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Maximum Principle for Vector Valued Minimizers

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We prove maximum principle for vector valued minimizers $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of some functionals

$$\mathcal{F}(u) = \int_\Omega f(x, Du(x))dx.$$

The main assumption on the density $f(x,z)$ is a kind of "monotonicity" with respect to the $N \times n$ matrix z. A model density is $f(z) = |z|^4 - (\det z)^2$, where $z \in \mathbb{R}^{2\times 2}$. We also consider relaxed functionals

$$\mathcal{RF}(u) = \inf\{\liminf_k \mathcal{F}(u_k) : \quad u_k \to u\}$$

and we prove a maximum principle under suitable assumptions.

Keywords: Calculus of variations, minimizers, rank-one convexity, maximum principle, relaxation

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1. Introduction

Let us consider vector valued mappings $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$; when $x \in \Omega$, it turns out that $Du(x)$ is a $n \times n$ matrix. For $i \in \{1, \dots, n\}$ we set $M_i(Du)$ to be the vector containing all the minors $i \times i$ taken from the $n \times n$ matrix Du. Thus $M_1(Du) = Du$ and $M_n(Du) = \det Du$. Let us consider the variational integral

$$\mathcal{I}(u) = \int_\Omega f(Du(x))dx, \tag{1}$$

where $f(Du) = g_1(M_1(Du)) + g_2(M_2(Du)) + \cdots + g_n(M_n(Du))$. For a suitable choice of g_i's, such an integral is a model functional in nonlinear elasticity. As long as regularity of minimizers for (1) is concerned, partial regularity of Du has been recently proved in [4] for degenerate convex g_i's. Maximum principle has been established in [7] when $g_i(M_i(Du)) = h_i(|\ M_i(Du)\ |)$ with increasing $h_i : [0, +\infty) \to [0, +\infty)$. The importance of maximum principle is highlighted in [5] where it is used in proving continuity of minimizers in the two dimensional case. Let us point out that, in general, the density f is not strictly convex, thus there may be more than one minimizer for (1). In this paper we select

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two conditions on f allowing for maximum principle: the first one ensures that for every minimizer u of (1) there exists another minimizer $\tilde{u}$ enjoying the maximum principle; the second one is stronger than the first one and it guarantees that *every* minimizer u of (1) satisfies the maximum principle. Next section contains precise statements and their proofs. In the last section we deal with relaxation: some integrals $\mathcal{I}$ show lack of semicontinuity that gives some trouble in proving existence of minimizers. To overcome this difficulty, we consider the relaxed functional

$$\mathcal{RI}(u) = \inf\left\{ \liminf_{k\to\infty} \mathcal{I}(u_k) : \quad \{u_k\}_k \subset Lip(\bar{\Omega}; \mathbb{R}^N) \quad \text{and} \right.$$
$$\left. u_k \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega; \mathbb{R}^N) \right\} \tag{2}$$

see [2]. Under suitable assumptions on f, $\mathcal{RI}$ turns out to be $W^{1,p}$ lower semicontinuous and p coercive, thus direct methods in the calculus of variations guarantee the existence of minimizers for $\mathcal{RI}$, provided a suitable boundary datum has been fixed. In Section 3 we give conditions for the validity of maximum principle for minimizers of the relaxed functional $\mathcal{RI}$.

2. Statements and proofs.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$; $n, N \geq 2$. We consider the functional

$$\mathcal{F}(u) = \int_\Omega f(x, Du(x))dx, \tag{3}$$

where $f : \Omega \times \mathbb{R}^{N\times n} \to \mathbb{R}$ is assumed to be measurable with respect to $x \in \Omega$ and continuous with respect to $z \in \mathbb{R}^{N\times n}$. We also require that

$$0 \leq f(x, z) \tag{4}$$

for every $x \in \Omega$, for each $z \in \mathbb{R}^{N\times n}$. When dealing with a matrix $z \in \mathbb{R}^{N\times n}$, we write $z^1, \ldots, z^N$ to denote the N rows; for each row z^α it results that $z^\alpha = (z_1^\alpha, \ldots, z_n^\alpha) \in \mathbb{R}^n$. Now we are ready to write the main assumption:

$$f(x, \tilde{z}) \leq f(x, z) \tag{5}$$

for every $x \in \Omega$, for every couple of matrices $\tilde{z}, z \in \mathbb{R}^{N\times n}$ such that there exists $\beta \in \{1, \ldots, N\}$ for which $\tilde{z}^\beta = 0 \neq z^\beta$ and $\tilde{z}^\alpha = z^\alpha$ for $\alpha \neq \beta$. We will refer to (5) as to "monotonicity".

A minimizer of functional (3) is a function $u \in W^{1,1}(\Omega, \mathbb{R}^N)$ such that $\mathcal{F}(u) < +\infty$ and

$$\mathcal{F}(u) \leq \mathcal{F}(v), \tag{6}$$

for every $v \in u + W_0^{1,1}(\Omega, \mathbb{R}^N)$.

For $\gamma \in \{1, \ldots, N\}$ and $b \in \mathbb{R}$ we define the truncation operator

$$T_{\gamma,b} : \mathbb{R}^N \to \mathbb{R}^N \tag{7}$$

as follows. For every $y = (y^1, \dots, y^N) \in \mathbb{R}^N$ we set $T_{\gamma,b}(y) = (T^1_{\gamma,b}(y), \dots, T^N_{\gamma,b}(y)) \in \mathbb{R}^N$ where $T^\alpha_{\gamma,b}(y) = y^\gamma \wedge b = \min\{y^\gamma; b\}$ if $\alpha = \gamma$, $T^\alpha_{\gamma,b}(y) = y^\alpha$ if $\alpha \neq \gamma$.

The first result of the present paper is the following

Theorem 2.1. *Let $u = (u^1, \dots, u^N) \in W^{1,1}(\Omega, \mathbb{R}^N)$ be a minimizer of functional* (3) *under* (4) *and* (5). *If there exist $\beta \in \{1, \dots, N\}$ and $k \in \mathbb{R}$ such that $u^\beta \leq k$ on $\partial\Omega$, then $T_{\beta,k}(u) \in u + W^{1,1}_0(\Omega, \mathbb{R}^N)$ and $T_{\beta,k}(u)$ minimizes* (3) *too.*

Remark 2.2. In general u^β may be greater than k in Ω, that is $T_{\beta,k}(u) \neq u$, as the following example shows, [10]. Take $n = N = 2$, $\Omega = \{x \in \mathbb{R}^2 :\mid x \mid < \pi/2\}$, $u^1(x) = \cos \mid x \mid$, $u^2(x) = 0$, $f(x,z) =\mid \det z \mid$. So, the second row of the matrix $Du(x)$ is zero, thus its determinant vanishes and $\mathcal{F}(u) = 0$. Since $0 \leq \mathcal{F}(v)$ for any v, it turns out that u is a minimizer of $\mathcal{F}$ with $u^1 = 0$ on $\partial\Omega$ but $u^1 > 0$ in Ω. On the other hand $T_{1,0}(u) = u^1 \wedge 0 = 0 < u^1$ in Ω. Note that $f(x,z) =\mid \det z \mid$ verifies (4) and (5).

In order to get the equality $T_{\beta,k}(u) = u$ we have to assume the following "strict monotonicity":

$$f(x, \tilde{z}) < f(x, z) \tag{8}$$

for every $x \in \Omega$, for every couple of matrices $\tilde{z}, z \in \mathbb{R}^{N \times n}$ such that there exists $\beta \in \{1, \dots, N\}$ for which $\tilde{z}^\beta = 0 \neq z^\beta$ and $\tilde{z}^\alpha = z^\alpha$ for $\alpha \neq \beta$. Under (8) we are able to prove that $T_{\beta,k}(u) = u$ in Theorem 2.1, that is, *every* minimizer enjoys the maximum principle: this is the second result of the present paper.

Theorem 2.3. *Let $u = (u^1, \dots, u^N) \in W^{1,1}(\Omega, \mathbb{R}^N)$ be a minimizer of functional* (3) *under* (4) *and* (8). *If there exist $\beta \in \{1, \dots, N\}$ and $k \in \mathbb{R}$ such that*

$$u^\beta \leq k \qquad \text{on } \partial\Omega$$

then

$$u^\beta \leq k \qquad \text{in } \Omega.$$

Remark 2.4. The assumption "$v \leq k$ on $\partial\Omega$" means that there exists a sequence $\{v_h\}_h \subset \mathrm{Lip}(\overline{\Omega})$ such that $v_h(x) \leq k$ for every $x \in \partial\Omega$, for each $h \in \mathbb{N}$ and $v_h \to v$ in $W^{1,1}(\Omega)$ as $h \to \infty$.

Remark 2.5. The statement "$v \leq k$ in Ω" means that $v(x) \leq k$ for almost every $x \in \Omega$.

Remark 2.6. Let us assume that there exists a function g such that

$$f(x, z) = g(x, |M_1(z)|, |M_2(z)|, \dots, |M_s(z)|), \tag{9}$$

where $s = \min\{n, N\}$ and $M_i(z)$ is the vector containing all the $i \times i$ minors taken from the $N \times n$ matrix z. About $g = g(x, p_1, p_2, \dots, p_s)$ we assume that

$$p_i \to g(x, p_1, \dots, p_i, \dots, p_s) \tag{10}$$

is increasing on $[0, +\infty)$ for every $i = 1, \dots, s$. Under (9) and (10) it is easy to prove that (5) holds true. A simple model is

$$f(x, z) =\mid \det z \mid$$

where $N = n = s$, $M_s(z) = \det z$ and $g(x, p_1, \dots, p_s) = p_s$.

Remark 2.7. Under (9) and (10), if we also assume that

$$p_1 \to g(x, p_1, p_2, \ldots, p_s) \tag{11}$$

is strictly increasing on $[0, +\infty)$, then it is easy to prove that (8) holds true. Variational integrals (3) under (9), (10), (11) have been considered in [7] and the model is

$$f(x, z) = \mid z \mid^p + \mid \det z \mid^q,$$

where $z \in \mathbb{R}^{n\times n}$, $p, q > 0$; see [5] for the case $n = 2 = p = q$.

Remark 2.8. Let us recall that f is said to be rank one convex if for all $x \in \Omega$ it results

$$f(x, tP + (1-t)Q) \leq tf(x, P) + (1-t)f(x, Q), \tag{12}$$

for every $t \in (0, 1)$, for every $P, Q \in \mathbb{R}^{N\times n}$ with $rank(P - Q) \leq 1$, see [3], page 99. Now we restrict ourselves to a special class of rank one matrices, those $P - Q$'s having all rows equal to zero but one: there exists $\beta \in \{1, \ldots, N\}$ such that $P^\beta - Q^\beta \neq 0$ and $P^\alpha - Q^\alpha = 0$ for $\alpha \neq \beta$. We say that f is *special rank one convex* if (12) holds true for P, Q such that $P - Q$ has all rows equal to zero but one. Convexity is not enough to ensure "monotonicity" (5) as the following example shows. Take $N = n = 2$, $f(z) = \mid z \mid^2 + \det z$. Then f is convex but it does not verify (5): set

$$z = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tilde{z} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

thus

$$f(z) = 3 < 4 = f(\tilde{z}).$$

In addition to special rank one convexity, we assume

$$z \to f(x, z) \in C^1(\mathbb{R}^{N\times n}) \tag{13}$$

for every $x \in \Omega$ and the following structure condition

$$\frac{\partial f}{\partial z_i^\beta} f(x, \xi) = 0, \tag{14}$$

for every $x \in \Omega$, for every $i = 1, \ldots, n$ for every $\beta = 1, \ldots, N$, for every $\xi \in \mathbb{R}^{N\times n}$ with $\xi^\beta = 0$. Under (13) and (14), any *special rank one convex* function $f(x, z)$ turns out to satisfy (5).

Remark 2.9. In order to satisfy the "strict monotonicity" (8), we have to require the strict inequality in (12). Precisely, we say that f is *strictly special rank one convex* if, for all $x \in \Omega$, it results that

$$f(x, tP + (1-t)Q) < tf(x, P) + (1-t)f(x, Q), \tag{15}$$

for every $t \in (0, 1)$, for every $P, Q \in \mathbb{R}^{N\times n}$ with $P - Q$ having all rows equal to zero but one. Under (13) and the structure condition (14), any *strictly special rank one convex* function $f(x, z)$ turns out to satisfy (8).

Example 2.10. Let us assume $n = N = 2$. We consider the function

$$f(z) = a|z|^4 - (\det z)^2$$

where $a \geq 1/2$. It turns out that f satisfies the "strict monotonicity" (8).

Proof of Theorem 2.1. Let $\beta \in \{1, \ldots, N\}$ and $k \in \mathbb{R}$ be such that $u^\beta \leq k$ on $\partial\Omega$. Set

$$\varphi^\beta = -\max\{u^\beta - k, 0\}.$$

Since $u^\beta \leq k$ on $\partial\Omega$, it turns out that $\varphi^\beta \in W_0^{1,1}(\Omega)$. If $\alpha \neq \beta$ we simply set $\varphi^\alpha = 0$. Then we have $\varphi \in W_0^{1,1}(\Omega; \mathbb{R}^N)$ and

$$\tilde{u} := u + \varphi \in u + W_0^{1,1}(\Omega; \mathbb{R}^N) \tag{16}$$

is a test function for the minimality condition (6). Set

$$\Omega_1 = \{x \in \Omega : u^\beta(x) \leq k\} \cup \{x \in \Omega : u^\beta(x) > k, \quad Du^\beta(x) = 0\}$$

and

$$\Omega_2 = \Omega \setminus \Omega_1.$$

Then

$$D\tilde{u} = Du \quad \text{ on } \Omega_1 \tag{17}$$

and

$$D\tilde{u}^\alpha = \begin{Bmatrix} Du^\alpha & \text{if } \alpha \neq \beta \\ 0 \neq Du^\beta & \text{if } \alpha = \beta \end{Bmatrix} \qquad \text{on } \Omega_2. \tag{18}$$

Thus

$$f(x, D\tilde{u}(x)) = f(x, Du(x)) \quad \text{ if } x \in \Omega_1 \tag{19}$$

and, using "monotonicity" (5),

$$f(x, D\tilde{u}(x)) \leq f(x, Du(x)) \quad \text{ if } x \in \Omega_2. \tag{20}$$

The previous (19), (20) and the positivity (4) merge into

$$0 \leq \mathcal{F}(\tilde{u}) \leq \mathcal{F}(u). \tag{21}$$

Since $\mathcal{F}(u) < +\infty$, it turns out that $\mathcal{F}(\tilde{u}) < +\infty$ too. Moreover, the minimality (6) of u gives

$$\mathcal{F}(u) \leq \mathcal{F}(\tilde{u})$$

thus

$$\mathcal{F}(\tilde{u}) = \mathcal{F}(u) = \min_{v \in u + W_0^{1,1}(\Omega; \mathbb{R}^N)} \mathcal{F}(v) \tag{22}$$

and $\tilde{u}$ turns out to be a minimizer too. Note that

$$\tilde{u} = T_{\beta,k}(u).$$

This ends the proof of Theorem 2.1. $\square$

Proof of Theorem 2.3. We argue as in the proof of Theorem 2.1 until we reach (22). Because of (17), the equality (22) reads as

$$\int_{\Omega_2} f(x, Du(x))dx = \int_{\Omega_2} f(x, D\tilde{u}(x))dx. \tag{23}$$

Now, the "strict monotonicity" (8) can be used with $\tilde{z} = D\tilde{u}(x)$ and $z = Du(x)$, because of (18):

$$f(x, D\tilde{u}(x)) < f(x, Du(x)) \qquad \text{if } x \in \Omega_2. \tag{24}$$

Comparing (23) with (24) gives that Ω_2 has zero measure. This means that $Du^\beta(x) = 0$ for almost every $x \in \{u^\beta > k\}$, thus $D\varphi^\beta = 0$ a.e. in Ω. Since $\varphi^\beta \in W_0^{1,1}(\Omega)$, by Poincaré inequality it follows that $\varphi^\beta(x) = 0$ for a.e. $x \in \Omega$. Since $\varphi^\beta = -\max[(u^\beta - k); 0] < 0$ on $\{u^\beta > k\}$, it turns out that $|\{u^\beta > k\}| = 0$, then

$$u^\beta(x) \leq k \qquad \text{for a.e. } x \in \Omega.$$

This ends the proof of Theorem 2.3. □

Remark 2.11. An inspection of the proof of Theorem 2.1 shows that

$$\mathcal{F}(T_{\beta,k}(u)) \leq \mathcal{F}(u) \tag{25}$$

for every $u \in W^{1,1}(\Omega; \mathbb{R}^N)$, for every $k \in \mathbb{R}$, for every $\beta \in \{1, \dots, N\}$: the inequality (25) does not require neither the minimality of u, nor the boundary condition $u^\beta \leq k$ on $\partial\Omega$. In order to force $T_{\beta,k}(u)$ to have the same boundary datum as u, we need $u^\beta \leq k$ on $\partial\Omega$. Thus the following results hold true.

Theorem 2.12. *Let us consider the functional* (3) *under* (4) *and* (5)*. For every $u \in W^{1,1}(\Omega, \mathbb{R}^N)$, for every $\beta \in \{1, \dots, N\}$, for every $k \in \mathbb{R}$ it results that*

$$\mathcal{F}(T_{\beta,k}(u)) \leq \mathcal{F}(u) \tag{26}$$

where $T_{\beta,k}(u)$ is introduced in (7)*.*

Theorem 2.13. *For every $u \in W^{1,p}(\Omega, \mathbb{R}^N)$, $1 \leq p < +\infty$, for every $\beta \in \{1, \dots, N\}$, for every $k \in \mathbb{R}$ such that*

$$u^\beta \leq k \text{ on } \partial\Omega \tag{27}$$

it results that

$$T_{\beta,k}(u) \in u + W_0^{1,p}(\Omega, \mathbb{R}^N) \tag{28}$$

where $T_{\beta,k}(u)$ is introduced in (7)*.*

Remark 2.14. In the previuos Theorem 2.13, the assumption "$u^\beta \leq k$ on $\partial\Omega$" means that there exists a sequence $\{v_h\}_h \subset \mathrm{Lip}(\overline{\Omega})$ such that $v_h(x) \leq k$ for every $x \in \partial\Omega$, for each $h \in \mathbb{N}$ and $v_h \to u^\beta$ in $W^{1,p}(\Omega)$ as $h \to \infty$: we remark that, starting from $u^\beta \in W^{1,p}(\Omega)$, we require the convergence of v_h in $W^{1,p}(\Omega)$ and it results that $T_{\beta,k}(u) \in u + W_0^{1,p}(\Omega, \mathbb{R}^N)$. Compare with Remark 2.4 where $p = 1$.

3. Relaxation.

Let us consider the model functional

$$\mathcal{G}(u) = \int_\Omega (|Du|^p + |\det Du|^q)\, dx, \tag{29}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, $1 < p$ and $q > 0$. Maximum principle for minimizers of (29) has been proven in Theorem 2.3, see also Remark 2.7. The case $1 < p < n$, $1 \leq q$ need to be dealt with more carefully: $\mathcal{G}$ is not sequentially weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^n)$, see [1]. Let us recall that, if we restrict ourselves to suitable subclasses of $W^{1,p}$ maps, then it is possible to have lower semicontinuity, provided $n - 1 \leq p$, see [8], [6] and their references. Let us remark that condition $n - 1 < p$ is important also for partial regularity of minimizers, [4]. Lack of semicontinuity gives some trouble in proving existence of minimizers. To overcome this difficulty, we consider the relaxed functional

$$\begin{aligned} \mathcal{RG}(u) = \inf \Big\{ \liminf_{k\to\infty} \mathcal{G}(u_k) : \quad & \{u_k\}_k \subset Lip(\bar\Omega; \mathbb{R}^N) \quad \text{and} \\ & u_k \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega; \mathbb{R}^N) \Big\} \end{aligned} \tag{30}$$

see [9], [2]; $\mathcal{RG}$ turns out to be $W^{1,p}$ lower semicontinuous; since $\mathcal{G}$ is p coercive, $\mathcal{RG}$ is p coercive too. Thus direct methods in the calculus of variations guarantee the existence of minimizers for $\mathcal{RG}$, provided a suitable boundary datum has been fixed. In this section we show that every minimizer of $\mathcal{RG}$ enjoys the maximum principle, provided $2 \leq p$. More generally, we consider the functional (3) under (4). We also assume p-coercivity: there exist $\nu > 0$, $m \geq 0$, $p > 1$ such that

$$\nu |z|^p - m \leq f(x, z) \tag{31}$$

for every $x \in \Omega$, for each $z \in \mathbb{R}^{N\times n}$, where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. We define the relaxation of (3) as follows

$$\begin{aligned} \mathcal{RF}(u) = \inf \Big\{ \liminf_{k\to\infty} \mathcal{F}(u_k) : \quad & \{u_k\}_k \subset Lip(\bar\Omega; \mathbb{R}^N) \quad \text{and} \\ & u_k \rightharpoonup u \quad \text{weakly in } W^{1,p}(\Omega; \mathbb{R}^N) \Big\}. \end{aligned} \tag{32}$$

In the following lemma we resume the properties of relaxed functional (32) that we will need later. Their proof is standard and we omit it. About relaxation we refer to [2].

Lemma 3.1. *Let $\mathcal{F}$ be the functional* (3) *under* (4) *and* (31)*. Let $\mathcal{RF}$ be its relaxation as defined in* (32)*. Then, the following properties hold true.*

1) $\nu \int_\Omega |Du|^p dx - m|\Omega| \leq \mathcal{RF}(u)$, *for every* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$.
2) $\mathcal{RF}(v) \leq \mathcal{F}(v)$, *for every* $v \in Lip(\bar\Omega; \mathbb{R}^N)$.
3) $0 \leq \mathcal{RF}(u) \leq \liminf_{k\to\infty} \mathcal{RF}(u_k)$, *for every* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$, *for every sequence* $\{u_k\}_k \subset W^{1,p}(\Omega; \mathbb{R}^N)$ *such that* $u_k \rightharpoonup u$ *weakly in* $W^{1,p}(\Omega; \mathbb{R}^N)$.
4) *For every* $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ *there exists* $\{u_k\}_k \subset Lip(\bar\Omega; \mathbb{R}^N)$ *such that* $u_k \rightharpoonup u$ *weakly in* $W^{1,p}(\Omega; \mathbb{R}^N)$ *and* $\mathcal{RF}(u) = \lim_{k\to\infty} \mathcal{F}(u_k)$.

We fix a function $u_0 \in Lip(\bar{\Omega}; \mathbb{R}^N)$ with $\mathcal{F}(u_0) < +\infty$. Property 2) of $\mathcal{RF}$ gives $\mathcal{RF}(u_0) \leq \mathcal{F}(u_0) < +\infty$. The left hand side in property 3) guarantees that $\mathcal{RF}$ is bounded below. Moreover, 1) and 3) in Lemma 3.1 tell us that $\mathcal{RF}$ is p-coercive and lower semicontinuous with respect to the weak convergence in $W^{1,p}$. Then, there exists at least one minimizer v of $\mathcal{RF}$ in the class $u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$. Our goal is to prove that $\mathcal{RF}$ admits a minimizer whose components enjoy the maximum principle. This is achieved in the following

Theorem 3.2. *Under the assumptions* (3), (4), (31), (5), *let* $u_0 = (u_0^1, \ldots, u_0^N) \in Lip(\bar{\Omega}; \mathbb{R}^N)$ *with* $\mathcal{F}(u_0) < +\infty$. *Then there exists* $u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$ *such that*

$$\mathcal{RF}(u) \leq \mathcal{RF}(v)$$

for every $v \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$*; moreover, every component* u^β *satisfies the maximum principle:*

$$u^\beta \leq \max_{\partial\Omega} u_0^\beta \qquad in \quad \Omega$$

for every $\beta = 1, \ldots, N$.

The previous Theorem 3.2 tells us that *at least one* minimizer enjoys the maximum principle. In order to ensure that *every* minimizer satisfies the maximum principle, we need an assumption stronger than (5): since the relaxed functional $\mathcal{RF}(u)$ is defined by means of limits taken over $\mathcal{F}(u_h)$, the strict monotonicity (8) does not seem to be strong enough (strict inequality might be equality in the limit). We need the following "uniform strict monotonicity": there exists $\mu > 0$ such that

$$\mu |z - \tilde{z}|^p + f(x, \tilde{z}) \leq f(x, z) \tag{33}$$

for every $x \in \Omega$, for every couple of matrices $\tilde{z}, z \in \mathbb{R}^{N \times n}$ such that there exists $\beta \in \{1, \ldots, N\}$ for which $\tilde{z}^\beta = 0 \neq z^\beta$ and $\tilde{z}^\alpha = z^\alpha$ for $\alpha \neq \beta$; in (33) the exponent p is the same as in (31) and in (32). Under (33) we are able to prove that *every* minimizer enjoys the maximum principle.

Theorem 3.3. *Under the assumptions* (3), (4), (31), (33), *let* $u_0 = (u_0^1, \ldots, u_0^N) \in Lip(\bar{\Omega}; \mathbb{R}^N)$ *with* $\mathcal{F}(u_0) < +\infty$. *If* $u \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$ *verifies*

$$\mathcal{RF}(u) \leq \mathcal{RF}(v)$$

for every $v \in u_0 + W_0^{1,p}(\Omega; \mathbb{R}^N)$, *then every component* u^β *satisfies the maximum principle:*

$$u^\beta \leq \max_{\partial\Omega} u_0^\beta \qquad in \quad \Omega$$

for every $\beta = 1, \ldots, N$.

Example 3.4. Let us assume $n = N \geq 2$. We consider the function

$$f(z) = |z|^p + |\det z|^q$$

where $p \geq 2$ and $q > 0$. It turns out that f satisfies the "uniform strict monotonicity" (33) with $\mu = 1$.

Proof of Theorem 3.2.

Step 1. Existence of minimizers.

Lemma 3.1 guarantees that $\mathcal{RF}$ is bounded below, p coercive and weak lower semicontinuous in $W^{1,p}$, thus direct methods in the calculus of variations give the existence of at least one $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ such that $\mathcal{RF}(u) \leq \mathcal{RF}(v)$ for every $v \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$.

Step 2. Approximation.

Let $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ minimize $\mathcal{RF}$ among all $v \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$. Because of 4) in Lemma 3.1, there is a sequence $\{u_k\}_k \subset Lip(\bar{\Omega};\mathbb{R}^N)$ such that

$$\mathcal{F}(u_k) \to \mathcal{RF}(u)$$

and $u_k \rightharpoonup u$ weakly in $W^{1,p}(\Omega;\mathbb{R}^N)$. Rellich Theorem gives strong convergence in $L^p(\Omega;\mathbb{R}^N)$, thus, up to a subsequence, we have pointwise convergence $u_k(x) \to u(x)$ for almost every $x \in \Omega$.

Step 3. Truncation and weak convergence.

For every $\beta \in \{1,\dots,N\}$ we consider the maximum on $\partial\Omega$ of the β component u_0^β of our boundary datum $u_0 = (u_0^1,\dots,u_0^N)$ and we set $b = \max_{\partial\Omega} u_0^\beta$. Then we consider the truncation operator $T_{\beta,b}$ defined in (7); it turns out that

$$T_{\beta,b}(u_k) \in Lip(\bar{\Omega};\mathbb{R}^N).$$

On the other hand, the $W^{1,p}$ norm of $T_{\beta,b}(u_k)$ is bounded, thus, up to a further subsequence, $T_{\beta,b}(u_k) \rightharpoonup w$ weakly in $W^{1,p}$, strongly in L^p, pointwise almost everywhere in Ω, for some $w \in W^{1,p}(\Omega;\mathbb{R}^N)$. Since $u_k(x) \to u(x)$, it turns out that $T_{\beta,b}(u_k(x)) \to T_{\beta,b}(u(x))$, thus $w = T_{\beta,b}(u)$ and

$$T_{\beta,b}(u_k) \rightharpoonup T_{\beta,b}(u) \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N).$$

Moreover, since $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ and $u_0^\beta \leq b$ on $\partial\Omega$, Theorem 2.13 guarantees that

$$T_{\beta,b}(u) \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N).$$

Step 4. Truncation and minimality.

The minimality of u with respect of $T_{\beta,b}(u)$ gives

$$\mathcal{RF}(u) \leq \mathcal{RF}(T_{\beta,b}(u));$$

the definition of $\mathcal{RF}$ guarantees that

$$\mathcal{RF}(T_{\beta,b}(u)) \leq \liminf_k \mathcal{F}(T_{\beta,b}(u_k));$$

Theorem 2.12 gives

$$\mathcal{F}(T_{\beta,b}(u_k)) \leq \mathcal{F}(u_k)$$

thus

$$\liminf_k \mathcal{F}(T_{\beta,b}(u_k)) \leq \liminf_k \mathcal{F}(u_k) = \mathcal{RF}(u),$$

where the last equality holds true by construction, see the beginning of Step 2. Thus

$$\mathcal{RF}(T_{\beta,b}(u)) = \mathcal{RF}(u).$$

Let us summarize Steps 2, 3 and 4 as follows. For every minimizer u of $\mathcal{RF}$, for every $\beta = 1, \dots, N$, the function $T_{\beta,b^\beta}(u)$ is a minimizer too, provided $b^\beta = \max\limits_{\partial\Omega} u_0^\beta$.

Step 5. Iterative truncation of the components of $(u^1, u^2, \dots, u^N)$.

We start from a minimizer u and the previous argument gives that $T_{1,b^1}(u)$ is a minimizer; thus $T_{2,b^2}((T_{1,b^1}(u))$ is a minimizer; we proceed in this way until we arrive at $T_{N,b^N}(\dots(T_{1,b^1}(u)))$ which is a minimizer for $\mathcal{RF}$. This ends the proof. □

Proof of Theorem 3.3. Let $u \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$ minimize $\mathcal{RF}$ among all $v \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N)$. The proof of Theorem 3.2 guarantees that

$$0 \le \mathcal{RF}(T_{\beta,b}(u)) = \mathcal{RF}(u) < +\infty \tag{34}$$

and

$$T_{\beta,b}(u) \in u_0 + W_0^{1,p}(\Omega;\mathbb{R}^N) \tag{35}$$

for every $\beta = 1, \dots, N$, where $b = b^\beta = \max\limits_{\partial\Omega} u_0^\beta$. We claim that, under (33), we have

$$\mu \int\limits_\Omega |Dv - D(T_{\beta,k}(v))|^p dx + \mathcal{RF}(T_{\beta,k}(v)) \le \mathcal{RF}(v) \tag{36}$$

for every $\beta = 1, \dots, N$, for every $k \in \mathbb{R}$, for every $v \in W^{1,p}(\Omega;\mathbb{R}^N)$, where μ is the same positive constant appearing in (33). Assume that (36) holds true; then we use it with $k = b = \max\limits_{\partial\Omega} u_0^\beta$ and $v = u$: equality (34) gives

$$\mu \int\limits_\Omega |Du - D(T_{\beta,b}(u))|^p dx + \mathcal{RF}(T_{\beta,b}(u)) \le \mathcal{RF}(u) = \mathcal{RF}(T_{\beta,b}(u)) \tag{37}$$

then

$$\mu \int\limits_\Omega |Du - D(T_{\beta,b}(u))|^p dx \le 0 \tag{38}$$

thus $D(T_{\beta,b}(u)) = Du$ almost everywhere in Ω; (35) tells us that $T_{\beta,b}(u)$ has the same boundary value as u, then Poincaré inequality gives

$$T_{\beta,b}(u) = u \tag{39}$$

almost everywhere in Ω, thus

$$u^\beta \le b = \max_{\partial\Omega} u_0^\beta$$

almost everywhere in Ω. So, we are left to prove (36). The case $\mathcal{RF}(v) = +\infty$ is straightforward, thus we assume that $\mathcal{RF}(v) < +\infty$. Property 4) in Lemma 3.1 tells us that there exists a sequence $\{v_h\} \subset Lip(\bar{\Omega};\mathbb{R}^N)$ such that

$$v_h \rightharpoonup v \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N) \tag{40}$$

and

$$\mathcal{F}(v_h) \to \mathcal{RF}(v). \tag{41}$$

Since $\mathcal{RF}(v) < +\infty$, we may assume that $\mathcal{F}(v_h) < +\infty$ too. Because of (40), we have

$$T_{\beta,k}(v_h) \rightharpoonup T_{\beta,k}(v) \text{ weakly in } W^{1,p}(\Omega;\mathbb{R}^N). \tag{42}$$

Moreover

$$T_{\beta,k}(v_h) \in Lip(\bar{\Omega};\mathbb{R}^N); \tag{43}$$

then, the definition of $\mathcal{RF}$ gives

$$\mathcal{RF}(T_{\beta,k}(v)) \leq \liminf_h \mathcal{F}(T_{\beta,k}(v_h)) \tag{44}$$

and the lower semicontinuity of the L^p norm gives

$$\mu \int_\Omega |Dv - D(T_{\beta,k}(v))|^p dx \leq \liminf_h \mu \int_\Omega |Dv_h - D(T_{\beta,k}(v_h))|^p dx. \tag{45}$$

We already know that

$$0 \leq \mathcal{F}(T_{\beta,k}(v_h)) \leq \mathcal{F}(v_h)$$

because of Theorem 2.12, thus

$$0 \leq \mathcal{RF}(T_{\beta,k}(v)) \leq \liminf_h \mathcal{F}(T_{\beta,k}(v_h)) \leq \liminf_h \mathcal{F}(v_h) = \mathcal{RF}(v) < +\infty \tag{46}$$

Now we want to improve on (46). We set

$$\Omega_1 = \{x \in \Omega : v_h^\beta(x) \leq k\} \cup \{x \in \Omega : v_h^\beta(x) > k, \quad Dv_h^\beta(x) = 0\}$$

and

$$\Omega_2 = \Omega \setminus \Omega_1.$$

Then

$$D(T_{\beta,k}(v_h)) = Dv_h \quad \text{ on } \Omega_1 \tag{47}$$

and

$$D(T^\alpha_{\beta,k}(v_h)) = \left\{ \begin{array}{ll} Dv_h^\alpha & \text{if } \alpha \neq \beta \\ 0 \neq Dv_h^\beta & \text{if } \alpha = \beta \end{array} \right\} \quad \text{ on } \Omega_2. \tag{48}$$

We use (33) and we get

$$\mu|Dv_h(x) - D(T_{\beta,k}(v_h(x)))|^p + f(x, D(T_{\beta,k}(v_h(x)))) \leq f(x, Dv_h(x)) \quad \text{ if } x \in \Omega_2 \tag{49}$$

and

$$\mu|Dv_h(x) - D(T_{\beta,k}(v_h(x)))|^p + f(x, D(T_{\beta,k}(v_h(x)))) = f(x, Dv_h(x)) \quad \text{ if } x \in \Omega_1. \tag{50}$$

Now we integrate with respect to $x \in \Omega_1 \cup \Omega_2 = \Omega$ and we obtain

$$\mu \int_\Omega |Dv_h - D(T_{\beta,k}(v_h))|^p dx + \mathcal{F}(T_{\beta,k}(v_h)) \leq \mathcal{F}(v_h). \tag{51}$$

This inequality and (41), (44), (45) give

$$
\begin{aligned}
& \mu \int_\Omega |Dv - D(T_{\beta,k}(v))|^p dx + \mathcal{RF}(T_{\beta,k}(v)) \\
\leq \; & \liminf_h \mu \int_\Omega |Dv_h - D(T_{\beta,k}(v_h))|^p dx + \liminf_h \mathcal{F}(T_{\beta,k}(v_h)) \\
\leq \; & \liminf_h \left(\mu \int_\Omega |Dv_h - D(T_{\beta,k}(v_h))|^p dx + \mathcal{F}(T_{\beta,k}(v_h)) \right) \\
\leq \; & \liminf_h \mathcal{F}(v_h) = \mathcal{RF}(v).
\end{aligned} \tag{52}
$$

This ends the proof of (36). Theorem 3.3 is completely proven. □

We end this section by gladly taking the opportunity to thank Paolo Marcellini for pointing us the lack of semicontinuity for the functional (29) and the related paper [1]: Marcellini's kind remark was the starting point for our research on relaxation and maximum principle.

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