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The Bilateral Minimal Time Function

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In this paper, we study the minimal time function as a function of two variables (the initial and the terminal points). This function, called the "*bilateral minimal time function*", plays a central role in the study of the Hamilton-Jacobi equation of minimal control in a domain which contains the target set, as shown in [11]. We study the regularity of the function, and characterize it as the unique (viscosity) solution of partial Hamilton-Jacobi equations with certain boundary conditions.

Keywords: Minimal time function, Hamilton-Jacobi equations, viscosity solutions, regularity of value functions, nonsmooth analysis, proximal analysis

1. Introduction

Let F be a multifunction mapping points x in $\mathbb{R}^N$ to subsets $F(x)$ of $\mathbb{R}^N$. Associated with F is the differential inclusion

$$\dot{x}(t) \in F(x(t)) \;\; a.e.\; t \in [0,T], \;\; x(0) = x_0. \tag{1}$$

A solution to (1) is an absolutely continuous function $x(\cdot)$ defined on the interval $[0,T]$ with initial value $x(0) = x_0$, in which case we say that $x(\cdot)$ is a trajectory of F that originates from x_0. The notation $\dot{x}(t)$ refers to the derivative of $x(\cdot)$ at t and is the right derivative if $t = 0$.

The minimal time control problem, associated to a nonempty subset S of $\mathbb{R}^N$ (called the *target* set), is a problem in which the goal is to steer an initial point α to S along a trajectory of the system F in minimal time. The minimal time value is denoted by $T(\alpha, S)$, which could be $+\infty$ if no trajectory from α can reach S.

The minimal time control problem is one of the most classical problems in control theory. It appears already in Carathéodory's book [8] and *it has a large literature*. The function $T(\cdot, S)$ is well studied and the property of *small time controllability* plays an important role in this study. Indeed, this property is equivalent to the continuity of $T(\cdot, S)$. There is a considerable literature devoted to local controllability, see for example [2], [15], [24] and more recently [16]. The Lipschitz continuity of $T(\cdot, S)$ is first studied in [19] for $S = \{0\}$. In this paper Petrov defined the *Petrov condition* and showed the equivalence

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between this condition and the Lipschitz continuity of $T(\cdot, S)$. This result was extended to arbitrary closed subsets by Soravia in [22]. In [25], Veliov gives a general result for the Lipschitz continuity of $T(\cdot, S)$, allowing the multifunction F to be nonautonomous and depend measurably on t. Hölder continuity of $T(\cdot, S)$ is also studied in the literature, see for example [20], [21] and [26]. On the other hand, simple examples show that $T(\cdot, S)$ fails to be everywhere differentiable, in general. Differentiability results for $T(\cdot, S)$ have been proved for linear systems if ∂S is smooth, see [5] and [14]. In [6], Cannarsa and Sinestrari study the semiconcavity of this function in analogy with the distance function $d_S(\cdot)$. We also find in this paper a semiconvexity result for the case where S is convex and the control system is linear. For a thorough history of such results, we invite the reader to see [2, Chapter 4].

Another important result for the minimal time function is its characterization as a solution of a Hamilton-Jacobi equation. Solving the Hamilton-Jacobi equation in some nonclassical sense has developed into an active research area, see for example [2] and [12]. In this paper, we are interested in *proximal solutions*. This concept of solution appeared in Clarke and Ledyaev [9], where the various concepts were also unified. In the context of this article, the proximal and viscosity solution concepts coincide. We can find in the literature many results concerning the characterization of $T(\cdot, S)$ as a solution of a Hamilton-Jacobi equation. The first result in this direction was found by Bardi in [1] using viscosity methods. In [23], Soravia extended these results to allow for noncontrollability and more general boundary conditions. Other related results are proved in [26] where Wolenski and Zhuang show using an invariance-based approach and without controllability assumptions that $T(\cdot, S)$ is the unique proximal solution of the Hamilton-Jacobi equation that satisfies certain boundary conditions, see [26, Theorem 3.2]. For more information about the possibility of characterizing $T(\cdot, S)$ as a solution of a Hamilton-Jacobi equation, see [2], [3], [4], and [13].

In [11] (see also [18]), Clarke and Nour study the Hamilton-Jacobi equation of the minimal time function in a domain which contains the target set (in this case, the minimal time function is *never* a solution of this equation). For the construction and the study of the regularity of solutions, Clarke and Nour used the minimal time function as a function of two variables. This new function, called the *bilateral minimal time function* and denoted by $T(\cdot, \cdot)$, is defined as follows. For $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$, $T(\alpha, \beta)$ is the minimum time taken by a trajectory to go from α to β (when no such trajectory exists, $T(\alpha, \beta)$ is taken to be $+\infty$)[1]. This kind of functions is also quite useful in geometric optics and the study of the eikonal equation, and it was used (under state constraints) in [17] to characterize the solvability of Dirichlet boundary value problems for Hamilton-Jacobi equations.

The purpose of this paper is to study the properties of $T(\cdot, \cdot)$ and its relation with the unilateral minimal function. We study the continuity, the Hölder continuity and the Lipschitz continuity of this function. Moreover, we give a semiconvexity result in the linear case. We calculate the proximal subgradient of this function and then we characterize it as a proximal solution of partial Hamilton-Jacobi equations.

In the next section we give some definitions and establish notation. We present some known results for the minimal time function in Section 3. Section 4 is devoted to the bilateral minimal time function $T(\cdot, \cdot)$.

[1]We remark that the (unilateral) minimal time function associated to $S := \{\beta\}$ is the function $T(\cdot, \beta)$.

2. Definitions and notations

We denote by $\|\cdot\|$, $\langle,\rangle$, B and $\bar{B}$, the Euclidean norm, the usual inner product, the open unit ball and the closed unit ball, respectively. For $\rho > 0$ and $x \in \mathbb{R}^N$, we set $B(x;\rho) := x + \rho B$ and $\bar{B}(x;\rho) := x + \rho\bar{B}$. For a set $S \subset \mathbb{R}^N$, $\operatorname{int} S$, $\bar{S}$ and $\operatorname{comp} S$ are the interior, closure and complement of S, respectively.

Let F be a multifunction mapping $\mathbb{R}^N$ to the subsets of $\mathbb{R}^N$. In this paper, we assume that F satisfies the following hypotheses:

- For every $x \in \mathbb{R}^N$, $F(x)$ is a nonempty compact convex set.
- The linear growth condition: For some positive constants γ and c, and for all $x \in \mathbb{R}^N$,

$$v \in F(x) \Longrightarrow \|v\| \leq \gamma\|x\| + c.$$

- F is locally Lipschitz; that is, every $x \in \mathbb{R}^N$ admits a neighborhood $U = U(x)$ and a positive constant $K = K(x)$ such that

$$x_1, x_2 \in U \Longrightarrow F(x_2) \subseteq F(x_1) + K\|x_1 - x_2\|\bar{B}.$$

We associate with F the following function h (resp. H), the *lower Hamiltonian* (resp. *upper Hamiltonian*):

$$h(x,p) := \min_{v\in F(x)} \langle p, v\rangle \quad (\text{resp. } H(x,p) := \max_{v\in F(x)} \langle p, v\rangle).$$

Now let $\Omega \subset \mathbb{R}^N$ be an open and let $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function such that $\operatorname{dom}\varphi \cap \Omega \neq \emptyset$ [2]. We say that:

- the system (φ, F) is *strongly increasing* on Ω if for any trajectory x on an interval $[a,b]$ for which $x([a,b]) \subset \Omega$, we have

$$\varphi(x(s)) \leq \varphi(x(t)) \;\; \forall s,t \in [a,b], s \leq t.$$

- the system (φ, F) is *weakly decreasing* on Ω if for every $\alpha \in \Omega$ there is a trajectory x on a nontrivial interval $[a,b]$ such that $x([a,b]) \subset \Omega$ and satisfying

$$x(a) = \alpha,\;\; \varphi(x(t)) \leq \varphi(\alpha) \quad \forall t \in [a,b].$$

Before giving a characterization of the preceding properties by proximal Hamilton-Jacobi inequalities, let us recall some definitions from proximal analysis. Given a lower semicontinuous function $f : \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ and a point x in the effective domain of f, we say that a vector $\zeta \in \mathbb{R}^N$ is a *proximal subgradient* of f at x if there exists $\sigma \geq 0$ such that

$$f(y) - f(x) + \sigma\|y - x\|^2 \geq \langle \zeta, y - x\rangle,$$

for all y in a neighborhood of x. The set of such ζ, which could be empty, is denoted by $\partial_P f(x)$ and referred to as the *proximal subdifferential*.
The following proposition is proven in [10, Chapter 4, §6].

Proposition 2.1 (Monotonicity of trajectories). *We have:*

$$(\varphi, F) \textit{ is strongly increasing on } \Omega \Longleftrightarrow h(x,\xi) \geq 0 \quad \forall \xi \in \partial_P\varphi(x) \quad \forall x \in \Omega.$$
$$(\varphi, F) \textit{ is weakly decreasing on } \Omega \Longleftrightarrow h(x,\xi) \leq 0 \quad \forall \xi \in \partial_P\varphi(x) \quad \forall x \in \Omega.$$

[2] $\operatorname{dom}\varphi := \{x \in \mathbb{R}^N : \varphi(x) < +\infty\}$ is the effective domain of φ.

The bilateral minimal time function $T(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow [0,+\infty]$ is defined as follows:

$$T(\alpha,\beta) := \inf\{T \geq 0 : \text{ some trajectory } x(\cdot) \text{ of } F \text{ has } x(0)=\alpha \text{ and } x(T)=\beta\}.$$

If no trajectory between α and β exists, then $T(\alpha,\beta) = +\infty$. Clearly we have $T(\alpha,\alpha) = 0$ for all $\alpha \in \mathbb{R}^N$. We define

$$\mathcal{R}_+^\beta(t) := \{\alpha \in \mathbb{R}^N \ : \ T(\beta,\alpha) < t\}, \quad t > 0,$$

the set of points reachable from β in time less than t.
Similarly, we introduce

- $\mathcal{R}_+^\beta := \bigcup_{t>0} \mathcal{R}_+^\beta(t) = \{\alpha \in \mathbb{R}^N \ : \ T(\beta,\alpha) < +\infty\}$,
- $\mathcal{R}_-^\beta(t) := \{\alpha \in \mathbb{R}^N \ : \ T(\alpha,\beta) < t\}, \quad t > 0$,
- $\mathcal{R}_-^\beta := \bigcup_{t>0} \mathcal{R}_-^\beta(t) = \{\alpha \in \mathbb{R}^N \ : \ T(\alpha,\beta) < +\infty\}$,
- $\mathcal{R}(t) := \{(\alpha,\beta) \in \mathbb{R}^N \times \mathbb{R}^N \ : \ T(\alpha,\beta) < t\}, \quad t > 0$,
- $\mathcal{R} := \bigcup_{t>0} \mathcal{R}(t) = \{(\alpha,\beta) \in \mathbb{R}^N \times \mathbb{R}^N \ : \ T(\alpha,\beta) < +\infty\}$.

We also define, for $x_0 \in \mathbb{R}^N$ and $T \geq 0$, the attainable set $A(x_0;T)$ by the set of all points of the form $x(T)$, where x is a trajectory for F on $[t_0,+\infty[$ satisfying $x(0) = x_0$.
It is easy to see that we have

- $T(\cdot,\cdot)$ and $T(\cdot,\beta)$ are lower semicontinuous.
- If $T(\alpha,\beta) < +\infty$, then the infimum defining $T(\alpha,\beta)$ is attained.
- For all (α,β,γ) we have the following triangle inequality:

$$T(\alpha,\beta) \leq T(\alpha,\gamma) + T(\gamma,\beta).$$

Now we give some definitions which play an important role in what follows. We characterize these properties in the next section.

Definition 2.2. Let $\beta \in \mathbb{R}^N$ and let $\lambda \in]0,1]$. We say that:

- F is β-LC (β*-locally controllable*), if $\beta \in \operatorname{int} \mathcal{R}_-^\beta$.
- F is β-STLC (β*-small-time locally controllable*), if $\beta \in \operatorname{int} \mathcal{R}_-^\beta(t)$ $\forall\ t > 0$; that is, $\forall$ $t > 0$ $\exists$ $\delta > 0$ such that $T(\cdot,\beta) < t$ on $B(\beta;\delta)$.
- F satisfies the "positive basis condition" at β, if $h(\beta,\gamma) < 0$ for any unit vector γ.
- F satisfies the hypothesis (H_λ) at β, if there exists $r > 0$ and $\delta > 0$ such that for any $\beta' \in B(\beta;r)$ and for any unit vector γ we have

$$h(\beta',\gamma) < \frac{-\delta\|\beta'-\beta\|^{1-\lambda}}{\lambda}.$$ [3]

Remark 2.3. We remark that if F satisfies the hypothesis (H_λ) at β then $-F$ also satisfies (H_λ) at β.

3. The (unilateral) minimal time function

In this section, we give some known results about the minimal time function $T(\cdot,\beta)$. Our principal reference is [2, Chapter 4], see also [25] and [26].[4]

[3]The hypothesis (H_λ) is known by the *Petrov λ-Hölder modulus condition*.

[4]The difference with these references is that our dynamic system is governed by a differential inclusion and our methods use the monotonicity of trajectories developed in [10, Chapter 4].

Proposition 3.1. *Let* $\beta \in \mathbb{R}^N$. *Then*

1. *F is β-LC iff $\mathcal{R}_-^\beta$ is open.*
2. *F is β-STLC $\Longrightarrow h(\beta,\gamma) \leq 0$ for any unit vector $\gamma \Longleftrightarrow 0 \in F(\beta)$.*
3. *The following statements are equivalent:*
   (*i*) *F satisfies the positive basis condition at β.*
   (*ii*) *F satisfies (H_1) at β.*
   (*iii*) *$0 \in int F(\beta)$.*

Proof. 1) Clearly we have: $\mathcal{R}_-^\beta$ is open $\Longrightarrow F$ is β-LC. For the converse, let $\alpha \in \mathcal{R}_-^\beta$. Then there exists a trajectory $\xi : [0,+\infty[\longrightarrow \mathbb{R}^N$ of F such that $\xi(0) = \alpha$ and $\xi(T(\alpha,\beta)) = \beta$ (ξ is the minimal-time trajectory between α and β). We invoke continuous dependence on the initial condition (see [10, Theorem 4.3.11]) to deduce the existence of $\rho > 0$ such that

$$A(\alpha;T(\alpha,\beta)) \subseteq A(\alpha';T(\alpha,\beta)) + \rho\|\alpha-\alpha'\|\bar{B}, \tag{2}$$

for all $\alpha' \in B(\alpha;1)$. Since F is β-LC, there exists $\varepsilon > 0$ such that $B(\beta;\varepsilon) \subset \mathcal{R}_-^\beta$. Then for $0 < \delta < \min\{1, \frac{\varepsilon}{2\rho}\}$ and by (2) we get that for all $\alpha' \in B(\alpha;\delta)$ there exists a trajectory $y : [0,+\infty[\longrightarrow \mathbb{R}^N$ of F such that $y(0) = \alpha'$ and $y(T(\alpha,\beta)) \in B(\beta;\varepsilon)$. The result follows.

2) It easy to show that: $[h(\beta,\gamma) \leq 0$ for any unit vector $\gamma] \Longleftrightarrow 0 \in F(\beta)$. Now we show the first implication. We reason by the absurd. Let $\beta \in \mathbb{R}^N$ and assume that there exist a unit vector γ and $\varepsilon > 0$ such that $h(\beta,\gamma) > \varepsilon$. We consider the sequence $\alpha_n := \beta + \frac{\gamma}{n}$. There exist a sequence $x_n : [0,+\infty[$ of trajectories of F and $T_n \geq 0$ such that $x_n(0) = \alpha_n$, $x_n(T_n) = \beta$ and $T(\alpha_n,\beta) \leq T_n < T(\alpha_n,\beta) + \frac{1}{n}$. We have

$$\frac{\gamma}{n} = \alpha_n - \beta = -\int_0^{T_n} \dot{x}_n(t)\,dt. \tag{3}$$

On the other hand, by the continuous dependence on the initial condition and since F is β-STLC ($T(\alpha_n,\beta) \longrightarrow 0$) there exists a constant $M > 0$ such that

$$\|x_n(t) - \beta\| \leq \frac{1}{n} + MT_n, \tag{4}$$

for all $t \in [0,T_n]$. Moreover, since F is locally Lipschitz, there exists $K > 0$ such that

$$\dot{x}_n(t) \in F(\beta) + K\|x(t)-\beta\|\bar{B}, \tag{5}$$

for all $t \in [0,T_n]$. Then by (3), (4), (5) and using the fact that $h(\beta,\gamma) > \varepsilon$ we get

$$\frac{1}{n} \leq T_n(K(\frac{1}{n} + MT_n) - \varepsilon),$$

but $T_n \longrightarrow 0$, hence the right-hand side can be made negative for n sufficiently large, which is a contradiction.

3) $(i) \Longleftrightarrow (ii)$: Follows by a simple continuity argument.

$(i) \Longrightarrow (iii)$: We reason by the absurd. Assume that $0 \notin \operatorname{int} F(\beta)$. Then there exists a sequence $v_n \in \mathbb{R}^N$ such that $v_n \longrightarrow 0$ and $v_n \notin F(\beta)$. Since $F(\beta)$ is a compact and convex set, there exists a sequence $p_n \in \mathbb{R}^N$ such that $\|p_n\| = 1$ and $h(\beta, p_n) > \langle p_n, v_n \rangle$, see [10, Exercise 4.1.15]. We can assume that $p_n \longrightarrow p$ such that $\|p\| = 1$. Hence $0 > h(\beta, p) \geq 0$, contradiction.

$(iii) \Longrightarrow (i)$: Let γ be a unit vector in $\mathbb{R}^N$. Since $0 \in \operatorname{int} F(\beta)$, there exists $r > 0$ such that $\bar{B}(0; r) \subset F(\beta)$ and then $-\gamma r \in F(\beta)$. Hence, $h(\beta, \gamma) \leq \langle -\gamma r, \gamma \rangle = -r < 0$. The result follows. □

The following proposition gives a standard necessary condition for the hypothesis (H_λ), which plays an important role in the study of the regularity of the minimal time function. See for example [26, Theorem 6.2] and [27, Theorem 2.1] for another and more general versions of this characterization[5]. We note that our proof of this proposition reposes on the monotonicity of trajectories.

Proposition 3.2. *Let $\beta \in \mathbb{R}^N$ and let $\lambda \in]0, 1]$. Assume that F satisfies (H_λ) at β. Then there exist $r > 0$ and $\delta > 0$ such that*

$$T(\alpha, \beta) \leq \frac{\|\alpha - \beta\|^\lambda}{\delta}, \quad \forall \alpha \in B(\beta; r).$$

Proof. Since F satisfies (H_λ) at β, there exist $r > 0$ and $\delta > 0$ such that for any $\beta' \in B(\beta; r)$ and for any unit vector γ we have $h(\beta', \gamma) < \frac{-\delta \|\beta' - \beta\|^{1-\lambda}}{\lambda}$. We set $V(\cdot) := \frac{\| \cdot - \beta\|^\lambda}{\delta}$. Then we have $h(\alpha, \partial_P V(\alpha)) < -1$ for all $\alpha \in B(\beta; r) \setminus \{\beta\}$. Hence the system $(t + V, \{1\} \times F)$ is weakly decreasing on $\mathbb{R} \times B(\beta; r) \setminus \{\beta\}$. Now let $\alpha \in B(\beta; r) \setminus \{\beta\}$, by Proposition 2.1 there exists a trajectory $x : [0, +\infty[\longrightarrow \mathbb{R}^N$ of F having the property that for any interval $[0, T]$ for which $x([0, T]) \subset B(\beta; r) \setminus \{\beta\}$, we have

$$V(\alpha) \geq t + V(x(t)) \ \forall t \in [0, T].$$

Let $\bar{T} := \inf\{t \geq 0 : x(t) \in \operatorname{comp}\{B(\beta; r) \setminus \{\beta\}\}\}$. We claim that $\bar{T} < +\infty$. Indeed, if not then for all $t \geq 0$ we have $x(t) \in B(\beta; r) \setminus \{\beta\}$. Hence

$$V(\alpha) \geq t + V(x(t))$$

for all $t \geq 0$, which gives a contradiction. Therefore $\bar{T} < +\infty$. Moreover we have:

- $\bar{T} \neq 0$.
- $x(\bar{T}) \in \operatorname{comp}\{B(\beta; r) \setminus \{\beta\}\}$.
- For all $T < \bar{T}$ we have $x(T) \in B(\beta; r) \setminus \{\beta\}$.

Hence

$$V(\alpha) \geq \bar{T} + V(x(\bar{T})). \tag{6}$$

We claim that $x(\bar{T}) = \beta$. Indeed, if not then $\|\beta - x(\bar{T})\| \geq r$. By (6) we have

$$\frac{r^\lambda}{\delta} > V(\alpha) \geq \bar{T} + V(x(\bar{T})) \geq \bar{T} + \frac{r^\lambda}{\delta}$$

[5]In [26, Theorem 6.2], the authors consider a C^2 modulus function $m(\cdot)$ for their characterization. In our case, $m(s) = \frac{s^\lambda}{\delta}$.

which gives a contradiction. Therefore $x(\bar{T}) = \beta$. Using again (6) we get that

$$V(\alpha) \geq \bar{T} + V(x(\bar{T})) = \bar{T} \geq T(\alpha, \beta)$$

which completes the proof. □

Remark 3.3. By Remark 2.3, we can deduce that in the preceding proposition we also get

$$T(\beta, \alpha) \leq \frac{\|\alpha - \beta\|^{\lambda}}{\delta}, \quad \forall \alpha \in B(\beta; r).$$

Proposition 3.4. *Let $\beta \in \mathbb{R}^{\mathrm{N}}$ and let $\lambda \in]0, 1]$. Then*

$$F \textit{ satisfies } (H_{\lambda}) \textit{ at } \beta \Longrightarrow F \textit{ is } \beta\textit{-STLC} \Longrightarrow F \textit{ is } \beta\textit{-LC}.$$

Proof. It is clear that: F is β-STLC $\Longrightarrow$ F is β-LC. The first implication follows from Proposition 3.2. □

The following proposition gives a necessary and sufficient condition for the continuity of $T(\cdot, \beta)$.

Proposition 3.5. *Let $\beta \in \mathbb{R}^{\mathrm{N}}$. Then the following statements are equivalent:*

(i) *F is β-STLC.*
(ii) *$T(\cdot, \beta)$ is continuous at β.*
(iii) *$\mathcal{R}_{-}^{\beta}$ is open, $T(\cdot, \beta)$ is continuous in $\mathcal{R}_{-}^{\beta}$ and for any $\alpha_0 \in \partial \mathcal{R}_{-}^{\beta}$ we have*

$$\lim_{\alpha \longrightarrow \alpha_0} T(\alpha, \beta) = +\infty.$$

Proof. Clearly we have $(i) \Longleftrightarrow (ii)$ and $(iii) \Longrightarrow (ii)$. Let us show that $(ii) \Longrightarrow (iii)$. We have that $\mathcal{R}_{-}^{\beta}$ is open since $(i) \Longleftrightarrow (ii)$. Let $\alpha \in \mathcal{R}_{-}^{\beta} \setminus \{\beta\}$. We will show that $T(\cdot, \beta)$ is continuous at α. By the continuous dependence of the initial condition there exists $\rho > 0$ such that for all α_1, $\alpha_2 \in B(\alpha; 1)$ we have

$$A(\alpha_1; T(\alpha, \beta)) \subseteq A(\alpha_2; T(\alpha, \beta)) + \rho \|\alpha_1 - \alpha_2\| \bar{B}. \tag{7}$$

Let α_n be a sequence such that $\alpha_n \longrightarrow \alpha$. By (7) ($c = T(\alpha, \beta)$) and for n sufficiently large, there exists a trajectory x_n of F on $[0, +\infty[$ such that $x_n(0) = \alpha_n$ and

$$\|\beta - x_n(T(\alpha, \beta))\| \leq \rho \|\alpha_n - \alpha\|.$$

We set $\beta_n = x_n(T(\alpha, \beta))$ then for n sufficiently large, we have

$$T(\alpha_n, \beta_n) \leq T(\alpha, \beta). \tag{8}$$

By the triangle inequality and using (8) we have

$$T(\alpha_n, \beta) \leq T(\alpha_n, \beta_n) + T(\beta_n, \beta) \leq T(\alpha, \beta) + T(\beta_n, \beta). \tag{9}$$

For $\varepsilon > 0$ and since $T(\cdot, \beta)$ is lower semicontinuous in $\mathbb{R}^{\mathrm{N}}$ and continuous at β and using (9) we get that

$$-\varepsilon + T(\alpha, \beta) \leq T(\alpha_n, \beta) \leq T(\alpha, \beta) + \varepsilon.$$

The continuity follows.
Now let $\alpha_0 \in \partial\mathcal{R}_-^\beta$ and suppose that there exist a constant M and a sequence $\alpha_n \in \mathcal{R}_-^\beta$ such that $\|\alpha_n - \alpha_0\| \leq \dfrac{1}{n}$ and $T(\alpha_n, \beta) \leq M$. We consider the minimal trajectory x_n between α_n and β. We have $x_n(0) = \alpha_n$ and $x_n(T(\alpha_n, \beta)) = \beta$. By the compactness of trajectories, there exist a trajectory $\bar{x}$ of F and a subsequence (we do not relabel) of x_n having the property that x_n converges uniformly to $\bar{x}$ on any interval $[0, b]$. Since $0 \leq T(\alpha_n, \beta) \leq M$ we can assume that $T(\alpha_n, \beta) \longrightarrow T \in [0, M]$. Then we have $\bar{x}(0) = \alpha_0$ and $\bar{x}(T) = \beta$ and hence $\alpha_0 \in \mathcal{R}_-^\beta$ which gives a contradiction since $\mathcal{R}_-^\beta$ is an open subset. $\square$

For the Hölder continuity we have the following.

Proposition 3.6. *Let $\beta \in \mathbb{R}^{\mathrm{N}}$ and let $\lambda \in]0, 1]$. Then if F satisfies (H_λ) at β then we have that $\mathcal{R}_-^\beta$ is open and $T(\cdot, \beta)$ is locally λ-Hölder continuous in $\mathcal{R}_-^\beta$.*

Proof. By Proposition 3.4 we have that $\mathcal{R}_-^\beta$ is open. By Proposition 3.2 there exist $r > 0$ and $\delta > 0$ such that $T(\beta', \beta) \leq \dfrac{\|\beta' - \beta\|^\lambda}{\delta}$ for all $\beta' \in B(\beta; r)$. Now let $\alpha \in \mathcal{R}_-^\beta$. By the continuous dependence of the initial condition, there exists ρ such that for all $c \in]-1, T(\alpha, \beta) + 1]$ and for all α_1, $\alpha_2 \in B(\alpha; 1)$ we have

$$A(\alpha_1; c) \subseteq A(\alpha_2; c) + \rho\|\alpha_1 - \alpha_2\|\bar{B}. \tag{10}$$

Since $T(\cdot, \beta)$ is continuous at α, there exists $\mu > 0$ such that on $B(\alpha; \mu) \subset \mathcal{R}_-^\beta$ we have $T(\cdot, \beta) \leq T(\alpha, \beta) + 1$.
Let $\nu := \min\{\dfrac{r}{4\rho}, \mu, 1\}$, and let α_1, $\alpha_2 \in B(\alpha; \nu)$ then by (10) $(c = T(\alpha_1, \beta))$, there exists a trajectory x of F such that $x(0) = \alpha_2$ and

$$\|\beta - x(T(\alpha_1, \beta))\| \leq \rho\|\alpha_1 - \alpha_2\| \leq \frac{r}{2}. \tag{11}$$

Then we have

$$\begin{aligned} T(\alpha_2, \beta) &\leq T(\alpha_1, \beta) + T(x(T(\alpha_1, \beta)), \beta) \\ &\leq T(\alpha_1, \beta) + \frac{\|x(T(\alpha_1, \beta)) - \beta\|^\lambda}{\delta} \\ &\leq T(\alpha_1, \beta) + \frac{\rho^\lambda}{\delta}\|\alpha_1 - \alpha_2\|^\lambda. \end{aligned}$$

By interchanging the role of α_1 and α_2, we get that $T(\cdot, \beta)$ is λ-Hölder continuous on $B(\alpha; \nu)$. The result follows. $\square$

We proceed to give a necessary and sufficient condition for the Lipschitz continuity of $T(\cdot, \beta)$.

Proposition 3.7. *Let $\beta \in \mathbb{R}^{\mathrm{N}}$, then the following statements are equivalent:*

(i) $\mathcal{R}_-^\beta$ *is open and* $T(\cdot, \beta)$ *is locally Lipschitz in* $\mathcal{R}_-^\beta$.

(ii) $T(\cdot, \beta)$ *is Lipschitz near* β.

(*iii*) $0 \in int F(\beta)$.

Proof. Clearly we have $(i) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (iii)$: We proceed as in the proof of 2) of Proposition 3.1. But here the assumption of the absurdity ($0 \notin \text{int}\, F(\beta)$ which is equivalent to F does not satisfy the positive basis condition at β) gives that $h(\beta, \gamma) \geq 0$. Moreover, since $T(\cdot, \beta)$ is Lipschitz near β, there exists $C > 0$ such that $T_n \leq \frac{1}{n}(C+1)$. Then by (3), (4) and (5) we get that

$$\frac{1}{n} \leq \frac{1}{n^2}(C+1)K(1+M(C+1)),$$

and then

$$1 \leq \frac{1}{n}(C+1)K(1+M(C+1))$$

which gives the required contradiction.

$(iii) \Longrightarrow (i)$: Follows from Proposition 3.6. □

4. The bilateral minimal time function

In this section, we give some properties of the function $T(\cdot, \cdot)$. First we study the regularity of this function. We show that if $T(\cdot, \cdot)$ is continuous (resp. locally Lipschitz) at every point of the diagonal $\mathcal{D} := \{(\alpha, \alpha) : \alpha \in \mathbb{R}^N\}$, then it is continuous (resp. locally Lipschitz) everywhere in $\mathcal{R}$. We also study the Hölder continuity of this function and we give a semiconvexity result in the linear case. We calculate the proximal subgradient and then we characterize this function as a proximal solution of partial Hamilton-Jacobi equations at the end of this section.

4.1. Regularity

We begin by the following proposition which gives a necessary and sufficient condition for $\mathcal{R}$ to be an open set.

Proposition 4.1. *We have the following statements:*

1. *Assume that $(\alpha, \beta) \in \mathcal{R}$ and that one of the following conditions holds:*
   (*i*) *F and $-F$ are respectively α-LC and β-LC.*
   (*ii*) *F and $-F$ are α-LC.*
   (*iii*) *F and $-F$ are β-LC.*
   Then $(\alpha, \beta) \in int \mathcal{R}$.
2. *The following statements are equivalent:*
   (*i*) *$\mathcal{R}$ is open.*
   (*ii*) *$\mathcal{D} \subset int \mathcal{R}$.*
   (*iii*) *F and $-F$ are α-LC, for all $\alpha \in \mathbb{R}^N$.*

Proof. 1) (*i*) Let $(\alpha, \beta) \in \mathcal{R}$ and assume that F and $-F$ are α-LC and β-LC respectively. Then by Proposition 3.5 we have $\mathcal{R}_-^\alpha$ and $\mathcal{R}_+^\beta$ are open. Then using the fact that $(\alpha, \beta) \in \mathcal{R}$ we get that $(\alpha, \beta) \in \mathcal{R}_-^\alpha \times \mathcal{R}_+^\beta \subset \mathcal{R}$. The result follows.

(*ii*) Follows since in this case $\mathcal{R}_-^\alpha$ and $\mathcal{R}_+^\alpha$ are open and $(\alpha, \beta) \in \mathcal{R}_-^\alpha \times \mathcal{R}_+^\alpha \subset \mathcal{R}$.

(iii) We proceed as in 2) and we find the result.

2) Clearly we have *(i)* $\Longrightarrow$ *(ii)*.

(ii) $\Longrightarrow$ *(iii)*: Assume that $\mathcal{D} \subset \operatorname{int} \mathcal{R}$ and let $\alpha \in \mathbb{R}^N$. Then $(\alpha, \alpha) \in \operatorname{int} \mathcal{R}$ and this gives the existence of $r > 0$ such that $(\alpha, \alpha) \in B(\alpha; r) \times B(\alpha; r) \subset \mathcal{R}$. Hence $\alpha \in B(\alpha; r) \subset \mathcal{R}_+^\alpha$ and $\alpha \in B(\alpha; r) \subset \mathcal{R}_-^\alpha$. Therefore F and $-F$ are α-LC.

(iii) $\Longrightarrow$ *(i)*: Follows from 1). □

Proposition 4.2. *Let $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$. Then we have:*

1. *$T(\cdot, \cdot)$ is continuous at $(\alpha, \alpha) \Longleftrightarrow F$ and $-F$ are α-STLC.*
2. *Assume that $(\alpha, \beta) \in \mathcal{R}$ and that one of the following conditions holds:*
   (i) F is α-STLC and $-F$ is β-STLC.
   (ii) F and $-F$ are α-STLC.
   (iii) F and $-F$ are β-STLC.
   Then $T(\cdot, \cdot)$ is continuous at (α, β).

Proof. 1) Let $\alpha \in \mathbb{R}^N$, by Proposition 3.5 we have

$$T(\alpha, \cdot) \text{ and } T(\cdot, \alpha) \text{ are continuous in } \alpha \Longleftrightarrow F \text{ and } -F \text{ are } \alpha\text{-STLC.}$$

But using the triangle inequality we also have

$$T(\alpha, \cdot) \text{ and } T(\cdot, \alpha) \text{ are continuous at } \alpha \Longleftrightarrow T(\cdot, \cdot) \text{ is continuous at } (\alpha, \alpha).$$

The result follows.

2) *(i)* Let $(\alpha, \beta) \in \mathcal{R}$ and suppose that F and $-F$ are respectively α-STLC and β-STLC. By (i) of Proposition 4.1, we have that $(\alpha, \beta) \in \operatorname{int} \mathcal{R}$. Now let (α_n, β_n) be a sequence such that $(\alpha_n, \beta_n) \longrightarrow (\alpha, \beta)$. By the triangle inequality we have

$$T(\alpha_n, \beta_n) \leq T(\alpha_n, \alpha) + T(\alpha, \beta) + T(\beta, \beta_n). \tag{12}$$

Then by the continuity of $T(\cdot, \alpha)$ and $T(\beta, \cdot)$ we get that $T(\cdot, \cdot)$ is upper semicontinuous and hence continuous.

(ii) By Proposition 4.1, we have $(\alpha, \beta) \in \operatorname{int} \mathcal{R}$. Now let (α_n, β_n) be a sequence such that $(\alpha_n, \beta_n) \longrightarrow (\alpha, \beta)$. By the triangle inequality we have

$$T(\alpha_n, \beta_n) \leq T(\alpha_n, \alpha) + T(\alpha, \beta_n).$$

Since $T(\cdot, \alpha)$ and $T(\alpha, \cdot)$ are continuous in $\mathcal{R}_-^\alpha$ and $\mathcal{R}_+^\alpha$ respectively, the result follows as above.

(iii) We proceed as in *(ii)* and we find the result. □

Now we give a necessary and sufficient condition for $T(\cdot, \cdot)$ to be continuous in $\mathcal{R}$.

Proposition 4.3. *The following statements are equivalent:*

(i) *$\mathcal{R}$ is open, $T(\cdot, \cdot)$ is continuous in $\mathcal{R}$ and for any $(\alpha_0, \beta_0) \in \partial\mathcal{R}$ we have*

$$\lim_{(\alpha,\beta) \longrightarrow (\alpha_0,\beta_0)} T(\alpha, \beta) = +\infty.$$

(*ii*) *$T(\cdot,\cdot)$ is continuous at (α,α) for all $\alpha \in \mathbb{R}^N$.*
(*iii*) *F and $-F$ are β-STLC for all $\beta \in \mathbb{R}^N$.*

Proof. Clearly we have $(i) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (iii)$: Follows from Proposition 4.2.

$(iii) \Longrightarrow (i)$: The first part ($\mathcal{R}$ is open and $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$) follows from Proposition 4.2 and the second part follows using the same idea as in the proof of Proposition 3.5. □

The following proposition gives a sufficient condition for $T(\cdot,\cdot)$ to be Hölder continuous near a given point.

Proposition 4.4. *Let $(\alpha,\beta) \in \mathcal{R}$ and let $\lambda \in]0,1]$. Then we have:*

$$F \text{ satisfies } (H_\lambda) \text{ at } \alpha \text{ or at } \beta \Longrightarrow T(\cdot,\cdot) \text{ is } \lambda\text{-Hölder continuous near } (\alpha,\beta).$$

Proof. Let $(\alpha,\beta) \in \mathcal{R}$ and assume that F satisfies (H_λ) at β (the case F satisfies (H_λ) at α follows by a similar argument, we only need to interchange the roles of α, β). By a simple continuity argument, there exists $r > 0$ such that F satisfies (H_λ) at β' for all $\beta' \in B(\beta,r)$. Then by Proposition 3.2 and Remark 3.3 there exist $s,\delta > 0$ such that $\forall \beta' \in B(\beta;r)$ we have:

$$T(\beta',\cdot) \le \frac{\|\cdot - \beta'\|^\lambda}{\delta} \text{ and } T(\cdot,\beta') \le \frac{\|\cdot - \beta'\|^\lambda}{\delta}, \text{ on } B(\beta';s).$$

By Proposition 4.1 we have $(\alpha,\beta) \in \operatorname{int}\mathcal{R}$ which gives that $\alpha \in \mathcal{R}_-^\beta$. Then using Proposition 3.6 we can assume that $T(\cdot,\beta)$ is λ-Hölder continuous in $B(\alpha;r)$ (the constant is $\ell := \frac{\rho^\lambda}{\delta}$).

Now we assume that $s < \min\{r,1\}$. Let $M := s^\lambda(\ell + \frac{1}{\delta}) + T(\alpha,\beta)$, $k \ge 1$ be a common Lipschitz constant for all trajectories of F on $[0,M]$ with initial-values in $B(\alpha;1)$, K be a Lipschitz constant for F on an appropriately large ball and $s' := \frac{s}{4k}e^{-KM}$. We claim that $T(\cdot,\cdot)$ is λ-Hölder continuous on $B((\alpha,\beta);s')$. Indeed, let (α',β'), $(\alpha'',\beta'') \in B((\alpha,\beta);s')$ and let $x(\cdot)$ be a minimal-time trajectory between α' and β', that is, $x(\cdot)$ is a trajectory of F on $[0,+\infty[$ which satisfies

$$x(0) = \alpha' \text{ and } x(T(\alpha',\beta')) = \beta'.$$

By the continuous dependence on initial condition and since

$$\begin{aligned} T(\alpha',\beta') &\le T(\alpha',\beta) + T(\beta,\beta') \\ &\le \ell\|\alpha' - \alpha\|^\lambda + T(\alpha,\beta) + T(\beta,\beta') \\ &\le \ell s^\lambda + T(\alpha,\beta) + \frac{1}{\delta}\|\beta' - \beta\|^\lambda \\ &\le s^\lambda(\ell + \frac{1}{\delta}) + T(\alpha,\beta) \\ &= M, \end{aligned}$$

there exists a trajectory $y(\cdot)$ of F on $[0,+\infty[$ which satisfies $y(0)=\alpha''$ and

$$\|y(T(\alpha',\beta'))-\beta'\|\le 2ke^{KM}\|\alpha'-\alpha''\|,$$

but $\|\alpha'-\alpha''\|<2s'$ then

$$y(T(\alpha',\beta'))\in B(\beta';s). \tag{13}$$

Using (13) and the fact that $\|\beta'-\beta''\|<2s'<s$, we get that

$$T(y(T(\alpha',\beta')),\beta')\le\frac{1}{\delta}\|y(T(\alpha',\beta'))-\beta'\|^{\lambda}$$

and

$$T(\beta',\beta'')\le\frac{1}{\delta}\|\beta'-\beta''\|^{\lambda}.$$

Moreover $T(\alpha'',\beta'')\le T(y(T(\alpha',\beta')),\beta'')+T(\alpha',\beta')$ then

$$\begin{aligned}T(\alpha'',\beta'')-T(\alpha',\beta') &\le T(y(T(\alpha',\beta')),\beta'')\\ &\le T(y(T(\alpha',\beta')),\beta')+T(\beta',\beta'')\\ &\le \frac{\|y(T(\alpha',\beta'))-\beta'\|^{\lambda}}{\delta}+\frac{\|\beta'-\beta''\|^{\lambda}}{\delta}\\ &\le (\frac{2ke^{KM}}{\delta})^{\lambda}\|\alpha'-\alpha''\|^{\lambda}+\frac{1}{\delta}\|\beta'-\beta''\|^{\lambda}\\ &\le K(\alpha,\beta)\|(\alpha'-\alpha'',\beta'-\beta'')\|^{\lambda}.\end{aligned}$$

By interchanging the roles of (α',β'), (α'',β''), the proof is completed. □

We proceed to study the Lipschitz continuity of $T(\cdot,\cdot)$. We begin by the following (local) proposition.

Proposition 4.5. *Let $(\alpha,\beta)\in\mathbb{R}^N\times\mathbb{R}^N$. Then we have:*

1. *If $(\alpha,\beta)\in\mathcal{R}$ then*
   $[\,0\in int\,F(\beta)$ *or* $0\in int\,F(\alpha)\,]\Longrightarrow T(\cdot,\cdot)$ *is Lipschitz near* (α,β). [6]
2. $0\in int\,F(\alpha)\Longleftrightarrow T(\cdot,\cdot)$ *is Lipschitz near* (α,α),

Proof. 1) Follows from Propositions 3.1 and 4.4.

2) The necessary condition follows from 1) and the sufficient condition follows from the fact that if $T(\cdot,\cdot)$ is Lipschitz near (α,α) then $T(\cdot,\alpha)$ is Lipschitz near α and hence by Proposition 3.7 we get that $0\in\operatorname{int}F(\alpha)$. □

The following proposition gives a necessary and sufficient conditions for $T(\cdot,\cdot)$ to be locally Lipschitz in $\mathcal{R}$.

Proposition 4.6. *The following statements are equivalent:*

(i) $\mathcal{R}$ *is open and* $T(\cdot,\cdot)$ *is locally Lipschitz in* $\mathcal{R}$.
(ii) $T(\cdot,\cdot)$ *is Lipschitz near* (α,α) *for all* $\alpha\in\mathbb{R}^N$.

[6] This is a slightly strengthened version of [11, Proposition 4.3, (1)] in which we have assumed that $[0\in int\,F(\beta)$ and $0\in int\,F(\alpha)]$.

(iii) $0 \in int F(\beta)$ *for all* $\beta \in \mathbb{R}^N$.

Proof. Clearly we have $(i) \Longrightarrow (ii)$.

$(ii) \Longrightarrow (iii)$: Follows from Proposition 4.5.

$(iii) \Longrightarrow (i)$: By Proposition 4.3 we have $\mathcal{R}$ is open, and by Proposition 4.5, $T(\cdot,\cdot)$ is locally Lipschitz in $\mathcal{R}$. □

We recall that a function $f : U \longrightarrow \mathbb{R}$ is semiconvex in the open set U provided that for all $x_0 \in U$ there exist δ, $C > 0$ such that

$$x \mapsto f(x) + \frac{C}{2}\|x\|^2 \text{ is convex on } B(x_0;\delta).$$

For more informations about the semconvexity, see [7]. It is well-known that in the linear case and under some hypotheses, the (unilateral) minimal time function $T(\cdot,\beta)$ is semiconvex, see [6]. In the following theorem, we show that in the linear case the function $T(\cdot,\cdot)$ is semiconvex if and only if it is locally Lipschitz.

Theorem 4.7. *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set. Assume that $T(\cdot,\cdot)$ is locally Lipschitz in an open set $\Omega \subset \mathcal{R}$. Then $T(\cdot,\cdot)$ is semiconvex in Ω.

Proof. Let $\Omega \subset \mathcal{R}$ in which $T(\cdot,\cdot)$ is locally Lipschitz and let $(\alpha,\beta) \in \Omega$. Then there exists $r, k_1 > 0$ such that $T(\cdot,\cdot)$ is k_1-Lipschitz on $B((\alpha,\beta);r) \subset \Omega \subset \mathcal{R}$. Let $0 < r' < r$ and let $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in B((\alpha,\beta);r')$. We suppose for instance that $T(\alpha_1,\beta_1) \leq T(\alpha_2,\beta_2)$ and $T(\alpha_2,\beta_2) \neq 0$[7]. We consider $y(\cdot)$ a trajectory which realizes the minimum time between α_2 and β_2. We set $z(\cdot) = (y(\cdot),\beta_2)$ and $w(\cdot)$ the solution of the following differential equation:

$$\dot{w}(t) = (A,0)w(t) + (u(2t),0), \quad w(0) = (\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}),$$

where $u(\cdot)$ is the optimal control which realizes the minimum time between α_2 and β_2. We define

- $(\alpha_3,\beta_2) = z(T(\alpha_2,\beta_2) - T(\alpha_1,\beta_1))$,
- $(\alpha_4, \frac{\beta_1+\beta_2}{2}) = w(\frac{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)}{2})$.

Then by the principle of optimality we have that

$$T(\alpha_3,\beta_2) = T(\alpha_1,\beta_1), \tag{14}$$

and

$$T(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}) \leq T(\alpha_4, \frac{\beta_1+\beta_2}{2}) + \frac{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)}{2}. \tag{15}$$

[7]If $T(\alpha_2,\beta_2) = 0$ then $T(\alpha_1,\beta_1) = T(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}) = 0$, and the desired inequality follows immediately.

Moreover,

$$\begin{aligned}\|2(\alpha_4, \frac{\beta_1+\beta_2}{2}) &- (\alpha_1,\beta_1)-(\alpha_3,\beta_2)\| \\ &= \|2\int_0^{\frac{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)}{2}} \dot{w}(t)\,dt - \int_0^{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)} \dot{z}(t)\,dt\| \\ &= \|\int_0^{T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)} (A,0)(w(\frac{t}{2})-(y(t),\beta_2))\,dt\| \\ &\le (T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1))\|A\|M,\end{aligned}$$

where $M := \max_{t\in[0,T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)]} \|x(\frac{t}{2})-y(t)\|$ with $x(\cdot)$ the solution of the following differential equation:

$$\dot{x}(t) = Ax(t)+u(2t), \quad x(0) = \frac{\alpha_1+\alpha_2}{2}.$$

Since $T(\cdot,\cdot)$ is k_1-Lipschitz on $B((\alpha,\beta);r)$ and $\dot{x}(\cdot)$, $\dot{y}(\cdot)$ are bounded on $[0,T(\alpha_2,\beta_2)-T(\alpha_1,\beta_1)]$, there exists a $k_2>0$ (depends only by r, k_1 and (α,β)) such that

$$M \le k_2\|(\alpha_1-\alpha_2,\beta_1-\beta_2)\|.$$

Hence

$$\|2(\alpha_4,\frac{\beta_1+\beta_2}{2})-(\alpha_1,\beta_1)-(\alpha_3,\beta_2)\| \le k_1k_2\|A\|.\|(\alpha_1-\alpha_2,\beta_1-\beta_2)\|^2.$$

Choosing r' very small, we can assume that $(\alpha_4,\frac{\beta_1+\beta_2}{2})$, $(\frac{\alpha_1+\alpha_3}{2},\frac{\beta_1+\beta_2}{2})$ are in the ball $B((\alpha,\beta);r)$ and then since $T(\cdot,\cdot)$ is k_1-Lipschitz on $B((\alpha,\beta);r)$ we get that:

$$T(\alpha_4,\frac{\beta_1+\beta_2}{2}) \le T(\frac{\alpha_1+\alpha_3}{2},\frac{\beta_1+\beta_2}{2}) + K\|(\alpha_1-\alpha_2,\beta_1-\beta_2)\|^2, \tag{16}$$

where $K := \frac{k_1^2k_2\|A\|}{2}$. By (14) and using the convexity of U we can easily show that

$$T(\frac{\alpha_3+\alpha_1}{2},\frac{\beta_1+\beta_2}{2}) \le T(\alpha_1,\beta_1).^8 \tag{17}$$

Then by (15), (16) and (17) we find that

$$T(\frac{\alpha_1+\alpha_2}{2},\frac{\beta_1+\beta_2}{2}) \le \frac{T(\alpha_1,\beta_1)+T(\alpha_2,\beta_2)}{2} + K\|(\alpha_1-\alpha_2,\beta_1-\beta_2)\|^2.$$

Then $T(\cdot,\cdot)$ is semiconvex near (α,β) which completes the proof. □

[8]We take $x_1(\cdot)$ (resp. $x_2(\cdot)$) a minimal trajectory between α_1 and β_1 (resp. α_3 and β_2). We define $x_3(\cdot) = \frac{x_1(\cdot)+x_2(\cdot)}{2}$. By the convexity of U, $x_3(\cdot)$ is a trajectory of F. Moreover, $x_3(0) = \frac{\alpha_1+\alpha_3}{2}$ and $x_3(T(\alpha_1,\beta_1)) = \frac{\beta_1+\beta_2}{2}$. The result follows.

Corollary 4.8. *Let F admit a representation of the form*

$$F(x) = \{Ax + u \,:\, u \in U\},$$

where A is an $n \times n$ matrix and U is a convex and compact set. Let $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$. Then we have the following statements:

1. *If $(\alpha, \beta) \in \mathcal{R}$ then*
   $[\, -A\alpha \in int U$ *or* $-A\beta \in int U\,] \Longrightarrow T(\cdot,\cdot)$ *is semiconvex near* (α, β),
2. $-A\alpha \in int U \Longleftrightarrow T(\cdot,\cdot)$ *is semiconvex near* (α, α),

Proof. Follows from Proposition 4.5 and Theorem 4.7. □

Example 4.9. For N $= 1$, let $F(x) = -x + [-1, 1]$. It is easy to prove that $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where

- $\mathcal{R}_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} :\, -1 < y \leq x\}$,
- $\mathcal{R}_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} :\, x \leq y < 1\}$.

We calculate $T(\cdot,\cdot)$ in $]-1,1[\times]-1,1[$ and we find that:

$$T(x,y) = \begin{cases} \ln(\dfrac{1+x}{1+y}) & \text{if } (x,y) \in \mathcal{R}_1, \\ \ln(\dfrac{1-x}{1-y}) & \text{if } (x,y) \in \mathcal{R}_2. \end{cases}$$

We remark that

$$T(x,y) = \max\{\ln(\frac{1+x}{1+y}), \ln(\frac{1-x}{1-y})\},$$

for all $(x, y) \in]-1,1[\times]-1,1[$. Hence $T(\cdot,\cdot)$ is the maximum of two C^2 functions and then it is semiconvex on $]-1,1[\times]-1,1[$, see [7]. We can easily deduce this from Corollary 4.8 since for all $x \in]-1,1[$ we have $x \in \text{int}\,([-1,1])$.

4.2. Proximal subgradients

In [26], Wolenski and Zhuang calculate the proximal subgradients of the (unilateral) minimal time function. In the following proposition we give an analogous result for the bilateral minimal function $T(\cdot,\cdot)$. This result will play an important role in the characterization of $T(\cdot,\cdot)$ as the solution of partial Hamilton-Jacobi equations.

Theorem 4.10. *We have:*

1. *For all $\alpha \in \mathbb{R}^N$, we have*

$$\partial_P T(\alpha,\alpha) = \{(\xi, -\xi) \in \mathbb{R}^N \times \mathbb{R}^N :\; h(\alpha,\xi) \geq -1\}.$$

2. *For all $(\alpha, \beta) \in \mathcal{R}$ with $\alpha \neq \beta$, we have*

$$\partial_P T(\alpha,\beta) = N^P_{\mathcal{A}(r)}(\alpha,\beta) \cap \{(\xi,\theta) \in \mathbb{R}^N \times \mathbb{R}^N :\; h(\alpha,\xi) = h(\beta,-\theta) = -1\},$$

where $r := T(\alpha,\beta)$ and $\mathcal{A}(r) := \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : T(x,y) \leq r\}$.

Proof. 1) Suppose $\alpha \in \mathbb{R}^{\mathrm{N}}$ and $(\xi, \theta) \in \partial_P T(\alpha, \alpha)$. Then there exist $\sigma > 0$ and $\nu > 0$ such that

$$T(\alpha', \beta') \geq -\sigma \|(\alpha' - \alpha, \beta' - \alpha)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \beta' - \alpha) \rangle,$$

for all $(\alpha', \beta') \in B((\alpha, \alpha); \nu)$. We take $\alpha' = \beta'$ and we get that

$$0 \geq -\sigma \|(\alpha' - \alpha, \alpha' - \alpha)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \alpha' - \alpha) \rangle,$$

for all $\alpha' \in B(\alpha; \nu)$. Let $v \in \mathbb{R}^{\mathrm{N}}$ and $\alpha_n = \alpha + \dfrac{v}{n}$ for all $n \in \mathbb{N}^*$. There exists n_0 such that for $n \geq n_0$ we have

$$0 \geq \frac{-\sigma}{n} \|(v, v)\|^2 + \langle (\xi, \theta), (v, v) \rangle,$$

hence

$$\langle (\xi, \theta), (v, v) \rangle \leq 0,$$

and this is true for all $v \in \mathbb{R}^{\mathrm{N}}$, then $\theta = -\xi$.
It is well-known that for $\zeta \in \partial_P T(\cdot, \alpha)(\alpha)$ we have $h(\alpha, \zeta) \geq -1$ ($(t + T(\cdot, \alpha), \{1\} \times F)$ is strongly increasing on $]0, +\infty[\times \mathbb{R}^{\mathrm{N}}$). But

$$\partial_P T(\alpha, \alpha) \subset \partial_P T(\cdot, \alpha)(\alpha) \times \partial_P T(\alpha, \cdot)(\alpha), \text{[9]}$$

hence $h(\alpha, \xi) \geq -1$. Therefore

$$\partial_P T(\alpha, \alpha) \subset \{(\xi, -\xi) \in \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}} : \ \ h(\alpha, \xi) \geq -1\}.$$

For the opposite inclusion, suppose now that $(\alpha, \xi) \in \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}}$ and $h(\alpha, \xi) \geq -1$. We will show that $(\xi, -\xi) \in \partial_P T(\alpha, \alpha)$. Suppose the contrary, then there exists a sequence $(\alpha_n, \beta_n) \in \mathbb{R}^{\mathrm{N}} \times \mathbb{R}^{\mathrm{N}}$ such that

$$(\alpha_n, \beta_n) \neq (\alpha, \alpha), \ \ (\alpha_n, \beta_n) \longrightarrow (\alpha, \alpha) \text{ and}$$

$$T_n = T(\alpha_n, \beta_n) < -n \|(\alpha_n - \alpha, \beta_n - \alpha)\|^2 + \langle (\xi, -\xi), (\alpha_n - \alpha, \beta_n - \alpha) \rangle \tag{18}$$

for all $n \in \mathbb{N}^*$.
Then we have

$$0 < T_n < 2\|\xi\|.\|(\alpha_n - \alpha, \beta_n - \alpha)\|. \tag{19}$$

Since $T_n < +\infty$ there exists a trajectory x_n of F on $[0, +\infty[$ such that $x_n(0) = \alpha_n$ and $x_n(T_n) = \beta_n$. Therefore

$$\beta_n - \alpha_n = \int_0^{T_n} \dot{x}_n(t)\, dt. \tag{20}$$

Let $p_n(t) := \mathrm{proj}_{F(\alpha)}(\dot{x}_n(t))$, then since $h(\alpha, \xi) \geq -1$ we have

$$\int_0^{T_n} \langle \xi, p_n(t) \rangle \, dt > -T_n. \tag{21}$$

By Gronwall's lemma (see [10, Proposition 4.1.4]) and since T_n is bounded there exists $M > 0$ such that $\forall n$, $\forall t \in [0, T_n]$ we have

$$\|x_n(t) - \alpha\| \leq \|x_n(t) - \alpha_n\| + \|\alpha_n - \alpha\| \leq M T_n + \|\alpha_n - \alpha\|,$$

[9] See [10, Exercise 1.2.9].

and then

$$\|x_n(t) - \alpha\| \leq MT_n + \|(\alpha_n - \alpha, \beta_n - \alpha)\|. \tag{22}$$

Moreover

$$\langle \xi, \int_0^{T_n} (\dot{x}_n(t) - p_n(t))\, dt \rangle \geq -K\|\xi\| \int_0^{T_n} \|x_n(t) - \alpha\|\, dt \tag{23}$$

where K is a Lipschitz constant for F on an appropriately large ball.
Using (19), (22) and (23) there exists $K' > 0$ such that

$$\langle \xi, \int_0^{T_n} (\dot{x}_n(t) - p_n(t))\, dt \rangle \geq -K'\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2. \tag{24}$$

By (20) and (21) we get that

$$T_n - \langle \xi, \alpha_n - \beta_n \rangle \geq -K'\|(\alpha_n - \alpha, \beta_n - \alpha)\|^2$$

and this contradicts (18) since $\langle (\xi, -\xi), (\alpha_n - \alpha, \beta_n - \alpha) \rangle = \langle \xi, \alpha_n - \beta_n \rangle$ and this finishes the proof.

2) Let $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\alpha \neq \beta$ and $r = T(\alpha, \beta)$. Let $(\xi, \theta) \in \partial_P T(\alpha, \beta)$. Then there exists $\sigma > 0$ and $\nu > 0$ such that

$$T(\alpha', \beta') \geq r - \sigma\|(\alpha' - \alpha, \beta' - \beta)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \rangle,$$

for all $(\alpha', \beta') \in B((\alpha, \beta); \nu)$. If we take $(\alpha', \beta') \in \mathcal{A}(r) \cup B((\alpha, \beta); \nu)$ we get

$$0 \geq -\sigma\|(\alpha' - \alpha, \beta' - \beta)\|^2 + \langle (\xi, \theta), (\alpha' - \alpha, \beta' - \beta) \rangle,$$

hence $(\xi, \theta) \in N^P_{\mathcal{A}(r)}(\alpha, \beta)$. It is well-known that for $\zeta \in \partial_P T(\cdot, \beta)(\alpha)$ with $\alpha \neq \beta$ we have $h(\alpha, \zeta) = -1$ ($(t + T(\cdot, \alpha), \{1\} \times F)$ is strongly increasing and weakly decreasing on $]0, +\infty[\times \mathbb{R}^N \setminus \{\alpha\}$). But

$$\partial_P T(\alpha, \beta) \in \partial_P T(\cdot, \beta)(\alpha) \times \partial_P T(\alpha, \cdot)(\beta),$$

then $h(\alpha, \xi) = h(\beta, -\theta) = -1$.
The proof of the opposite inclusion is similar to that of 1). □

Remark 4.11. We deduce from the preceding theorem that the function $T(\cdot, \cdot)$ is *not* differentiable at any point of $\mathcal{D}$.

4.3. The Hamilton-Jacobi equation

The following theorem gives a characterization of $T(\cdot, \cdot)$ as the solution of partial Hamilton-Jacobi equations.

Theorem 4.12. *$T(\cdot, \cdot)$ is the unique lower semicontinuous function bounded below on $\mathbb{R}^N \times \mathbb{R}^N$ and satisfying the following:*

1. $\forall \alpha \in \mathbb{R}^N$, $T(\alpha, \alpha) = 0$.
2. $\forall \alpha \neq \beta \in \mathbb{R}^N$, $\forall (\xi, \theta) \in \partial_P T(\alpha, \beta)$

$$h(\alpha, \xi) = -1.$$

3. $\forall \alpha \in \mathbb{R}^N$, $\forall (\xi, \theta) \in \partial_P T(\alpha, \alpha)$

$$h(\alpha, \xi) \geq -1.$$

Proof. For all $\alpha \in \mathbb{R}^N$, $T(\alpha, \alpha) = 0$ and by Proposition 4.10, $T(\cdot, \cdot)$ satisfies 2) and 3). To prove uniqueness, suppose $\psi : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, bounded below and satisfies 1), 2) and 3), and let $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$. Then for all $(\xi, \theta) \in \partial_P \psi(\alpha, \beta)$ we have

$$1 + h(\alpha, \xi) \geq 0.$$

This gives that the system $(t + \psi, \{1\} \times F \times \{0\})$ is strongly increasing on $\mathbb{R} \times \mathbb{R}^{2n}$. Hence for $(\alpha, \beta) \in \mathcal{R}$ and for $\bar{x}(\cdot)$ a trajectory which realizes the minimal time from α to β we have

$$0 + \psi(z(0)) \leq T(\alpha, \beta) + \psi(z((T(\alpha, \beta))), \tag{25}$$

where $z(\cdot)$ is the trajectory of $F \times \{0\}$ on $[0, T(\alpha, \beta)]$ defined by $z(t) = (\bar{x}(t), \beta)$. By (25) we get that $\psi(\alpha, \beta) \leq T(\alpha, \beta)$. Therefore $\psi(\cdot, \cdot) \leq T(\cdot, \cdot)$ on $\mathcal{R}$ and then on $\mathbb{R}^N \times \mathbb{R}^N$.
Now we show the reverse inequality. Let $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \mathcal{D}$. Then for all $(\xi, \theta) \in \partial_P \psi(\alpha, \beta)$ we have

$$1 + h(\alpha, \xi) \leq 0.$$

Hence the system $(t + \psi, \{1\} \times F \times \{0\})$ is weakly decreasing on $\mathbb{R} \times \mathbb{R}^{2n} \setminus \mathcal{D}$. Then for $(\alpha, \beta) \in \mathbb{R}^N \times \mathbb{R}^N$ with $\alpha \neq \beta$ there exists a trajectory $z(\cdot)$ of $F \times \{0\}$ on $[0, +\infty[$ such that $z(0) = (\alpha, \beta)$ and

$$t + \psi(z(t)) \leq 0 + \psi(\alpha, \beta) \quad \forall t \in [0, b],$$

where $[0, b]$ is any subinterval of $[0, +\infty[$ upon which $z(t) \notin \mathcal{D}, \forall t \in [0, b]$.
There are two cases to consider.

Case 1: $z(t) \notin \mathcal{D}, \forall t \in]0, +\infty[$.
Then $\psi(\alpha, \beta) \geq t + \psi(z(t)), \forall t \geq 0$. Since ψ is bounded below we get that $\psi(\alpha, \beta) = +\infty$ hence $\psi(\alpha, \beta) \geq T(\alpha, \beta)$.

Case 2: There exists $a \in]0, +\infty[$ such that $z(a) \in \mathcal{D}$.
Let $\bar{a} := \inf\{a \in]0, +\infty[: z(a) \in \mathcal{D}\}$. Since $z(0) = (\alpha, \beta) \notin \mathcal{D}$ we have $\bar{a} \neq 0$. Then $\forall t \in [0, \bar{a}[, \quad t + \psi(z(t)) \leq \psi(\alpha, \beta)$. Therefore

$$\psi(\alpha, \beta) \geq \liminf_{t \longrightarrow \bar{a}} t + \psi(z(t)) \geq \bar{a} \geq T(\alpha, \beta),$$

hence

$$T(\alpha, \beta) \leq \psi(\alpha, \beta)$$

which completes the proof since $\psi(\cdot, \cdot) = T(\cdot, \cdot) = 0$ on $\mathcal{D}$. □

Remark 4.13. Arguing as in the preceding proof, we can replace conditions 2) and 3) in Theorem 4.12 by

- $\forall \alpha \neq \beta \in \mathbb{R}^N$, $\forall (\xi, \theta) \in \partial_P T(\alpha, \beta)$

$$h(\beta, -\theta) = -1.$$

- $\forall \alpha \in \mathbb{R}^N$, $\forall (\xi, \theta) \in \partial_P T(\alpha, \alpha)$ we have

$$h(\alpha, -\xi) \geq -1.$$

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