.

Remark 3.3. For $p = N$, we have only $u \in W_0^{1,q}(\Omega)$, $\forall q \in [1, N[$. The smoothness result $\Phi_A(u) \in W_0^{1,q_0}(\Omega)$ is not true here for all Φ_A in $\mathcal{F}_N$. In fact, for $p = N = 2$, denoting by $\mathcal{U}$ the unit Euclidian ball of $\mathbb{R}^2$, it is easily seen that the function $u(x,y) = -\frac{1}{2\pi}\ln\sqrt{x^2+y^2}$ is solution of $-\Delta u = \delta$ in $\mathcal{U}$ with $u = 0$ on $\partial\mathcal{U}$, δ is the Dirac distribution supported by the origin, and that $\Phi_0(u) = u/\ln^\beta(2+|u|)$ $(\beta > 1)$, is not in $H_0^1(\mathcal{U}) = W_0^{1,2}(\mathcal{U})$.

4. Summability of solution of Problem (P)

Our approach to the parabolic Problem (P), will follows the one of Feng-Quan and Guang-Wei [6]; we will merely replace the function Φ_0 by a more general one Φ_A belonging to the class $\mathcal{F}_p$.

Theorem 4.1. *If Conditions* (1–3) *are satisfied, for* $2 - 1/(N+1) < p$, *the Problem* (P) *has a solution* u *such that*

$$\Phi_A(u) \in L^{\tilde{q}_0}(0,T; W_0^{1,\tilde{q}_0}(\Omega)),\ \forall \Phi_A \in \mathcal{F}_p,$$

where $\tilde{q}_0 = p - N/(N+1)$.

Proof. As in the elliptic case, the proof needs, here also, four steps.

Step 1: Approximation. The given problem is replaced by a family of approximate ones:

$$(P_k) \qquad \begin{cases} u_k' - \operatorname{div}(\hat{a}(x,t,u_k,\nabla u_k)) = \mu_k & \text{in} \quad Q \\ u_k = 0 & \text{on} \quad \partial\Omega\times]0,T[\\ u_k(\cdot,0) = u_{0k} & \text{in} \quad \Omega, \end{cases}$$

where $\{\mu_k\}_{k=1}^\infty$ ($\{u_{0k}\}_{k=1}^\infty$) is a sequence of function in $L^\infty(Q)$ ($L^\infty(\Omega)$) converging to μ (μ_0) in the weak-$\star$ topology of $M(Q)$ ($M(\Omega)$) and there exists a positive constant C_0 such that $\|\mu_k\|_{L^1(Q)} \leq C_0$ ($\|u_{0k}\|_{L^1(\Omega)} \leq C_0$), $\forall k \geq 1$ (respectively). The existence of a weak solution u_k to the Problem (P_k) is guaranteed by classical results, see [8]. This solution is in the space $L^p(0,T; W_0^{1,p}(\Omega)) \cap C([0,T]; L^2(\Omega))$.

Step 2: Uniform estimates on $\{u_k\}_{k=1}^\infty$.

Lemma 4.2. *Let* A *a function satisfying Conditions* (a)–(d) *of Definition 2.1. Then, there exists a constant* $C > 0$ *such that*

$$\int_{B_n} |\nabla u_k|^p\,dxdt \leq C, \forall k \geq 1, \quad B_n = \{(x,t) \in Q \mid n \leq |u_k(x,t)| < n+1\}; \tag{8}$$

$$\int_{\{|u_k| \geq n_0\}} \frac{|\nabla u_k|^p}{A(u_k)}\,dxdt \leq C,\ \forall k \geq 1; \tag{9}$$

$$\|u_k\|_{L^\infty(0,T;L^1(\Omega))} \leq C,\ \forall k \geq 1; \tag{10}$$

$$\|\Phi_A(u_k)\|_{L^\infty(0,T;L^1(\Omega))} \leq C,\ \forall k \geq 1. \tag{11}$$

Proof of Lemma 4.2. To prove the Estimate (8), we take $\varphi_n(u_k)$, φ_n being the real function used above in the elliptic case, as a test function in the variational formulation of Problem (P_k). Integrating by parts the term containing the time derivative, using the Condition (1) and the boundedness of $\{\mu_k\}_{k=1}^{\infty}$ in $L^1(Q)$ and of $\{u_{0k}\}_{k=1}^{\infty}$ in $L^1(\Omega)$, we get the result.

The Estimate (9) can be proved in the same way as in the proof of Lemma 3.2.

The proof of Estimate (10) can be performed in a classical way by taking, for $\tau \in]0,T]$, as test function $\psi = T_1(u_k)\chi_{]0,\tau[}$ with $T_1(t) = \frac{1}{2}\{|t+1| - |t-1|\}$ $(t \in \mathbb{R})$ and $\chi_{]0,\tau[}$ the characteristic function of the open interval $]0,\tau[$, and then use an integration by parts formula.

The Estimate (11) is a consequence of the Estimate (10) and the fact that $|\Phi_A(t)| \le K|t|$, $\forall t \in \mathbb{R}$.

Step 3: Uniform estimates on $\{\nabla\Phi_A(u_k)\}_{k=1}^{\infty}$. We have

$$\int_Q |\nabla\Phi_A(u_k)|^{\tilde{q}_0}\, dxdt \;=\; \int_Q |\Phi_A'(u_k)|^{\tilde{q}_0}|\nabla u_k|^{\tilde{q}_0}\, dxdt = J_k^1 + J_k^2,$$

where

$$J_k^1 = \int_{\{|u_k|<n_0\}} |\Phi_A'(u_k)|^{\tilde{q}_0}|\nabla u_k|^{\tilde{q}_0}\, dxdt \quad\text{and}\quad J_k^2 = \int_{\{|u_k|\ge n_0\}} |\Phi_A'(u_k)|^{\tilde{q}_0}|\nabla u_k|^{\tilde{q}_0}\, dxdt.$$

To see the boundedness of sequence $\{J_k^1\}_{k=1}^{\infty}$, we note first that $\tilde{q}_0 < p$, then use the Estimate (8), and boundedness of the derivative Φ_A'.

To estimate the term J_k^2, we use Hölder inequality, Lemma 4.2, and Equation (5) to write:

$$\begin{aligned} J_k^2 &= \int_{\{|u_k|\ge n_0\}} \frac{|\nabla u_k|^{\tilde{q}_0}}{A(u_k)^{\tilde{q}_0/p}} |\Phi_A'(u_k)|^{\tilde{q}_0} A(u_k)^{\tilde{q}_0/p}\, dxdt \\ &\le \Big[\int_{\{|u_k|\ge n_0\}} \frac{|\nabla u_k|^p}{A(u_k)}\, dxdt\Big]^{\frac{\tilde{q}_0}{p}} \Big[\int_{\{|u_k|\ge n_0\}} |\Phi_A'(u_k)|^{\frac{p\tilde{q}_0}{p-\tilde{q}_0}} A(u_k)^{\frac{\tilde{q}_0}{p-\tilde{q}_0}}\, dxdt\Big]^{1-\frac{\tilde{q}_0}{p}} \\ &\le C_1\Big[\int_{\{|u_k|\ge n_0\}} |\Phi_A(u_k)|^{\frac{\tilde{q}_0}{p-\tilde{q}_0}}\, dxdt\Big]^{1-\frac{\tilde{q}_0}{p}} \le C_3\Big[\int_Q |\Phi_A(u_k)|^{\frac{\tilde{q}_0}{p-\tilde{q}_0}}\, dxdt\Big]^{1-\frac{\tilde{q}_0}{p}}. \end{aligned}$$

Thus, we have

$$\int_Q |\nabla\Phi_A(u_k)|^{\tilde{q}_0}\, dxdt \le C_0 + C_1\Big[\int_0^T \|\Phi_A(u_k(t))\|_{L^{\tilde{q}_0/(p-\tilde{q}_0)}(\Omega)}^{\tilde{q}_0/(p-\tilde{q}_0)}\, dt\Big]^{1-\frac{\tilde{q}_0}{p}}. \tag{12}$$

Now, using Gagliardo-Nirenberg embedding theorem (see [9] or [6]), we can write

$$\|\Phi_A(u_k(t))\|_{L^{\tilde{q}_0/(p-\tilde{q}_0)}(\Omega)} \le C_2\|\nabla\Phi_A(u_k(t))\|_{L^{\tilde{q}_0}(\Omega)}^{p-\tilde{q}_0}\|\Phi_A(u_k(t))\|_{L^1(\Omega)}^{p-\tilde{q}_0}, \quad \text{a.e.} \quad t \in]0,T[.$$

This, with Estimates (12) and (11), allow us to write

$$\int_Q |\nabla\Phi_A(u_k)|^{\tilde{q}_0}\, dxdt \le C_0 + C_3\Big(\int_Q |\nabla\Phi_A(u_k)|^{\tilde{q}_0}\, dxdt\Big)^{1-\frac{\tilde{q}_0}{p}}.$$

As $0 < \frac{\tilde{q}_0}{p} < 1$, the above estimate implies the existence of a constant $C_4 > 0$ such that

$$\int_Q |\nabla\Phi_A(u_k)|^{\tilde{q}_0}\, dxdt \le C_4, \ \forall k \ge 1.$$

Now, the application of Poincaré inequality gives

$$\int_0^T \|\Phi_A(u_k)\|^{\tilde{q}_0}_{W_0^{1,\tilde{q}_0}(\Omega)}\, dt \le C_5, \forall k \ge 1. \tag{13}$$

Step 4: Passage to the limit. As in the elliptic case, we use a pointwise compactness result (see [11]) on the sequence $\{u_k\}_{k=1}^{\infty}$ of approximate solutions and their gradients to construct a function u in $L^q(0,T;W_0^{1,q}(\Omega))$, $\forall q \in [1,\tilde{q}_0[$, solution of Problem (P). Using this convergence, we can make k goes to infinity in the Estimate (13) to conclude that $\Phi_A(u) \in L^{\tilde{q}_0}(0,T;W_0^{1,\tilde{q}_0}(\Omega))$.

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