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Variational Formulations on Thin Elastic Plates with Constraints

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We derive variational formulations for thin elastic plates from bulk energies by dimensional reduction. The main feature is to consider a family of problems with internal constraints on normal deformations. Our approach consists of two stages. First we obtain an abstract variational convergence result. Then we study the integral representation of the limit functional.

1. Introduction

The goal of this paper[1] is to provide variational formulations on some thin elastic plates by dimensional reduction (see for instance [1, 5, 12]). The distinguishable feature is its capability of taking account of internal constraints on normal deformations in original three dimensional problems.
Let $\omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. Let $\Omega_\varepsilon = \omega\times]-\varepsilon/2;\varepsilon/2[$ be the reference configuration of an hyperelastic three dimensional body, with $\varepsilon \in]0,1]$ a small paramater which describes the small thickness of the geometric structure. Let $\hat{W} : \mathbb{M}^{3\times 3} \to [0,+\infty[$ be a continuous function which is the stored energy function associated with the material. We consider the family of constrained minimization problems

$$(\mathcal{P}_\varepsilon) \quad \inf\Big\{\frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \hat{W}(\nabla u(x,x_3))dxdx_3 : u \in A_\varepsilon \text{ and } \frac{\partial u}{\partial x_3}(x,x_3) \in \Lambda(x)\Big\},$$

where $A_\varepsilon = \{u \in \mathcal{C}^\infty(\overline{\Omega}_\varepsilon;\mathbb{R}^3) : u(x,x_3) = (x,x_3) \text{ on } \partial\omega\times]-\varepsilon/2;\varepsilon/2[\}$ is the set of admissible deformations. The lower semicontinuous[2] convex closed valued[3] multifunction $\Lambda : \overline{\omega} \rightrightarrows \mathbb{R}^3$ plays the role of constraints. To avoid the trivial case in which the set of admissible deformations is empty, we will assume that for every $x \in \overline{\omega}$

$$e \in \Lambda(x) \text{ where } e = (0,0,1).$$

To handle the constraints Λ in our analysis, we set, for each $F = (\xi \mid \zeta) \in \mathbb{M}^{3\times 3}$ with $\xi \in \mathbb{M}^{3\times 2}$ and $\zeta \in \mathbb{R}^3$

$$W(x,F) = \begin{cases} \hat{W}(\xi \mid \zeta) & \text{if } \zeta \in \Lambda(x) \\ +\infty & \text{otherwise.} \end{cases}$$

[1]This paper is a revised version of [3].
[2]A multifunction $\Lambda : \overline{\omega} \rightrightarrows \mathbb{R}^3$ is said to be lower semicontinuous if for every closed set $G \subset \mathbb{R}^3$ and converging sequence $\overline{\omega} \ni x_n \to x$ satisfying $\Lambda(x_n) \subset G$, we have $\Lambda(x) \subset G$.
[3]$\Lambda(x)$ is a nonempty convex closed subset of $\mathbb{R}^3$ for all $x \in \overline{\omega}$.

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Thus, we rewrite $(\mathcal{P}_\varepsilon)$ as $\inf\Big\{E_\varepsilon(u) : u \in A_\varepsilon\Big\}$ where

$$A_\varepsilon \ni u \mapsto E_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} W(x, \nabla u(x, x_3))dxdx_3;$$

we will consider this formulation in the rest of the paper.

Our objective is then to provide a variational limit of the problems $(\mathcal{P}_\varepsilon)$, i.e., to find a two-dimensional minimization problem which is the limit of infima of $(\mathcal{P}_\varepsilon)$. In addition, we need that "limit" points of ε-minimizing sequences to be minimizers of the limit problem. A serious difficulty is that the set of admissible deformations depends on the thickness ε. To avoid it, we "immerse" original functionals E_ε in $L^p(\omega;\mathbb{R}^3)$ with $p \in [1, +\infty[$, by considering functionals

$$L^p(\omega;\mathbb{R}^3) \ni v \mapsto \mathcal{E}_\varepsilon(v) = \inf\left\{E_\varepsilon(u) : u \in A_\varepsilon,\ ⨍_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} u(\cdot, x_3)dx_3 = v(\cdot)\right\}.$$

These functionals will be justified by the Anzellotti-Baldo-Percivale's variational convergence notion [5] (see Section 2). Then, our approach is divided in two steps. We first show an abstract variational convergence theorem under coercivity conditions on $\hat{W}$ (see Section 3 for precise assumptions). Indeed, we show (Theorem 3.2 (a)) that there exists a functional F given by the Γ-limit of functionals $\{\mathcal{E}_\varepsilon\}_\varepsilon$ (with respect to the strong topology of $L^p(\omega;\mathbb{R}^3)$), which satisfies (Theorem 3.2 (b)) that: any ε-minimizing sequence $\{u_\varepsilon\}_\varepsilon$ of $\{E_\varepsilon\}_\varepsilon$ admit a subsequence such that

$$⨍_{-\frac{\varepsilon_k}{2}}^{\frac{\varepsilon_k}{2}} u_{\varepsilon_k}(\cdot, x_3)dx_3 \to \hat{v}(\cdot) \text{ in } L^p(\omega;\mathbb{R}^3),$$

and

$$\lim_{\varepsilon\to 0}\inf_{u\in A_\varepsilon} E_\varepsilon(u) = \inf_{v\in L^p(\omega;\mathbb{R}^3)} F(v) = F(\hat{v}).$$

In fact, the functional F is nothing but the relaxed functional (with respect to the strong topology of $L^p(\omega;\mathbb{R}^3)$) of

$$\mathcal{C}^\infty_*(\omega;\mathbb{R}^3) \ni v \mapsto I(v) = \inf\left\{\int_\omega W(x, (\nabla v(x) \mid \phi(x)))dx : \phi \in \mathcal{C}^\infty_e(\omega;\Lambda)\right\}.$$

Therefore, in the second step, assuming growth conditions on $\hat{W}$, we give integral representations for F. More precisely, with the help of interchange of infimum and integral arguments introduced in [4], we obtain (Proposition 5.2) the following representation for I

$$I(v) = \int_\omega W_\Lambda(x, \nabla v(x))dx, \quad \text{where } W_\Lambda(x,\xi) = \inf_{\zeta\in\Lambda(x)} W(x, (\xi \mid \zeta)).$$

Then the integral representations for F in $L^p(\omega;\mathbb{R}^3)$ are achieved (Theorem 3.4) by using standard relaxation theorems.

This approach seems rather flexible because if, now, we relax I in the space of Young measures instead of Lebesgue's spaces then we can also obtain a variational formulation in terms of Young measures (Theorem 3.6).

The paper is organized as follows. In Section 2 we state the main results of Anzellotti-Baldo-Percivale's variational convergence. Section 3 is devoted to statements of main results. In Sections 4, 5 and 6 we give the proofs.

Notation. Let $\Omega \subset \mathbb{R}^3$ be an open subset and $u : \Omega \to \mathbb{R}^3$ be a differentiable function, we write $Du = \left(\frac{\partial u}{\partial x} \,\middle|\, \frac{\partial u}{\partial x_3} \right)$ with

$$\frac{\partial u}{\partial x} = \begin{pmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} \end{pmatrix} \quad \text{and} \quad \frac{\partial u}{\partial x_3} = \begin{pmatrix} \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_3} \end{pmatrix}.$$

Let $\omega \subset \mathbb{R}^2$ be an open subset. A function v belongs to $\mathcal{C}^\infty(\overline{\omega}; \mathbb{R}^3)$ if there exists $w \in \mathcal{C}^\infty(\mathbb{R}^2; \mathbb{R}^3)$ such that $v = w\lfloor_{\overline{\omega}}$. We denote

$$\mathcal{C}^\infty_*(\omega; \mathbb{R}^3) = \{v \in \mathcal{C}^\infty(\overline{\omega}; \mathbb{R}^3) : v(x) = (x, 0) \text{ on } \partial\omega\},$$

and similarly for $W^{1,p}_*(\omega; \mathbb{R}^3)$. We denote

$$\mathcal{C}^\infty_e(\omega; \Lambda) = \Big\{ \phi \in \mathcal{C}^\infty(\overline{\omega}; \mathbb{R}^3) : \phi(x) = e \text{ on } \partial\omega \text{ and } \phi(x) \in \Lambda(x) \text{ a.e. in } \omega \Big\}.$$

For a topological space X and a function $G : X \to [0, +\infty]$, $\mathrm{cl}(G)$ will denote the lower semicontinuous envelope of G. In particular if $X = L^p(\omega; \mathbb{R}^3)$ is endowed with the strong topology, we denote $\mathrm{cl}_{\mathrm{p}}(G)$.

2. Preliminaries: $\Gamma(\pi)$-convergence

Since $\{E_\varepsilon\}_\varepsilon$ is defined on varying domains $\{\mathrm{A}_\varepsilon\}_\varepsilon$, we need to make precise definition of variational limit. In [5] Anzellotti-Baldo-Percivale introduced an adapted notion of variational convergence for these kind of problems: the $\Gamma(q)$-convergence. In the following, we state the main results of this convergence.
Let X be a topological space. Let $\{\mathrm{A}_\varepsilon\}_\varepsilon$ be a family of arbitrary sets and $E_\varepsilon : \mathrm{A}_\varepsilon \to [0, +\infty]$ be a family of functionals. Let $\{\pi_\varepsilon\}_\varepsilon$, $\pi_\varepsilon : \mathrm{A}_\varepsilon \to X$ be a family of maps.

Definition 2.1. We say that $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$-converges, if for every $v \in X$ and $\{\varepsilon_n\}_n \subset]0, 1]$ such that $\varepsilon_n \to 0$ when $n \to +\infty$ we have

$$\sup_{U \in \mathcal{V}_v} \liminf_{n \to +\infty} \inf E_{\varepsilon_n}(\pi^{-1}_{\varepsilon_n}(U)) = \sup_{U \in \mathcal{V}_v} \limsup_{n \to +\infty} \inf E_{\varepsilon_n}(\pi^{-1}_{\varepsilon_n}(U)),$$

where $\mathcal{V}_v$ is the set of all open neighborhoods of v in X.

The following lemma makes precise the fact that this definition is not far from Γ-convergence. For each $\varepsilon \in]0, 1]$ we consider $\mathcal{E}_\varepsilon : X \to [0, +\infty]$ given by

$$\mathcal{E}_\varepsilon(v) = \inf \Big\{ E_\varepsilon(w) : w \in \pi^{-1}_\varepsilon(v) \Big\}.$$

Lemma 2.2. *For every $v \in X$ and $\{\varepsilon_n\}_n \subset]0, 1]$ such that $\varepsilon_n \to 0$ when $n \to +\infty$, we have*

(a) $\displaystyle\sup_{U \in \mathcal{V}_v} \liminf_{n \to +\infty} \inf E_{\varepsilon_n}(\pi^{-1}_{\varepsilon_n}(U)) = (\Gamma\text{-}\liminf_{n \to +\infty} \mathcal{E}_{\varepsilon_n})(v)$;

(*b*) $\sup_{U\in\mathcal{V}_v}\limsup_{n\to+\infty}\inf E_{\varepsilon_n}(\pi_{\varepsilon_n}^{-1}(U)) = (\Gamma\text{-}\limsup_{n\to+\infty}\mathcal{E}_{\varepsilon_n})(v)$.

Furthermore, assume that for every $\varepsilon\in]0,1]$ *we have* $\mathrm{A}_\varepsilon=\mathrm{A}$, $\pi_\varepsilon=\pi$ *and* $E_\varepsilon=E$ *then*

(*c*) $\sup_{U\in\mathcal{V}_v}\inf E(\pi^{-1}(U)) = \mathrm{cl}(\mathcal{E})(v)$,
where $\mathcal{E}(v)=\inf\Big\{E(w):w\in\pi^{-1}(v)\Big\}$ *for all* $v\in X$.

Proof. It is sufficient to note that for every $v\in X$, $U\in\mathcal{V}_v$ and $\varepsilon\in]0,1]$ we have

$$\inf E_\varepsilon(\pi_\varepsilon^{-1}(U)) = \inf_{w\in U}\inf_{u\in\pi_\varepsilon^{-1}(w)} E(u).$$

Note also that (*c*) is a particular case of (*a*). □

Remark 2.3.

i) $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$-converges to $\mathcal{E}$ if and only if $\{\mathcal{E}_\varepsilon\}_\varepsilon$ Γ-converges to $\mathcal{E}$.

ii) The Definition 2.1 covers the case of Γ-convergence. For instance, if $\mathrm{A}_\varepsilon=X$ and $\pi_\varepsilon=\pi$ for all $\varepsilon\in]0,1]$, with π the identity map, then $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$-converges if and only if $\{E_\varepsilon\}_\varepsilon$ Γ-converges. If moreover $E_\varepsilon=E$ for all $\varepsilon\in]0,1]$, then $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$-converges to $\mathrm{cl}(E)$ (the lower semicontinuous envelope of E).

iii) If $\mathrm{A}\subset X$ and $\pi:\mathrm{A}\to X$ is the natural embedding then $E:\mathrm{A}\to[0,+\infty]$ $\Gamma(\pi)$-converges to $\mathrm{cl}(\hat{E})$, where $\hat{E}:X\to[0,+\infty]$ is given by

$$\hat{E}(v)=\begin{cases}E(v) & \text{if } v\in\mathrm{A}\\ +\infty & \text{otherwise.}\end{cases}$$

A sequence $\{u_\varepsilon\}_\varepsilon$ is said to be an ε-minimizing sequence of $\{E_\varepsilon\}_\varepsilon$, if for each $\varepsilon\in]0,1]$

$$u_\varepsilon\in\mathrm{A}_\varepsilon \text{ and } E_\varepsilon(u_\varepsilon)\le\inf\{E_\varepsilon(u):u\in\mathrm{A}_\varepsilon\}+\varepsilon.$$

The following result is an analogue of convergence of infima and minimizers result in the theory of Γ-convergence. Under the notations of Definition 2.1, we have

Proposition 2.4. *Assume that* X *is a metric space. Assume that*

(*i*) $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$*-converges to* F*;*

(*ii*) *for any* ε*-minimizing sequence* $\{u_\varepsilon\}_\varepsilon$ *of* $\{E_\varepsilon\}_\varepsilon$, $\{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon\subset X$ *is relatively compact.*

Then for every ε*-minimizing sequence* $\{u_\varepsilon\}_\varepsilon$ *of* $\{E_\varepsilon\}_\varepsilon$, *there exist a sequence* $\{\varepsilon_k\}_k\subset]0,1]$ *going to zero, and* $\hat{v}\in X$ *such that*

$$\pi_{\varepsilon_k}(u_{\varepsilon_k})\to\hat{v} \quad\textit{and}\quad \lim_{\varepsilon\to0}\inf\{E_\varepsilon(u):u\in\mathrm{A}_\varepsilon\}=\inf\{F(v):v\in X\}=F(\hat{v}).$$

Proof. Let $\varepsilon\in]0,1]$. We begin by noticing that there is conservation of infima between E_ε and $\mathcal{E}_\varepsilon$. Indeed, since $\pi_\varepsilon^{-1}(X)=\bigcup\{\pi_\varepsilon^{-1}(v):v\in X\}=\mathrm{A}_\varepsilon$, then

$$\inf_{u\in\mathrm{A}_\varepsilon}E_\varepsilon(u)=\inf_{\bigcup\{\pi_\varepsilon^{-1}(v):v\in X\}}E_\varepsilon(u)=\inf_{v\in X}\inf_{u\in\pi_\varepsilon^{-1}(v)}E_\varepsilon(u)=\inf_{v\in X}\mathcal{E}_\varepsilon(v). \tag{1}$$

Let $\{u_\varepsilon\}_\varepsilon$ be an ε-minimizing sequence of $\{E_\varepsilon\}_\varepsilon$. From (1), we can deduce the following inequalities

$$\mathcal{E}_\varepsilon(\pi_\varepsilon(u_\varepsilon))\le E_\varepsilon(u_\varepsilon)\le\inf_{u\in\mathrm{A}_\varepsilon}E_\varepsilon(u)+\varepsilon=\inf_{v\in X}\mathcal{E}_\varepsilon(v)+\varepsilon,$$

which proves that $\{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon$ is an ε-minimizing sequence of $\{\mathcal{E}_\varepsilon\}_\varepsilon$. By (ii) $\{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon$ is relatively compact. From (i) and according to Remark 2.3 i), we have that $\{\mathcal{E}_\varepsilon\}_\varepsilon$ Γ-converges to F. Thus by property of Γ-convergence (see [10]), we deduce that there exist a sequence $\{\varepsilon_k\}_k \subset]0,1]$ which satisfies $\varepsilon_k \to 0$ as $k \to +\infty$, and $\hat{v} \in X$ such that

$$\pi_{\varepsilon_k}(u_{\varepsilon_k}) \to \hat{v} \quad \text{and} \quad \lim_{\varepsilon\to 0} \inf_{v\in X} \mathcal{E}_\varepsilon(v) = \inf_{v\in X} F(v) = F(\hat{v}).$$

From (1) it follows

$$\lim_{\varepsilon\to 0} \inf_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) = \inf_{v\in X} F(v) = F(\hat{v}).$$

This completes the proof. □

Remark 2.5. Since the $\Gamma(\pi)$-convergence depends on the choice of the topological space X and the "projections" maps $\{\pi_\varepsilon\}_\varepsilon$, we should pay attention to choose them to avoid losing informations on deformations.

3. Statements of main results

Throughout the paper, we consider three conditions on $\hat{W}$.
There exists $p \in [1; +\infty[$ such that:

(H$_0$) there exists $\alpha > 0$ such that for every $x \in \overline{\omega}$ and $F \in \mathbb{M}^{3\times 2} \times \Lambda(x)$

$$\alpha|F|^p \le \hat{W}(F);$$

(H$_1$) there exists $\beta > 0$ such that for every $F \in \mathbb{M}^{3\times 2} \times \{e\}$

$$\hat{W}(F) \le \beta(1 + |F|^p).$$

Moreover, we will suppose that:

(H$_2$) for every $v \in \mathcal{C}^\infty_*(\omega;\mathbb{R}^3)$ the function

$$x \mapsto \max\left\{\hat{W}(\nabla v(x) \mid r\phi(x) + (1-r)\hat{\phi}(x)) : r \in [0,1]\right\}$$

belongs to $L^1_{loc}(\omega)$ for some $\hat{\phi} \in \mathcal{C}^\infty_e(\omega;\Lambda)$ and for all $\phi \in \mathcal{C}^\infty_e(\omega;\Lambda)$.

Remark 3.1. Note that if $\hat{W}$ satisfies standard growth conditions, i.e.,

$$c|F|^p \le \hat{W}(F) \le C(1 + |F|^p)$$

for some $c > 0$, $C > 0$, $p \ge 1$, and for all $F \in \mathbb{M}^{3\times 3}$, then (H$_1$) and (H$_2$) are satisfied. Assumption (H$_2$) is satisfied if, for instance, we suppose that the function

$$\zeta \mapsto \hat{W}(\cdot \mid \zeta)$$

is convex.

3.1. Abstract variational convergence result

Let $\varepsilon \in]0,1]$. Set $X = L^p(\omega;\mathbb{R}^3)$ endowed with the strong topology and $\pi_\varepsilon : \mathrm{A}_\varepsilon \to L^p(\omega;\mathbb{R}^3)$ the average operator defined by

$$\pi_\varepsilon(u)(\cdot) = \fint_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} u(\cdot, x_3)dx_3 \ \Big(= \frac{1}{\varepsilon}\int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} u(\cdot, x_3)dx_3\Big).$$

Note that, since the thickness of the structure is small, roughly speaking, we don't lose informations on original deformations by replacing them with their mean value. Let $\mathcal{E}_\varepsilon : L^p(\omega;\mathbb{R}^3) \to [0+\infty]$ be defined by

$$\mathcal{E}_\varepsilon(v) = \inf\{E_\varepsilon(u) : u \in \pi_\varepsilon^{-1}(v)\}.$$

Let us define the functional[4] $I : L^p(\omega;\mathbb{R}^3) \to [0,+\infty]$ by

$$I(v) = \begin{cases} \inf\left\{\displaystyle\int_\omega W(x,(\nabla v(x) \mid \phi(x)))dx : \phi \in \mathcal{C}_e^\infty(\omega;\Lambda)\right\} & \text{if } v \in \mathcal{C}_*^\infty(\omega;\mathbb{R}^3) \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{C}_e^\infty(\omega;\Lambda) = \Big\{\phi \in \mathcal{C}^\infty(\overline{\omega};\mathbb{R}^3) : \phi(x) = e \text{ on } \partial\omega \text{ and } \phi(x) \in \Lambda(x) \text{ a.e. in } \omega\Big\}.$$

Now, we can give one main result of the paper.

Theorem 3.2. *Assume that* (H$_0$) *holds, then*

(*a*) $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$*-converges to* F *given by*

$$F = \Gamma\text{-}\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon = \mathrm{cl_p}(I);$$

(*b*) *for every* ε*-minimizing sequence* $\{u_\varepsilon\}_\varepsilon$ *of* $\{E_\varepsilon\}_\varepsilon$*, there exist a sequence* $\{\varepsilon_k\}_k \subset]0,1]$ *going to zero, and* $\hat{v} \in L^p(\omega;\mathbb{R}^3)$*, such that*

$$\pi_{\varepsilon_k}(u_{\varepsilon_k}) \to \hat{v} \text{ in } L^p(\omega;\mathbb{R}^3) \quad \text{and} \quad \lim_{\varepsilon\to 0}\inf_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) = \inf_{v\in L^p(\omega;\mathbb{R}^3)} F(v) = F(\hat{v}).$$

Remark 3.3. Assertion (*b*) is an abstract variational formulation in the sense that we have not identified the "limit" stored energy function associated with the plate.

3.2. Integral representations in Lebesgue's spaces

Now we deal with the problem of giving an integral representation for $F = \mathrm{cl_p}(I)$.
Let us introduce some notation. Let $f : \overline{\omega}\times\mathbb{M}^{3\times 2} \to [0,+\infty[$ be a Carathéodory function, following Morrey [14], f is quasiconvex if for every $x \in \overline{\omega}$

$$f(x,\xi) \le \fint_A f(x,\xi+\nabla\phi(z))dz$$

for all $\xi \in \mathbb{M}^{3\times 2}$, $A \subset \mathbb{R}^N$ bounded open set, and for all $\phi \in \mathcal{C}_c^\infty(A;\mathbb{R}^3)$. The quasiconvex envelope of f is the greatest quasiconvex function $g : \overline{\omega}\times\mathbb{M}^{3\times 2} \to [0,+\infty[$ which satisfies $g(x,\cdot) \le f(x,\cdot)$ for all $x \in \overline{\omega}$. We denote $Qf = g$. The recession function f^∞ associated with f is given by

$$f^\infty(x,\xi) = \limsup_{t\to+\infty} \frac{f(x,t\xi)}{t}.$$

If $u \in L^1(\omega;\mathbb{R}^3)$ is such that the distributional derivative Du is a bounded Radon measure, then u belongs to the space of bounded variation functions $BV(\omega;\mathbb{R}^3)$. We denote by $\gamma : BV(\omega;\mathbb{R}^3) \to L^1(\partial\omega)$ the trace operator, and $\nu(x)$ the outward unit normal at $\mathcal{H}_2$ almost all x on $\partial\omega$.
Now we can state the integral representations for F.

[4]This functional has relationships with geometric integrals, see [3, 4, 13].

Theorem 3.4. *Assume that* (H_0), (H_1) *and* (H_2) *hold.*

(a) *If* $p > 1$ *then*

$$F(v) = \begin{cases} \displaystyle\int_\omega QW_\Lambda(x, \nabla v(x))dx & \text{if } v \in W^{1,p}_*(\omega; \mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

(b) *If* $p = 1$, *we assume moreover that*

- *there exist* $C, L > 0$ *and* $m \in]0; 1[$ *such that*

$$\left| W^\infty_\Lambda(x, \xi) - \frac{W_\Lambda(x, t\xi)}{t} \right| \leq \frac{C}{t^m}$$

for all $(x, \xi) \in \overline{\omega} \times \mathbb{M}^{3\times 2}$ *satisfying* $|\xi| = 1$ *and* $t > L$;

- Λ *is a closed*[5] *multifunction.*

then

$$F(v) = \begin{cases} \displaystyle\int_\omega QW_\Lambda(x, \nabla v(x))dx + \int_\omega (QW_\Lambda)^\infty(x, D_s v) & \\ \qquad \displaystyle + \int_{\partial\omega} (QW_\Lambda)^\infty(x, \gamma(v)(x) \otimes \nu(x))d\mathcal{H}_2(x) & \text{if } v \in BV(\omega; \mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

Now, from Theorem 3.2 and 3.4 we deduce easily variational formulations.

Corollary 3.5. *Let assumptions of Theorem* 3.4 *hold. For every* ε*-minimizing sequence* $\{u_\varepsilon\}_\varepsilon$ *of* $\{E_\varepsilon\}_\varepsilon$, *there exist a sequence* $\{\varepsilon_k\}_k \subset]0, 1]$ *going to zero, and* $\hat{v} \in L^p(\omega; \mathbb{R}^3)$ *such that*

$$\pi_{\varepsilon_k}(u_{\varepsilon_k}) \to \hat{v} \quad \text{in } L^p(\omega; \mathbb{R}^3),$$

and if $p > 1$

$$\lim_{\varepsilon\to 0} \inf_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) = \inf_{v\in W^{1,p}_*(\omega;\mathbb{R}^3)} \int_\omega QW_\Lambda(x, \nabla v(x))dx = \int_\omega QW_\Lambda(x, \nabla \hat{v}(x))dx;$$

if $p = 1$

$$\begin{aligned} \lim_{\varepsilon\to 0} \inf_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) &= \inf_{v\in BV(\omega;\mathbb{R}^3)} \Big\{ \int_\omega QW_\Lambda(x, \nabla v(x))dx + \int_\omega (QW_\Lambda)^\infty(x, D_s v) \\ &\qquad + \int_{\partial\omega} (QW_\Lambda)^\infty(x, \gamma(v)(x) \otimes \nu(x))d\mathcal{H}_2(x) \Big\} \\ &= \int_\omega QW_\Lambda(x, \nabla \hat{v}(x))dx + \int_\omega (QW_\Lambda)^\infty(x, D_s \hat{v}) \\ &\qquad + \int_{\partial\omega} (QW_\Lambda)^\infty(x, \gamma(\hat{v})(x) \otimes \nu(x))d\mathcal{H}_2(x). \end{aligned}$$

3.3. Variational formulation in Young measures space

Here, we establish a variational formulation theorem[6] in Young measures space for problems $(\mathcal{P}_\varepsilon)$. The main interest is that there is no quasiconvexification of the density in

[5] Λ is a closed multifunction if the set $\{(x, \zeta) \in \overline{\omega} \times \mathbb{R}^3 : \zeta \in \Lambda(x)\}$ which is the graph of Λ, is closed.

[6] This result has already proved in [3].

the relaxation step. Thus limit minimizing sequences may capture oscillations of gradient minimizing sequences due to possibly multiwell structure of the density W_Λ by means of probability measures whose barycenter is a limit gradient, solution of the classical limit problem treated in previous sections (see Corollary 6.3). Indeed, if W exhibits potential wells, it will be the same for W_Λ, thus this model may account for microstructures in thin films as for instance, possible untwinned austenite/martinsite interfaces for thin structures, predict by Battacharya and James in [6].
Following [16], a Young measure $\nu \in \mathcal{Y}(\overline{\omega}; \mathbb{M}^{3\times 2})$ is a positive measure on $\overline{\omega} \times \mathbb{M}^{3\times 2}$ such that for every borel set $A \subset \overline{\omega}$, $\nu(A \times \mathbb{M}^{3\times 2}) = |A|$, where $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}^2$. The space of Young measures $\mathcal{Y}(\overline{\omega}; \mathbb{M}^{3\times 2})$ is endowed with the narrow topology which is the weakest topology on $\mathcal{Y}(\overline{\omega}; \mathbb{M}^{3\times 2})$ for which the maps

$$\nu \mapsto \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi d\nu$$

are continuous for all bounded Carathéodory integrand ϕ. Let $\mathcal{W}^{1,p}_*$ be the subset of the space of Young measures generated by gradients. More precisely, $\nu \in \mathcal{W}^{1,p}_*$ if there exists a bounded sequence $\{v_k\}_k \subset W^{1,p}_*(\omega; \mathbb{R}^3)$ such that $\delta_{\{\nabla v_k(\cdot)\}} \overset{nar}{\rightharpoonup} \nu$, i.e., $\delta_{\{\nabla v_k(\cdot)\}}$ narrowly converges to ν.

Theorem 3.6. *Let* $p \in]1, +\infty[$. *Assume that* (H_0), (H_1) *and* (H_2) *hold. Let* $G : \mathcal{Y}(\overline{\omega}; \mathbb{M}^{3\times 2}) \to [0, +\infty]$ *be defined by*

$$G(\nu) = \begin{cases} \displaystyle\int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} W_\Lambda(x,\xi) d\nu(x,\xi) & \text{if } \nu \in \mathcal{W}^{1,p}_* \\ +\infty & \text{otherwise.} \end{cases}$$

For any ε*-minimizing sequence* $\{u_\varepsilon\}_\varepsilon$ *of* $\{E_\varepsilon\}_\varepsilon$, *there exist a sequence* $\{\varepsilon_k\}_k \subset]0,1]$ *going to zero, and* $\hat{\nu} \in \mathcal{Y}(\overline{\omega}; \mathbb{M}^{3\times 2})$, *such that*

$$\delta_{\{\nabla \pi_{\varepsilon_k}(u_{\varepsilon_k})(\cdot)\}} \overset{nar}{\rightharpoonup} \hat{\nu} \text{ and } \lim_{\varepsilon\to 0} \inf_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) = \inf_{\nu \in \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})} G(\nu) = G(\hat{\nu}).$$

4. Proof of Theorem 3.2

Proof of (a). By Lemma 2.2, it suffices to prove that $\{\mathcal{E}_\varepsilon\}_\varepsilon$ Γ-converges to $\mathrm{cl_p}(I)$. The proof will be divided into 2 steps.
Step 1. Let $v \in L^p(\omega; \mathbb{R}^3)$, we begin by proving

$$(\Gamma\text{-}\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon)(v) \le \mathrm{cl_p}(I)(v).$$

Let $\varepsilon \in]0, 2^{-1}]$. Note that if u is of the form $u(x, x_3) = w(x) + x_3\phi(x)$ with $w \in \mathcal{C}^\infty_*(\omega; \mathbb{R}^3)$, $\phi \in \mathcal{C}^\infty_e(\omega; \Lambda)$, then $u \in \pi_\varepsilon^{-1}(w)$. From that, we define $\mathcal{S}_\varepsilon : L^p(\omega; \mathbb{R}^3) \to [0, +\infty]$ by

$$\mathcal{S}_\varepsilon(w) = \begin{cases} \displaystyle\inf_{\phi\in\mathcal{C}^\infty_e(\omega;\Lambda)} \Big\{ \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} W(x, (\nabla w(x) \\ \qquad + x_3 \nabla\phi(x) \mid \phi(x))) dx dx_3 \Big\} & \text{if } w \in \mathcal{C}^\infty_*(\omega; \mathbb{R}^3) \\ +\infty & \text{otherwise.} \end{cases}$$

It follows easily that $\mathcal{E}_\varepsilon(v) \le \mathcal{S}_\varepsilon(v)$. Assume that

$$\limsup_{\varepsilon\to 0} \mathcal{S}_\varepsilon(v) \le I(v). \tag{2}$$

Then we will conclude by the following inequalities

$$(\Gamma\text{-}\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon)(v) \le \mathrm{cl_p}(\limsup_{\varepsilon\to 0}\mathcal{E}_\varepsilon)(v) \le \mathrm{cl_p}(\limsup_{\varepsilon\to 0}\mathcal{S}_\varepsilon)(v) \le \mathrm{cl_p}(I)(v).$$

We are reduced to prove (2). Without loss of generality, assume $I(v) < +\infty$. Let $\phi \in \mathcal{C}_e^\infty(\omega;\Lambda)$, then

$$\mathcal{S}_\varepsilon(v) \le \frac{1}{\varepsilon}\int_{\Omega_\varepsilon} \hat{W}(\nabla v(x) + x_3\nabla\phi(x) \mid \phi(x))dx_3dx.$$

Note that the function $\Psi : \overline{\Omega}\times[0,1] \to \mathbb{M}^{3\times 3}$ defined by

$$\Psi(x, x_3, \lambda) = (\nabla v(x) + \lambda x_3\nabla\phi(x) \mid \phi(x))$$

is continuous and so its range $\Psi(\overline{\Omega}\times[0,1])$ is compact. From continuity assumption on $\hat{W}$, it follows that for every $(x, x_3, \varepsilon) \in \overline{\Omega}\times]0,1]$

$$\hat{W}(\nabla v(x) + \varepsilon x_3\nabla\phi(x) \mid \phi(x)) \le \sup(\hat{W}\circ\Psi)(\overline{\Omega}\times[0,1]) < +\infty,$$

and

$$\lim_{\varepsilon\to 0}\hat{W}(\nabla v(x) + \varepsilon x_3\nabla\phi(x) \mid \phi(x)) = \hat{W}(\nabla v(x) \mid \phi(x)).$$

By Lebesgue's dominated convergence theorem, we deduce that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\hat{W}(\nabla v(x) + x_3\nabla\phi(x) \mid \phi(x))dx_3dx\\ =\ &\lim_{\varepsilon\to 0}\int_{\omega\times]-\frac12,\frac12[}\hat{W}(\nabla v(x) + \varepsilon x_3\nabla\phi(x) \mid \phi(x))dx = \int_\omega \hat{W}(\nabla v(x) \mid \phi(x))dx,\end{aligned}$$

and therefore

$$\limsup_{\varepsilon\to 0}\mathcal{S}_\varepsilon(v) \le \inf_{\phi\in\mathcal{C}_e^\infty(\omega;\Lambda)}\int_\omega W(x, (\nabla v(x) \mid \phi(x)))dx,$$

which completes the first step.

Step 2. Let us show that for every $v \in L^p(\omega;\mathbb{R}^3)$

$$(\Gamma\text{-}\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon)(v) \ge \mathrm{cl_p}(I)(v).$$

Let $v \in L^p(\omega;\mathbb{R}^3)$ and $\{v_\varepsilon\}_\varepsilon \subset \mathcal{C}_*^\infty(\omega;\mathbb{R}^3)$ such that $v_\varepsilon \to v$ in $L^p(\omega;\mathbb{R}^3)$. Let $\{\varepsilon_n\}_n \subset]0,1]$ be such that $\varepsilon_n \to 0$ when $n\to\infty$. There is no loss of generality in assuming

$$\liminf_{n\to\infty}\mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}) < +\infty.$$

There exists a subsequence still denoted $\{v_{\varepsilon_n}\}_n$ such that

$$\sup\{\mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}) : n\in\mathbb{N}\} < +\infty \quad \text{and} \quad \lim_{n\to\infty}\mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}) = \liminf_{n\to\infty}\mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}).$$

To simplify notation, we write $\mathcal{E}_{\varepsilon_n} = \mathcal{E}_n$, $v_{\varepsilon_n} = v_n$ and $E_{\varepsilon_n} = E_n$ for all $n\in\mathbb{N}$. Let $n\in\mathbb{N}$. There exists $u_n \in \pi_{\varepsilon_n}^{-1}(v_n)$ such that

$$\frac{\partial u_n}{\partial x_3}(x, x_3) \in \Lambda(x)$$

for all $(x,x_3)\in\overline{\Omega}_{\varepsilon_n}$ and moreover $E_n(u_n)\le\mathcal{E}_n(v_n)+\varepsilon_n$. Let $w_n(x,x_3)=u_n(x,\varepsilon_n x_3)$ with $(x,x_3)\in\overline{\Omega}$. This gives $E_n(u_n)=\mathcal{G}_n(w_n)$ where

$$\mathcal{G}_n(w_n)=\int_\Omega W\left(x,\left(\frac{\partial w_n}{\partial x}\;\Big|\;\frac{1}{\varepsilon_n}\frac{\partial w_n}{\partial x_3}\right)\right)dxdx_3.$$

According to coercivity condition (H_0) on $\hat{W}$, there exists a constant $C>0$, such that $\left\|\frac{\partial w_n}{\partial x_3}\right\|_{L^p(\Omega;\mathbb{R}^3)}\le C\varepsilon_n^p\sup\{\mathcal{G}_n(v_n):n\in\mathbb{N}\}$. By Poincaré-Wirtinger's inequality, we have

$$\|w_n-v_n\|_{L^p(\Omega;\mathbb{R}^3)}\le C'\left\|\frac{\partial w_n}{\partial x_3}\right\|_{L^p(\Omega;\mathbb{R}^3)},$$

where $C'>0$ depends on k and p only. From above, it follows that $w_n\to v$ in $L^p(\Omega;\mathbb{R}^3)$. Define $\tilde{w}_n^{x_3}(x)=w_n(x,x_3)$ for all $(x,x_3)\in\overline{\Omega}$, it is easy to see that, up to a subsequence, $\tilde{w}_n^{x_3}\to v$ in $L^p(\omega;\mathbb{R}^3)$ for almost all $x_3\in]-\frac{1}{2},\frac{1}{2}[$. By Fatou's lemma we obtain

$$\begin{aligned}
\lim_{n\to\infty}\mathcal{E}_{\varepsilon_n}(v_{\varepsilon_n}) &\ge \liminf_{n\to\infty}\mathcal{G}_n(w_n)\\
&= \liminf_{n\to\infty}\int_\Omega W\left(x,\left(\nabla\tilde{w}_n^{x_3}(x)\;\Big|\;\frac{\partial u_n}{\partial x_3}(x,\varepsilon_n x_3)\right)\right)dxdx_3\\
&\ge \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\liminf_{n\to\infty}\int_\omega W\left(x,\left(\nabla\tilde{w}_n^{x_3}(x)\;\Big|\;\frac{\partial u_n}{\partial x_3}(x,\varepsilon_n x_3)\right)\right)dx\right)dx_3\\
&\ge \int_{-\frac{1}{2}}^{\frac{1}{2}}\left(\liminf_{n\to\infty}\inf_{\phi\in\mathcal{C}_e^\infty(\omega;\Lambda)}\int_\omega W\left(x,\left(\nabla\tilde{w}_n^{x_3}(x)\mid\phi(x)\right)\right)dx\right)dx_3\\
&\ge \mathrm{cl_p}(I)(v).
\end{aligned}$$

Consequently, for every $v\in L^p(\Omega;\mathbb{R}^3)$

$$\mathrm{cl_p}(I)(v)\le(\Gamma\text{-}\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon)(v)\le(\Gamma\text{-}\limsup_{\varepsilon\to0}\mathcal{E}_\varepsilon)(v)\le\mathrm{cl_p}(I)(v),$$

which establishes that $\{\mathcal{E}_\varepsilon\}_\varepsilon$ Γ-converges to $\mathrm{cl_p}(I)$, and the proof of (a) is complete.

Proof of (b). We begin by proving a compactness lemma.

Lemma 4.1. *Assume that* (H_0) *holds. Let* $\{v_\varepsilon\}_\varepsilon\subset\mathcal{C}_*^\infty(\omega;\mathbb{R}^3)$ *such that*

$$\sup\{\mathcal{E}_\varepsilon(v_\varepsilon):\varepsilon\in]0,1]\}<+\infty.$$

Then there exist a subsequence $\{v_{\varepsilon_k}\}_k$ *and* $v\in L^p(\omega;\mathbb{R}^3)$ *such that* $v_{\varepsilon_k}\to v$ *in* $L^p(\omega;\mathbb{R}^3)$.

Proof. Let $\{v_\varepsilon\}_\varepsilon\subset\mathcal{C}_*^\infty(\omega;\mathbb{R}^3)$ such that $\sup\{\mathcal{E}_\varepsilon(v_\varepsilon):\varepsilon\in]0,1]\}<+\infty$. Let $\{\varepsilon_n\}_n\subset]0,1]$ such that $\varepsilon_n\to0$ when $n\to+\infty$. Set $\mathcal{E}_{\varepsilon_n}=\mathcal{E}_n$, $v_{\varepsilon_n}=v_n$ and $\pi_{\varepsilon_n}=\pi_n$ for all $n\in\mathbb{N}$. Let $n\in\mathbb{N}$. According to coercivity (H_0) condition and Jensen's inequality, we have

$$\begin{aligned}
\mathcal{E}_n(v_n) &= \inf\Big\{\int_\omega\Big(\int_{-\frac{\varepsilon_n}{2}}^{\frac{\varepsilon_n}{2}}W(x,\nabla u(x,x_3)dx_3\Big)dx:u\in\pi_n^{-1}(v_n)\Big\}\\
&\ge \beta\inf\Big\{\int_\omega\Big(\int_{-\frac{\varepsilon_n}{2}}^{\frac{\varepsilon_n}{2}}|\nabla u(x,x_3)|^pdx_3\Big)dx:u\in\pi_n^{-1}(v_n)\Big\}\\
&\ge \beta\int_\omega|\nabla v_n(x)|^pdx.
\end{aligned}$$

Poincaré's inequality gives $\sup\{||v_n||_{L^p(\omega;\mathbb{R}^3)}+||\nabla v_n||_{L^p(\omega;\mathbb{R}^3)}:n\in\mathbb{N}\}<+\infty$. The Rellich compactness embedding theorem leads to the existence of a subsequence $\{v_{n_k}\}_k$ and $v\in L^p(\omega;\mathbb{R}^3)$ such that $v_{n_k}\to v$ in $L^p(\omega;\mathbb{R}^3)$. This completes the proof of the lemma. □

By (a), $\{E_\varepsilon\}_\varepsilon$ $\Gamma(\pi)$-converges to F, thus (i) of Proposition 2.4 is satisfied. Let $\{u_\varepsilon\}_\varepsilon$ be an ε-minimizing sequence of $\{E_\varepsilon\}_\varepsilon$. Take $u_0(x,x_3)=(x,x_3)$ then

$$\begin{cases} u_0 \in \mathrm{A}_\varepsilon \text{ for all } \varepsilon \in]0,1], \; \dfrac{\partial u_0}{\partial x_3}(x,x_3) = e \in \Lambda(x); \\ \sup\limits_{\varepsilon\in]0,1]} \inf\limits_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) \le |\omega| \hat{W}(\mathrm{Id}) < +\infty, \text{ where } \mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{cases}$$

Thus, (1) of proof of Proposition 2.4 leads to

$$\sup_{\varepsilon\in]0,1]} \mathcal{E}_\varepsilon(\pi_\varepsilon(u_\varepsilon)) < +\infty,$$

and by the previous compactness Lemma 4.1, (ii) of Proposition 2.4 holds. The proof is then complete. □

5. Proof of Theorem 3.4

The proof of Theorem 3.4 is divided in two steps shown in the following subsections. In the first step, we obtain an integral representation of I by using interchange of infimum and integral argument. In the second step, by classical relaxation results, integral representations of $\mathrm{cl_p}(I)$ are obtained.

5.1. Integral representation of I

We recall in a less general form a result of interchange of infimum and integral (for more details see [4]). Let $p \in [1,+\infty[$ and $f : \overline{\omega}\times\mathbb{R}^3 \to [0,+\infty]$ be a Carathéodory integrand. Let $\mathcal{X} \subset L^p(\omega;\mathbb{R}^3)$ be a $\mathcal{C}_c^\infty(\omega;[0,1])$-decomposable set, i.e. $\varphi w + (1-\varphi)\hat{w} \in \mathcal{X}$ whenever $w,\ \hat{w} \in \mathcal{X}$ and $\varphi \in \mathcal{C}_c^\infty(\omega;[0,1])$. Suppose that there exists $\hat{\phi} \in \mathcal{X}$ such that

$$\int_\omega f(x,\hat{\phi}(x))dx < +\infty,$$

and such that the function $x \mapsto \max\{f(x,\alpha\phi(x)+(1-\alpha)\hat{\phi}(x)) :\ 0 \le \alpha \le 1\}$ belongs to $L^1_{loc}(\omega)$, for all $\phi \in \mathcal{X}$. Then we have the interchange of infimum and integral

$$\inf_{\phi\in\mathcal{X}} \int_\omega f(x,\phi(x))dx = \int_\omega \inf_{\zeta\in\Gamma(x)} f(x,\zeta)dx,$$

where Γ is *the essential supremum* of $\mathcal{X}$, i.e.

- $\phi(x) \in \Gamma(x)$ a.e. for all $\phi \in \mathcal{X}$;
- and $\Gamma(x) \subset \Gamma'(x)$ a.e. whenever Γ' is a closed valued measurable[7] multifunction satisfying $\phi(x) \in \Gamma'(x)$ a.e. for all $\phi \in \mathcal{X}$.

We will need the following lemma which we recall in our context, see [7] Sect. 2.2 for the general case. This one gives some characterizations of the essential supremum.

Characterization lemma. Let $p \in [1,+\infty[$ and let Γ be the essential supremum of $\mathcal{X} \subset L^p(\omega;\mathbb{R}^3)$. Then:

[7]A multifunction $\Sigma : \overline{\omega} \rightrightarrows \mathbb{R}^3$ is measurable if $\{x \in \overline{\omega} : \Sigma(x)\cap U \neq \emptyset\}$ is measurable for all open subset $U \subset \mathbb{R}^3$.

(1) there exists a countable subset $\mathcal{D} \subset \mathcal{X}$ such that $\Gamma(x) = \mathrm{cl}\{w(x) : w \in \mathcal{D}\}$ a.e.;
(2) if $\mathcal{X} \subset \mathcal{C}_e(\omega; \mathbb{R}^3)$, then $\Gamma(x) = \mathrm{cl}\{w(x) : w \in \mathcal{X}\}$ a.e..

The following lemma allows us to determine the essential supremum of $\mathcal{C}_e^\infty(\omega; \Lambda)$.

Lemma 5.1. *Let* $\Gamma : \overline{\omega} \rightrightarrows \mathbb{R}^3$ *be defined by* $\Gamma = \operatorname{ess\,sup}(\mathcal{C}_e^\infty(\omega; \Lambda))$ *then* $\Gamma = \Lambda$ *a.e..*

Proof. From (1) of the Characterization lemma, there exists a countable subset $\mathcal{D} \subset \mathcal{C}_e^\infty(\omega; \Lambda)$ such that $\Gamma(x) = \mathrm{cl}\{\phi(x) : \phi \in \mathcal{D}\}$ a.e. Hence $\Gamma(x) \subset \Lambda(x)$ a.e. Let $x \in \overline{\omega}$ and $\zeta \in \Lambda(x)$. Let $\phi \in \mathcal{C}^\infty(\overline{\omega}; \Lambda)$ be defined by $\phi(z) = \zeta$ for all $z \in \overline{\omega}$. Let $\varphi_x \in \mathcal{C}_c^\infty(\omega; [0; 1])$ defined by $\varphi_x(z) = 0$ if $z \in \overline{\omega} \setminus B_{2\delta}(x)$ and $\varphi_x(z) = 1$ if $z \in \overline{B}_\delta(x)$, with δ small enough for $\overline{B}_{2\delta}(x) \subset \omega$. Set $v_x = \varphi_x \phi + (1 - \varphi_x)e$. Since $v_x(z) = \varphi_x(z)\phi(z) + (1 - \varphi_x(z))e$ is a convex combination of elements of $\Lambda(z)$, then $v_x(z) \in \Lambda(z)$ for all $z \in \overline{\omega}$. Thus $v_x \in \mathcal{C}_e^\infty(\omega; \Lambda)$ satisfies $v_x(x) = \zeta$, therefore $\Lambda(x) \subset \{\phi(x) : \phi \in \mathcal{C}_e^\infty(\omega; \Lambda)\} \subset \Gamma(x)$ for all $x \in \overline{\omega}$, where the last inclusion follows from (2) of the Characterization lemma. This proves the lemma. □

Proposition 5.2. *Assume that* (H_2) *holds, then*

$$I(v) = \begin{cases} \displaystyle\int_\omega W_\Lambda(x, \nabla v(x))dx & \text{if } v \in \mathcal{C}_*^\infty(\omega; \mathbb{R}^3) \\ +\infty & \text{otherwise,} \end{cases}$$

where $W_\Lambda : \overline{\omega} \times \mathbb{M}^{3\times 2} \to [0, +\infty[$ *is defined by*

$$W_\Lambda(x, \xi) = \inf\{W(x, (\xi \mid \zeta)) : \zeta \in \Lambda(x)\} = \inf\left\{\hat{W}(\xi \mid \zeta) : \zeta \in \Lambda(x)\right\}.$$

Proof. It is easy to see that $\mathcal{C}_e^\infty(\omega; \Lambda)$ is a $\mathcal{C}_c^\infty(\omega; [0, 1])$-decomposable set. Let $v \in \mathcal{C}_*^\infty(\omega; \mathbb{R}^3)$ and $f : \overline{\omega} \times \mathbb{R}^3 \to [0, +\infty]$ be defined by $f(x, \zeta) = \hat{W}(\nabla v(x) \mid \zeta)$. Obviously f is a Carathéodory integrand. Let $\mathcal{C}_e^\infty(\omega; \Lambda) \ni \hat{\phi} = e$, then

$$\int_\omega f(x, \hat{\phi}(x))dx = \int_\omega \hat{W}(\nabla v(x) \mid e)dx \le \sup_{x \in \overline{\omega}} \hat{W}(\nabla v(x) \mid e).|\omega| < +\infty.$$

Assumption (H_2) and interchange of infimum and integral argument together with Lemma 5.1 lead us to

$$I(v) = \int_\omega W_\Lambda(x, \nabla v(x))dx,$$

which completes the proof. □

5.2. Integral representations of $F = \mathrm{cl}_\mathrm{p}(I)$

The function W_Λ inherites continuity, growth and coercivity conditions from W. More precisely, we have

Lemma 5.3. *Let assumptions* (H_0) *and* (H_1) *holds. Then* W_Λ *is a Carathéodory function and satisfies growth and coercivity condition of order* $p \in [1, +\infty[$*, i.e., there exists* $\alpha_0, \beta_0 > 0$ *such that for every* $(x, \xi) \in \overline{\omega} \times \mathbb{M}^{3\times 2}$

$$\alpha_0 |\xi|^p \le W_\Lambda(x, \xi) \le \beta_0(1 + |\xi|^p).$$

Moreover if Λ *is a closed multifunction, then* W_Λ *is continuous.*

Proof. Let us first prove that W_Λ satisfies growth and coercivity conditions. It is easy to see that there exists α', β' depending on p, such that for every $(\xi \mid \zeta) \in \mathbb{M}^{3\times 2} \times \mathbb{R}^3$

$$\alpha'(|\xi|^p + |\zeta|^p) \leq |(\xi \mid \zeta)|^p \leq \beta'(|\xi|^p + |\zeta|^p).$$

Let $x \in \overline{\omega}$, $\overline{\xi} \in \mathbb{M}^{3\times 2}$ and set $\alpha_0 = \alpha\alpha'$ and $\beta_0 = \beta(1+\beta')$. By (H_0), we have

$$W_\Lambda(x, \overline{\xi}) \geq \alpha \inf_{\zeta\in\Lambda(x)} |(\overline{\xi} \mid \zeta)|^p \geq \alpha\alpha' \left(|\overline{\xi}|^p + \inf_{\zeta\in\Lambda(x)} |\zeta|^p \right) \geq \alpha_0 |\overline{\xi}|^p.$$

By (H_1), we have

$$W_\Lambda(x, \overline{\xi}) \leq W(x, (\overline{\xi} \mid e)) \leq \beta \max(1 + \beta'|e|^p, \beta')(1 + |\overline{\xi}|^p) = \beta_0(1 + |\overline{\xi}|^p).$$

Now let us show that W_Λ is a Carathéodory function. Classical arguments (see for instance [9]) lead to the measurability of $x \mapsto W_\Lambda(x, \cdot)$. Let $\overline{\xi}$, $\{\xi_n\}_n \subset \mathbb{M}^{3\times 2}$ such that $\xi_n \to \xi$. Let $x \in \overline{\omega}$. For every $n \in \mathbb{N}^*$ there exists $\zeta_n \in \Lambda(x)$ such that

$$W_\Lambda(x, \xi_n) = \inf_{\zeta\in\Lambda(x)} \hat{W}(\xi_n \mid \zeta) \geq \hat{W}(\xi_n \mid \zeta_n) - \frac{1}{n}.$$

According to coercivity condition (H_0) on $\hat{W}$, it follows, up to a subsequence, that $\zeta_n \to \overline{\zeta} \in \Lambda(x)$. From continuity of $\hat{W}$ we deduce that

$$\liminf_{n\to+\infty} W_\Lambda(x, \xi_n) \geq \lim_{n\to+\infty} \hat{W}(\xi_n \mid \zeta_n) = \hat{W}(\overline{\xi} \mid \overline{\zeta}) \geq W_\Lambda(x, \xi).$$

This proves the l.s.c of W_Λ. Now for the u.s.c of W_Λ, it is sufficient to remark that for each $\zeta \in \Lambda(x)$, we have

$$\limsup_{n\to+\infty} W_\Lambda(x, \xi_n) \leq \lim_{n\to+\infty} \hat{W}(\xi_n | \zeta) = \hat{W}(\overline{\xi} \mid \zeta).$$

Therefore $\xi \mapsto W_\Lambda(x, \xi)$ is continuous.

Assume now, that Λ is a closed multifunction. Let $(\overline{x}, \overline{\xi})$, $\{(x_n, \xi_n)\}_n \subset \overline{\omega} \times \mathbb{M}^{3\times 2}$ such that $x_n \to \overline{x}$ and $\xi_n \to \overline{\xi}$. We first prove the lower semicontinuity of W_Λ. Up to a subsequence (not relabelled), we have

$$\liminf_{n\to+\infty} W_\Lambda(x_n, \xi_n) = \lim_{n\to+\infty} W_\Lambda(x_n, \xi_n).$$

For every $n \in \mathbb{N}^*$ there exists $\zeta_n \in \Lambda(x_n)$ such that

$$W_\Lambda(x_n, \xi_n) \geq \hat{W}(\xi_n \mid \zeta_n) - \frac{1}{n}.$$

According to coercivity condition (H_0) on $\hat{W}$, it follows that there exist a subsequence $\{n_k\}_k$ going to infinity and $\overline{\zeta}$ such that $\zeta_{n_k} \to \overline{\zeta}$. Since the graph of Λ is closed, $\overline{\zeta} \in \Lambda(\overline{x})$. From the continuity of $\hat{W}$, we obtain

$$\begin{aligned} \liminf_{n\to+\infty} W_\Lambda(x_n, \xi_n) = \lim_{k\to+\infty} W_\Lambda(x_{n_k}, \xi_{n_k}) &\geq \lim_{k\to+\infty} \hat{W}(\xi_{n_k} \mid \zeta_{n_k}) \\ &= \hat{W}(\overline{\xi} \mid \overline{\zeta}) \geq W_\Lambda(\overline{x}, \overline{\xi}). \end{aligned}$$

The second step is to prove the upper semicontinuity of W_Λ. Up to a subsequence (not relabelled), we have

$$\limsup_{n\to+\infty} W_\Lambda(x_n,\xi_n) = \lim_{n\to+\infty} W_\Lambda(x_n,\xi_n).$$

Let $\overline{\zeta} \in \Lambda(\overline{x})$. From l.s.c of Λ, we can find a subsequence $\{x_{n_k}\}_k$ and a sequence $\{\zeta_k\}_k \subset \mathbb{R}^3$, satisfying

$$\forall k\in\mathbb{N}\ \ \zeta_k \in \Lambda(x_{n_k}) \ \text{ and } \ \lim_{k\to+\infty}\zeta_k = \overline{\zeta}.$$

By continuity of $\hat{W}$, we have

$$\begin{aligned}\limsup_{n\to+\infty} W_\Lambda(x_n,\xi_n) &= \lim_{n\to+\infty} W_\Lambda(x_n,\xi_n)\\ &= \lim_{k\to+\infty} W_\Lambda(x_{n_k},\xi_{n_k})\\ &\le \lim_{k\to+\infty} \hat{W}(\xi_{n_k},\zeta_{n_k}) = \hat{W}(\overline{\xi}\mid\overline{\zeta}).\end{aligned}$$

Since $\overline{\zeta} \in \Lambda(\overline{x})$ is arbitrary, the u.s.c of W_Λ follows. Which proves the lemma. □

We recall some results about relaxation of multiple integrals of the Calculus of Variations in our context. Let $J : L^p(\omega;\mathbb{R}^3) \to [0,+\infty]$ be defined by

$$J(v) = \begin{cases} \displaystyle\int_\omega f(x,\nabla v(x))dx & \text{if } v\in\mathcal{C}^\infty_*(\omega;\mathbb{R}^3)\\ +\infty & \text{otherwise,}\end{cases}$$

where $f : \overline{\omega}\times\mathbb{M}^{3\times2} \to [0,+\infty[$ is a Carathéodory function, such that there exist $c, C > 0$ and $p \in [1;+\infty[$ such that for all $(x,\xi) \in \overline{\omega}\times\mathbb{M}^{3\times2}$

$$c|\xi|^p \le f(x,\xi) \le C(1+|\xi|^p).$$

For a proof of the following relaxation theorem see for instance [2].

Theorem. If $p > 1$, the lower semicontinuous envelope $\mathrm{cl_p}(J)$ for the strong topology of $L^p(\omega;\mathbb{R}^3)$ is given by

$$\mathrm{cl_p}(J)(v) = \begin{cases} \displaystyle\int_\omega Qf(x,\nabla v(x))dx & \text{if } u\in W^{1,p}_*(\omega;\mathbb{R}^3)\\ +\infty & \text{otherwise.}\end{cases}$$

From [8] we have a relaxation result in the case $p = 1$.

Theorem. Assume that f is continuous and that there exist $C > 0$, $m \in]0,1[$, $L > 0$ such that

$$|f^\infty(x,\xi) - \frac{f(x,t\xi)}{t}| \le \frac{C}{t^m}$$

for all $(x,\xi) \in \overline{\omega}\times\mathbb{M}^{3\times2}$ such that $||\xi|| = 1$, and for all $t > L$. Then the lower semicontinuous envelope $\mathrm{cl}_1(J)$ for the strong topology of $L^1(\omega;\mathbb{R}^3)$ is given by

$$\mathrm{cl}_1(J)(v) = \begin{cases} \displaystyle\int_\omega Qf(x,\nabla v(x))dx + \int_\omega (Qf)^\infty(x,D_s v) & \\ \qquad\displaystyle + \int_{\partial\omega} Qf^\infty(x,\gamma(v)(x)\otimes\nu(x))d\mathcal{H}_2(x) & \text{if } v\in BV(\omega;\mathbb{R}^3)\\ +\infty & \text{otherwise.}\end{cases}$$

Proof of Theorem 3.4. By Proposition 5.2 and Lemma 5.3 together with the previous relaxation theorems, the proof follows. □

6. Proof of Theorem 3.6

We recall, in our context, two convergence results in the theory of Young measures (see [16]). Let $\phi : \overline{\omega} \times \mathbb{M}^{3\times 2} \to [0,+\infty[$ be a Carathéodory function. Let $\{w_\varepsilon\}_\varepsilon$ a sequence of mesurable functions of $\overline{\omega}$ to $\mathbb{M}^{3\times 2}$ and $\{\nu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ their corresponding Young measures, i.e. $\delta_{\{w_\varepsilon(\cdot)\}} = \nu_\varepsilon$ for all $\varepsilon \in]0,1]$. Assume that $\nu_\varepsilon \overset{nar}{\rightharpoonup} \nu$. Then

(Lsc) $\displaystyle\liminf_{\varepsilon\to 0} \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu_\varepsilon(x,\xi) \geq \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu(x,\xi);$

(Con) if moreover $\{\phi(\cdot, w_\varepsilon(\cdot))\}_\varepsilon$ is uniformly integrable then

$$\lim_{\varepsilon\to 0} \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu_\varepsilon(x,\xi) = \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu(x,\xi).$$

For the proof of Theorem 3.6, we need of the following two lemma which are adapted from [15]. Before, let us introduce the setting. Assume that there exist $c, C > 0$ and $p \in]1;+\infty[$ such that

$$c|\xi|^p \leq \phi(x,\xi) \leq C(1+|\xi|^p) \tag{3}$$

for all $(x,\xi) \in \overline{\omega} \times \mathbb{M}^{3\times 2}$. Let $J : \mathcal{C}^\infty_*(\omega;\mathbb{R}^3) \to [0,+\infty[$ be defined by

$$J(v) = \int_\omega \phi(x,\nabla v(x))dx.$$

Lemma 6.1. *Let $\pi : \mathcal{C}^\infty_*(\omega;\mathbb{R}^3) \to \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ be defined by $\pi(v) = \delta_{\{\nabla v(\cdot)\}}$. The integral functional J $\Gamma(\pi)$-converges to S given by*

$$S(\nu) = \begin{cases} \displaystyle\int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu(x,\xi) & \textit{if } \nu \in \mathcal{W}^{1,p}_* \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. By Lemma 2.2 we are reduced to prove that the lower semicontinuous envelope for the narrow topology of $\mathcal{J} : \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2}) \to [0,+\infty]$ defined by

$$\mathcal{J}(\nu) = \inf\{I(v) : v \in \pi^{-1}(\nu)\}$$

is S. It is easy to see that π is injective. Therefore we have the representation

$$\mathcal{J}(\nu) = \begin{cases} \displaystyle\int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu(x,\xi) & \text{if } \nu \in \pi(\mathcal{C}^\infty_*(\omega;\mathbb{R}^3)) \\ +\infty & \text{otherwise.} \end{cases}$$

Let ν , $\{\nu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ such that $\nu_\varepsilon \overset{nar}{\rightharpoonup} \nu$. Without restriction of generality, we can assume

$$\liminf_{\varepsilon\to 0} \mathcal{J}(\nu_\varepsilon) < +\infty.$$

Then $\{\nu_\varepsilon\}_\varepsilon \subset \pi(\mathcal{C}^\infty_*(\omega;\mathbb{R}^3))$ and there exists $\{v_\varepsilon\}_\varepsilon \subset \mathcal{C}^\infty_*(\omega;\mathbb{R}^3)$ bounded in $W^{1,p}_*(\omega;\mathbb{R}^3)$ such that $\nu_\varepsilon = \delta_{\{\nabla v_\varepsilon(\cdot)\}} \overset{nar}{\rightharpoonup} \nu$. By (Lsc) it follows that

$$\liminf_{\varepsilon\to 0} \mathcal{J}(\nu_\varepsilon) = \liminf_{\varepsilon\to 0} \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu_\varepsilon(x,\xi) \geq \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi d\nu = S(\nu).$$

Now, let $\nu \in \mathcal{W}^{1,p}_*$. Then there exists a bounded sequence $\{v_\varepsilon\}_\varepsilon \subset W^{1,p}_*(\omega;\mathbb{R}^3)$ such that $\delta_{\{\nabla v_\varepsilon(\cdot)\}} \overset{nar}{\rightharpoonup} \nu$. By coercivity condition (3) on ϕ and a well known decomposition lemma, see for instance [15, 11], there exists a sequence $\{v_\varepsilon\}_\varepsilon \subset W^{1,\infty}(\omega;\mathbb{R}^3)$ bounded in $W^{1,p}(\omega;\mathbb{R}^3)$, such that $\{|\nabla v_\varepsilon|^p\}_\varepsilon$ is equi-integrable, $\nu_\varepsilon = \delta_{\{\nabla v_\varepsilon(\cdot)\}} \overset{nar}{\rightharpoonup} \nu$. Growth condition (3) on ϕ then gives that $\{x \mapsto \phi(x, \nabla v_\varepsilon(x))\}_\varepsilon$ is uniformly integrable. According to (Con) we have

$$\lim_{\varepsilon\to 0}\int_\omega \phi(x,\nabla v_\varepsilon(x))dx = \int_{\overline{\omega}\times\mathbb{M}^{3\times 2}} \phi(x,\xi)d\nu(x,\xi),$$

and the proof is finished. □

A subset $\mathcal{H} \subset \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ is tight if for every $\varepsilon > 0$ there exists a compact subset $K_\varepsilon \subset \mathbb{M}^{3\times 2}$ such that $\sup\{\nu(\overline{\omega}\times(\mathbb{M}^{3\times 2}\setminus K_\varepsilon)) : \nu \in \mathcal{H}\} < \varepsilon$. We recall Prokhorov's theorem (see for instance [16]): if $\mathcal{H} \subset \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ is tight and $\{\nu_\varepsilon\}_\varepsilon \subset \mathcal{H}$, then there exist a subsequence $\{\nu_{\varepsilon_k}\}_k$ and $\hat{\nu} \in \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ such that $\nu_{\varepsilon_k} \overset{nar}{\rightharpoonup} \hat{\nu}$.

Lemma 6.2. *Let $\{\nu_\varepsilon\}_\varepsilon \subset \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ such that $\sup\{\mathcal{J}(\nu_\varepsilon) : \varepsilon \in]0,1]\} < +\infty$. Then there exist $\nu \in \mathcal{W}^{1,p}_*$ and a subsequence $\{\nu_{\varepsilon_k}\}_k$ such that $\nu_{\varepsilon_k} \overset{nar}{\rightharpoonup} \nu$.*

Proof. We just give a sketch of the proof here. Since $\sup\{\mathcal{J}(\nu_\varepsilon) : \varepsilon \in]0,1]\} < +\infty$, coercivity condition (3) on ϕ gives

$$\sup_{\varepsilon\in]0,1]}\int_\omega |\nabla u_\varepsilon(x)|^p dx < +\infty,$$

where $\nu_\varepsilon = \delta_{\{\nabla u_\varepsilon(\cdot)\}}$ for all $\varepsilon \in]0,1]$. By Poincaré 's inequality we have that $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,p}_*(\omega;\mathbb{R}^3)$. Holdër's inequality gives

$$\sup_{\varepsilon\in]0,1]}\int_\omega |\nabla u_\varepsilon(x)| dx < +\infty,$$

and then the sequence $\{\nu_\varepsilon\}_\varepsilon$ is tight. Prokhorov's theorem gives the desired conclusion. □

Proof of Theorem 3.6. From Lemma 5.3, W_Λ satisfies growth and coercivity conditions. By Proposition 5.2, $\tilde{I} = I\lfloor_{\mathcal{C}^\infty_*(\omega;\mathbb{R}^3)}$ admit an integral representation. Then by Lemma 6.1 $\tilde{I}$ $\Gamma(\pi)$-converges to G. Let $\{u_\varepsilon\}_\varepsilon$ be an ε-minimizing sequence of $\{E_\varepsilon\}_\varepsilon$. It is easy to see that $\{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon$ is an ε-minimizing sequence of F. By Proposition 5.2 $\{\pi_\varepsilon(u_\varepsilon)\}_\varepsilon$ is a minimizing sequence of I and then of $\tilde{I}$. Lemma 6.2 implies that there exist a subsequence $\{\varepsilon_k\}_k \subset]0,1]$ going to zero, and $\hat{\nu} \in \mathcal{W}^{1,p}_*$ such that $\delta_{\{\nabla\pi_{\varepsilon_k}(u_{\varepsilon_k})(\cdot)\}} \overset{nar}{\rightharpoonup} \hat{\nu}$, and moreover by Proposition 2.4 we have

$$\inf_{v\in\mathcal{C}^\infty_*(\omega;\mathbb{R}^3)} \tilde{I}(v) = \inf_{\nu\in\mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})} G(\nu) = G(\hat{\nu}).$$

Since $\inf\{I(v) : v \in L^p(\omega;\mathbb{R}^3)\} = \inf\{\tilde{I}(v) : v \in \mathcal{C}^\infty_*(\omega;\mathbb{R}^3)\}$, we deduce

$$\lim_{\varepsilon\to 0}\inf_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) = \inf_{\nu\in\mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})} G(\nu) = G(\hat{\nu}),$$

which completes the proof. □

Now, it is interesting to compare Theorem 3.2 (b) and Theorem 3.6.

Corollary 6.3. *Let assumptions of Theorem* 3.6 *hold. Let* $\{u_\varepsilon\}_\varepsilon$ *is an* ε*-mini-mizing sequence of* $\{E_\varepsilon\}_\varepsilon$. *Then there exist a sequence* $\{\varepsilon_k\}_k \subset]0,1]$ *going to zero,* $\hat{\nu} \in \mathcal{W}^{1,p}_*$ *and* $\hat{v} \in W^{1,p}_*(\omega;\mathbb{R}^3)$, *such that*

i) $\delta_{\{\nabla \pi_{\varepsilon_k}(u_{\varepsilon_k})(\cdot)\}} \overset{nar}{\rightharpoonup} \hat{\nu}$ *and* $\pi_{\varepsilon_k}(u_{\varepsilon_k}) \rightharpoonup \hat{v}$ *in* $W^{1,p}(\omega;\mathbb{R}^3)$;

ii) $\lim\limits_{\varepsilon\to 0} \inf\limits_{u\in \mathrm{A}_\varepsilon} E_\varepsilon(u) = \begin{cases} \inf\limits_{\nu\in\mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})} G(\nu) = G(\hat{\nu}) \\ \inf\limits_{v\in L^p(\omega;\mathbb{R}^3)} F(v) = F(\hat{v}); \end{cases}$

iii) $\nabla \hat{v}(x) = \displaystyle\int_{\mathbb{M}^{3\times 2}} \xi d\hat{\nu}_x(\xi)$ *a.e..*

Where $\{\hat{\nu}_x\}_{x\in\overline{\omega}}$ *is a family of probabilities on* $\mathbb{M}^{2\times 3}$ *corresponding to the disintegration of* $\hat{\nu}$.

Proof. In our context, we recall a result on gradient Young measures, see for instance [15, 11]. Let $\{w_\varepsilon\}_\varepsilon$ be a bounded sequence in $W^{1,p}(\omega;\mathbb{R}^3)$ and $p > 1$. Then there exist a sequence $\{\varepsilon_k\}_k \subset]0,1]$ going to zero, $\nu \in \mathcal{Y}(\overline{\omega};\mathbb{M}^{3\times 2})$ and $w \in W^{1,p}(\omega;\mathbb{R}^3)$ such that

$$\delta_{\{\nabla w_{\varepsilon_k}(\cdot)\}} \overset{nar}{\rightharpoonup} \nu \text{ and } w_{\varepsilon_k} \rightharpoonup w \text{ in } W^{1,p}(\omega;\mathbb{R}^3)$$

and

$$\nabla w(x) = \int_{\mathbb{M}^{3\times 2}} \xi d\nu_x(\xi) \text{ a.e..}$$

On account of the previous result together with Theorem 3.2 (b) and Theorem 3.6, it is easy to deduce the desired conclusion. □

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