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From Linear to Convex Systems: Consistency, Farkas' Lemma and Applications*

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This paper analyzes inequality systems with an arbitrary number of proper lower semicontinuous convex constraint functions and a closed convex constraint subset of a locally convex topological vector space. More in detail, starting from well-known results on linear systems (with no constraint set), the paper reviews and completes previous works on the above class of convex systems, providing consistency theorems, two new versions of Farkas' lemma, and optimality conditions in convex optimization. A new closed cone constraint qualification is proposed. Suitable counterparts of these results for cone-convex systems are also given.

1. Introduction

This paper mainly deals with systems of the form

$$\sigma := \{f_t(x) \leq 0, t \in T;\ x \in C\},$$

where T is an arbitrary (possibly infinite) index set, $C \subset X$, X is a locally convex Hausdorff topological vector space, and $f_t : X \to \mathbb{R} \cup \{+\infty\}$ for all $t \in T$. We assume that σ satisfies the following mild condition:

(**A**) C is a nonempty closed convex subset of X and f_t is a proper lower semicontinuous (l.s.c., in brief) convex function, for all $t \in T$.

In many applications $C = X$, in which case we write $\sigma := \{f_t(x) \leq 0,\ t \in T\}$. The system σ is called semi-infinite if either the dimension of X or the number of constraints ($|T|$) is finite. If both cardinal numbers are finite, then σ is called ordinary or finite. Observe that when all the functions f_t, $t \in T$, are finite valued, σ can be reformulated as $\{g(x) \in -\mathbb{R}^T_+;\ x \in C\}$, where $\mathbb{R}^T_+$ is the positive cone in $\mathbb{R}^T$ and $g : X \to \mathbb{R}^T$ is defined as $(g\,(x))\,(t) := f_t(x)$ for $x \in X$ and $t \in T$. It can easily be observed that g satisfies

$$g(\alpha x_1 + (1-\alpha)x_2) - \alpha g(x_1) - (1-\alpha)g(x_2) \in -\mathbb{R}^T_+,$$

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for every $x_1, x_2 \in X$ and every $\alpha \in [0, 1]$. This will be an example of the class of systems that we introduce next.

Let X and Y be locally convex Hausdorff topological vector spaces, and let S be a convex cone in Y, not necessarily solid (i.e., with nonempty interior). The mapping $g : X \to Y$ is called S-convex if

$$g(\alpha x_1 + (1-\alpha)x_2) - \alpha g(x_1) - (1-\alpha)g(x_2) \in -S,$$

for every $x_1, x_2 \in X$ and every $\alpha \in [0, 1]$. We associate with S, g, and a constraint set $C \subset X$, the cone-convex system

$$\sigma^* := \{g(x) \in -S;\ x \in C\}.$$

The focus of the paper is on systems satisfying (**A**). Our approach is based on linearization; i.e., the original system is replaced by a linear equivalent one obtained via the Fenchel conjugates of all the involved functions. In this way it is possible to apply well-known consistency and Farkas-like theorems for linear systems. More in detail, Section 2 contains the necessary notations, recalls some basic results on convexity, and states the required results on linear systems. In Section 3 some consistency theorems are given, Section 4 provides a new nonasymptotic version of Farkas' lemma for σ under a new weak regularity condition, and Section 5 yields a Kuhn-Tucker optimality condition for convex programs in which only a finite number of constraints are present. The results in Sections 3, 4 and 5, which are valid under very general regularity conditions, are applied to an important class of cone-convex systems thanks to the fact that they can be reformulated as convex systems satisfying (**A**).

Most results in the paper involve either the topological closure of certain subsets of the topological dual of X, X^*, endowed with the weak*-topology, or closures of subsets of the product space $X^* \times \mathbb{R}$, or the cone $\mathbb{R}_+^{(T)}$ of the so-called generalized sequences $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}_+$, for each $t \in T$, and with only finitely many λ_t different from zero.

2. Preliminaries

For a set $D \subset X$, the closure of D will be denoted $\operatorname{cl} D$. The convex hull of D will be represented by $\operatorname{conv}D$, and the convex cone generated by $D \cup \{0\}$ by $\operatorname{cone}D$. In the sequel, and for the sake of convenience, the closure with respect to the weak*-topology of a subset A of the dual space X^* will be denoted by $\operatorname{cl} A$ as well (which specially makes sense when A is convex).

Let further I be an arbitrary index set, $\{X_i, i \in I\}$ be a family of subsets of X, and let $\Im$ be the collection of all the nonempty finite subsets of I. Then

$$\operatorname{cone}\left(\bigcup_{i\in I} X_i\right) = \bigcup_{J\in\Im} \operatorname{cone}\left(\bigcup_{j\in J} X_j\right) = \bigcup_{J\in\Im}\left(\sum_{j\in J} \operatorname{cone}X_j\right). \tag{1}$$

Lemma 2.1. *Let A be a nonempty subset of X and let B be a convex cone of X containing the vector zero. Then*

$$\operatorname{cone}(A+B) \subset \operatorname{cone}(A \cup B) \subset \operatorname{cl}\operatorname{cone}(A+B). \tag{2}$$

Proof. We have to prove only the second inclusion. Since $0 \in B$, $A \subset A+B$. It remains to be proved that $B \subset \operatorname{cl}\operatorname{cone}(A+B)$. Let $b \in B$. If we take an arbitrary $a \in A$, and from the assumption on B, $a+nb \in A+B$ for any $n \in \mathbb{N}$. It follows that $n^{-1}a+b \in \operatorname{cone}(A+B)$ for any $n \in \mathbb{N}$. Letting $n \to \infty$ we get $b \in \operatorname{cl}\operatorname{cone}(A+B)$. $\square$

It is worth noting, from (2), that if $\operatorname{cone}(A+B)$ is closed then $\operatorname{cone}(A\cup B) = \operatorname{cone}(A+B)$ is closed. The converse is not true as the following simple example shows.

Example 2.2. Let $X = \mathbb{R}^2$, $B = \{0\} \times \mathbb{R}_+$, and $A = \{1\} \times \mathbb{R}_+$. Then $\operatorname{cone}(A\cup B) = \mathbb{R}^2_+$ is closed whereas $\operatorname{cone}(A+B) = (\mathbb{R}_{++} \times \mathbb{R}_+) \cup \{(0,0)\}$, which is not a closed subset of $\mathbb{R}^2_+$.

For a nonempty closed convex set C in X, the recession cone of C, denoted by C^∞, is defined in [17] as

$$C^\infty := \bigcap_{\varepsilon>0} \left[\operatorname{cl} \bigcup_{0<\lambda<\varepsilon} \lambda C \right],$$

where $\lambda C := \{\lambda c \mid c \in C\}$. According to [17, Theorem 2A], C^∞ can be characterized algebraically as

$$\begin{aligned} C^\infty &= \{z \in X \mid C + z \subset C\} \\ &= \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists some } c \in C \text{ such that} \\ c + \lambda z \in C \text{ for every } \lambda \geq 0 \end{array} \right\} \\ &= \{z \in X \mid c + \lambda z \in C \text{ for all } c \in C \text{ and every } \lambda \geq 0\}. \end{aligned} \tag{3}$$

For a set $D \subset X$, the indicator function δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. If D is nonempty, closed and convex, then δ_D is a proper l.s.c. convex function. The normal cone of D at x is given by

$$N_D(x) = \{u \in X^* \mid u(y-x) \leq 0 \text{ for all } y \in D\},$$

when $x \in D$, and $N_D(x) = \varnothing$, otherwise.

Now let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. convex function. The effective domain, the graph, and the epigraph of f are

$$\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\},$$

$$\operatorname{gph} f = \{(x, f(x)) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f\},$$

and

$$\operatorname{epi} f = \{(x,\gamma) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f,\ f(x) \leq \gamma\},$$

respectively, whereas the conjugate function of f, $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) \mid x \in \operatorname{dom} f\}.$$

In particular, it is obvious that the support function of $D \subset X$ is the conjugate of the indicator function of D, and

$$\delta^*_D(u) = \delta^*_{\operatorname{cl}(\operatorname{conv} D)}(u) = \sup_{x\in D} u(x),\ u \in X^*.$$

It is well-known that f^* is also a proper l.s.c. convex function and its conjugate, denoted by f^{**}, coincides with f. We also define the subdifferential of f at $x \in \operatorname{dom} f$ as

$$\partial f(x) = \{u \in X^* \mid f(y) \geq f(x) + u(y-x) \ \ \forall y \in X\},$$

and the recession function of f, denoted by f^∞, as the proper l.s.c. sublinear function verifying

$$\operatorname{epi} f^\infty = (\operatorname{epi} f)^\infty.$$

For $z \in X$ and $\mu \in \mathbb{R}$, (4) gives rise to the following equivalence:

$$f^\infty(z) \leq \mu \Leftrightarrow f(x+\lambda z) \leq f(x) + \lambda\mu, \text{ for all } x \in X \text{ and all } \lambda \geq 0.$$

Thus, as a consequence of the so-called property of increasing slopes of the convex functions, we have

$$f^\infty(z) \leq \mu \Leftrightarrow \sup_{\lambda>0} \frac{f(x+\lambda z)-f(x)}{\lambda} = \lim_{\lambda\to\infty} \frac{f(x+\lambda z)-f(x)}{\lambda} \leq \mu, \text{ for all } x \in X,$$

so that

$$f^\infty(z) = \lim_{\lambda\to\infty} \frac{f(x+\lambda z)-f(x)}{\lambda}, \text{ for all } x \in X.$$

Hence

$$\{z \in X \mid f^\infty(z) \leq 0\} = \{x \in X \mid f(x) \leq \eta\}^\infty, \tag{4}$$

for every η such that the lower sublevel set $\{x \in X \mid f(x) \leq \eta\}$ is nonempty. Consequently, f has bounded lower sublevel sets when $\{z \in X \mid f^\infty(z) \leq 0\} = \{0\}$ and $\dim X < \infty$, but this statement is no longer true in the infinite-dimensional setting. Moreover, [17, Corollary 3D] establishes the following useful identity

$$f^\infty = \delta^*_{\operatorname{cl}(\operatorname{dom} f^*)}. \tag{5}$$

The following lemma was established in [3, Theorem 3.1] for proper l.s.c. convex functions defined on a Banach space. However, the result still holds for locally convex vector spaces without any change in the proof.

Lemma 2.3 (Convex subdifferential sum formulae). *Let $g, h : X \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c. convex functions. If* $\operatorname{epi} g^* + \operatorname{epi} h^*$ *is weak*-closed then for each $a \in \operatorname{dom} g \cap \operatorname{dom} h$,*

$$\partial(g+h)(a) = \partial g(a) + \partial h(a).$$

It is worth noting that the conclusion of Lemma 2.3 still holds if one of the functions g or h is continuous at one point in $\operatorname{dom} g \cap \operatorname{dom} h$. In fact, if, for instance, g is continuous at $c \in \operatorname{dom} g$, it is clear that $c \in \operatorname{int}(\operatorname{dom} g) \cap \operatorname{dom} h$, and this implies $0 \in \operatorname{core}(\operatorname{dom} g - \operatorname{dom} h)$, which, in turn, entails that $\operatorname{cone}(\operatorname{dom} g - \operatorname{dom} h)$ is a closed space. It then follows from [3, Proposition 3.1] that the set $\operatorname{epi} g^* + \operatorname{epi} h^*$ is weak*-closed.

Finally, in this introduction, we consider linear systems of the form

$$\sigma := \{a_t(x) \leq b_t, \ t \in T\},$$

where $a_t \in X^*$ and $b_t \in \mathbb{R}$, for all $t \in T$ (observe that σ satisfies condition (**A**), with $C = X$). We say that the system σ is consistent if there exists $z \in X$ satisfying all the inequalities in σ. If σ is consistent, an inequality $a(x) \leq b$, $a \in X^*$ and $b \in \mathbb{R}$, is a consequence of σ if $a(z) \leq b$ for all $z \in X$ solution of σ. Now we recall two well-known results characterizing the consistency and the consequent inequalities of σ in terms of $(a_t, b_t) \in X \times \mathbb{R}$, $t \in T$.

Lemma 2.4 (Consistency theorem). *The following statements are equivalent to each other:*

(*i*) $\sigma = \{a_t(x) \leq b_t,\ t \in T\}$ *is consistent;*
(*ii*) $(0, -1) \notin \operatorname{cl}\operatorname{cone}\{(a_t, b_t),\ t \in T\}$;
(*iii*) $\operatorname{cl}\operatorname{cone}\{(a_t, b_t),\ t \in T;\ (0, 1)\} \neq \operatorname{cl}\operatorname{cone}\{a_t,\ t \in T\} \times \mathbb{R}$.

Lemma 2.5 (Farkas' lemma). *If* $\sigma = \{a_t(x) \leq b_t,\ t \in T\}$ *is consistent,* $v \in X^*$ *and* $\alpha \in \mathbb{R}$, *then the following statements are equivalent:*

(*i*) $v(x) \leq \alpha$ *is a consequence of* σ*;*
(*ii*) $(v, \alpha) \in \operatorname{cl}\operatorname{cone}\{(a_t, b_t),\ t \in T;\ (0, 1)\}$.

[(*i*)⇔(*ii*)] and [(*i*)⇔(*iii*)] in Lemma 2.4 are equivalent to [5, Theorem 1] and [8, Theorem 4.2], respectively, whereas Lemma 2.5 is equivalent to [4, Theorem 2] (actually these papers consider systems of the form $\{x(a_t) \leq b_t,\ t \in T\}$, where $a_t \in X$ and the space of the unknown x is X^*). Since the consistency is preserved by the aggregation of a trivial inequality (with $a = 0$ and $b \geq 0$), it is obvious that the cone in Lemma 2.4(*ii*) can be replaced with the cone in Lemma 2.5(*ii*) (this is [4, Theorem 1]).

3. Consistency

Assume that $\sigma = \{f_t(x) \leq 0, t \in T;\ x \in C\}$ satisfies the condition (**A**). Since f_t is a proper l.s.c. convex function, we have $f_t^{**} = f_t$ for all $t \in T$. Therefore, for each $t \in T$, we have

$$\begin{aligned} f_t(x) \leq 0 \quad &\Longleftrightarrow\ f_t^{**}(x) \leq 0 \\ &\Longleftrightarrow\ u_t(x) - f_t^*(u_t) \leq 0,\ \forall u_t \in \operatorname{dom} f_t^* \\ &\Longleftrightarrow\ u_t(x) \leq f_t^*(u_t),\ \forall u_t \in \operatorname{dom} f_t^* \\ &\Longleftrightarrow\ u_t(x) \leq f_t^*(u_t) + \alpha,\ \forall u_t \in \operatorname{dom} f_t^* \text{ and } \forall \alpha \in \mathbb{R}_+. \end{aligned}$$

On the other hand $x \in C$ can be expressed as $\delta_C(x) \leq 0$, with δ_C proper, l.s.c. and convex, so that

$$\begin{aligned} \delta_C(x) \leq 0 \quad &\Longleftrightarrow\ u(x) \leq \delta_C^*(u),\ \forall u \in \operatorname{dom}\delta_C^* \\ &\Longleftrightarrow\ u(x) \leq \delta_C^*(u) + \beta,\ \forall u \in \operatorname{dom}\delta_C^* \text{ and } \forall \beta \in \mathbb{R}_+. \end{aligned}$$

Then the following linear systems have the same solutions in X as σ (so that they are called linearizations of σ):

$$\sigma_1 := \left\{ \begin{array}{l} u_t(x) \leq f_t^*(u_t),\ u_t \in \operatorname{dom} f_t^*,\ t \in T \\ u(x) \leq \delta_C^*(u),\ u \in \operatorname{dom}\delta_C^* \end{array} \right\},$$

and

$$\sigma_2 := \left\{ \begin{array}{l} u_t(x) \leq f_t^*(u_t) + \alpha,\ \ u_t \in \mathrm{dom} f_t^*,\ \ t \in T,\ \ \alpha \in \mathbb{R}_+ \\ u(x) \leq \delta_C^*(u) + \beta,\ \ u \in \mathrm{dom}\delta_C^*,\ \ \beta \in \mathbb{R}_+ \end{array} \right\}.$$

Theorem 3.1. *Let $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ be a convex system satisfying condition* (**A**). *Then the following statements are equivalent to each other:*

(i) σ *is consistent;*
(ii) $(0,-1) \notin \mathrm{cl\,cone}\left\{\bigcup_{t\in T} \mathrm{gph} f_t^* \cup \mathrm{gph}\delta_C^*\right\}$;
(iii) $(0,-1) \notin \mathrm{cl\ cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\}$;
(iv) $\mathrm{cl\,cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\} \neq \mathrm{cl\,cone}\left\{\bigcup_{t\in T} \mathrm{dom} f_t^* \cup \mathrm{dom}\delta_C^*\right\} \times \mathbb{R}$.

Proof. The equivalence between (i) and (ii) is straightforward consequence of $[(i)\Longleftrightarrow(ii)]$ in Lemma 2.4, taking into account that the set of coefficient vectors of σ_1 is

$$\{(u_t, f_t^*(u_t)), u_t \in \mathrm{dom} f_t^*,\ \ t \in T; (u, \delta_C^*(u)), u \in \mathrm{dom}\delta_C^*\} = \bigcup_{t\in T} \mathrm{gph} f_t^* \cup \mathrm{gph}\delta_C^*.$$

Now observe that the set of coefficient vectors of σ_2 is

$$\begin{aligned} &\{(u_t, f_t^*(u_t) + \alpha), u_t \in \mathrm{dom} f_t^*, t \in T, \alpha \geq 0; (u, \delta_C^*(u) + \beta), u \in \mathrm{dom}\delta_C^*, \beta \geq 0\} \\ = &\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*. \end{aligned}$$

Hence, by the same argument as before, σ is consistent if and only if

$$(0,-1) \notin \mathrm{cl\ cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\},$$

so that $[(i)\Longleftrightarrow(iii)]$ holds.

Finally, $[(i)\Longleftrightarrow(iv)]$ follows from $[(i)\Longleftrightarrow(iii)]$ in Lemma 2.4, applied to σ_1, taking into account the identity

$$\mathrm{cone}\left[\bigcup_{t\in T} \mathrm{gph} f_t^* \cup \mathrm{gph}\delta_C^* \cup \{(0,1)\}\right] = \mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\}. \tag{6}$$

□

Observe that, according to Lemma 2.1 (since $\mathrm{epi}\delta_C^*$ is a convex cone containing zero), we have

$$\mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* + \mathrm{epi}\delta_C^*\right\} \subset \mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\}, \tag{7}$$

and

$$\mathrm{cl\ cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\} = \mathrm{cl\ cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* + \mathrm{epi}\delta_C^*\right\}. \tag{8}$$

From (8), it is possible to replace, in statements (iii) and (iv) of Theorem 3.1, $\mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\}$ with $\mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* + \mathrm{epi}\delta_C^*\right\}$. In particular, if $C = X = \mathbb{R}^n$, then

$\text{epi}\delta_C^* = \{0\} \times \mathbb{R}_+$ and $[(i) \Longleftrightarrow (iii)]$ means that $\sigma = \{f_t(x) \le 0, t \in T\}$ is consistent if and only if

$$(0,-1) \notin \text{cl cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup (\{0\} \times \mathbb{R}_+)\right\} = \text{cl cone}\left(\bigcup_{t\in T} \text{epi} f_t^*\right)$$

(this is [6, Proposition 3.1]). Similarly, from $[(i) \Longleftrightarrow (ii)]$, it is easy to prove that, if $C = X$, then $\sigma = \{f_t(x) \le 0, t \in T\}$ is consistent if and only if $(0,-1) \notin \text{cl cone}\left(\bigcup_{t\in T} \text{gph} f_t^*\right)$ (this is [9, Theorem 3]).

We have observed that, if K is either

$$\text{cone}\left\{\bigcup_{t\in T} \text{gph} f_t^* \cup \text{gph}\delta_C^*\right\} \text{ or cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\},$$

then

$$\{v(x) \le \alpha, (v,\alpha) \in K\}$$

is a linearization of σ. The same is true for

$$\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* + \text{epi}\delta_C^*\right\},$$

by (7), (8), and Lemma 2.5. These assertions come from the fact that the aggregation of constraints which are consequent relations of a consistent system does not modify its solution set.

The following results involve two desirable properties of $\sigma = \{f_t(x) \le 0, t \in T; x \in C\}$ and a certain convex cone, $K \subset X^* \times \mathbb{R}$, such that $\{v(x) \le \alpha, (v,\alpha) \in K\}$ is a linearization of σ:

(**C**) K is weak*-closed;

(**D**) K is solid if X is infinite dimensional, and

$$C^\infty \cap \{x \in X \mid f_t^\infty(x) \le 0,\ t \in T\} = \{0\}. \tag{9}$$

Notice that, by (7) and (8), if (**C**) holds for $\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* + \text{epi}\delta_C^*\right\}$, then it also holds for $\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\}$, but the converse statement is not true; i.e., the closed cone constraint qualification (**C**) is strictly weaker for $\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\}$.

Example 3.2. Consider $C = X = \mathbb{R}$ and $\sigma = \{f(x) = x \le 0\}$. Then $f^* = \delta_{\{1\}}$ and $\delta_C^* = \delta_{\{0\}}$, so that (**C**) holds for $\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\}$ whereas it fails for $\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* + \text{epi}\delta_C^*\right\}$ (recall Example 2.2). On the other hand, since $C^\infty = \mathbb{R}$ and $f^\infty(x) = x$, we have $\{x \in C^\infty \mid f^\infty(x) \le 0\} =]-\infty, 0] \neq \{0\}$ so that (**D**) cannot hold independently of K.

Concerning the couple of cones formed by $\text{cone}\left\{\bigcup_{t\in T} \text{gph} f_t^* \cup \text{gph}\delta_C^*\right\}$ and $\text{cone}\left\{\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\}$, we can have that (**C**) holds for exactly one of them. The following example shows the nontrivial part of this statement.

Example 3.3. Consider $C = X = \mathbb{R}^2$ and the inconsistent system

$$\sigma = \left\{ f_t\left(x\right) = tx_1 + t^2 x_2 + 1 \leq 0, t \in [-1,1] \right\}.$$

Then

$$\operatorname{cone}\left\{ \bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^* \right\} = \operatorname{cone}\left\{ \left(t, t^2, -1\right), t \in [-1,1] \right\}$$

is closed whereas

$$\operatorname{cone}\left\{ \bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \delta_C^* \right\} = \operatorname{cone}\left[\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^* \cup \{(0,0,1)\} \right]$$

is not closed, so that (**C**) holds for $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^*\right\}$ but not for $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \delta_C^*\right\}$. Let us observe that (**D**) also fails since

$$\begin{aligned} \{x \in C^\infty \mid f_t^\infty(x) \leq 0, t \in [-1,1]\} &= \left\{x \in C^\infty \mid tx_1 + t^2 x_2 \leq 0, t \in [-1,1]\right\} \\ &= \{0\} \times \left]-\infty, 0\right]. \end{aligned}$$

It is worth noting that the system in Example 3.3 is inconsistent. The following proposition shows that if σ is consistent and (**C**) holds for $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^*\right\}$, then it also holds for $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \delta_C^*\right\}$.

Proposition 3.4. *If σ is consistent and* $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^*\right\}$ *is weak*-closed, then* $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \delta_C^*\right\}$ *is also weak*-closed.*

Proof. In fact, since $(0,-1) \notin \operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^*\right\}$, this cone being weak*-closed by hypothesis, and since $\operatorname{cone}\{(0,1)\}$ is weak*-closed and locally compact (because it is finite-dimensional) and (6) holds, we get the conclusion from the well-known Dieudonné theorem (see, for instance, [21, Theorem 1.1.8]). □

The regularity condition (**C**), with $K := \operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^* + \operatorname{epi} \delta_C^*\right\}$, was introduced in [13] for the case where X is a Banach space and all the functions involved are finite valued, and it is called the *closed cone constraint qualification.* It is worth emphasizing that this regularity condition is strictly weaker than several known interior type regularity conditions (for more details, see [13]). In the particular case that $X = \mathbb{R}^n$, condition (**C**) is called *Farkas-Minkowski constraint qualification* and plays a crucial role in convex semi-infinite optimization (see [7]). When σ is linear and $X = \mathbb{R}^n$, $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^*\right\}$ and $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \delta_C^*\right\}$ are called 2nd moment cone and characteristic cone of σ, respectively (see, e.g. [7]). The recession condition (9) appeared in [2], in relation with the so-called limiting Lagrangian. Another constraint qualification based on the use of recession directions was introduced in [14, Theorem 3.2].

Theorem 3.5 (Generalized Fan's theorem). *Suppose that $\sigma = \{f_t(x) \leq 0, t \in T;\ x \in C\}$ satisfies* (**A**) *and let K be either* $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{gph} f_t^* \cup \operatorname{gph} \delta_C^*\right\}$ *or* $\operatorname{cone}\left\{\bigcup_{t\in T} \operatorname{epi} f_t^* \cup \operatorname{epi} \delta_C^*\right\}$. *If either* (**C**) *or* (**D**) *holds for K, then the following statements are equivalent:*

(*i*) *σ is consistent;*

(*ii*) $(0,-1) \notin K$;

(iii) *For any* $\lambda \in \mathbb{R}_+^{(T)}$, *there exists* $x_\lambda \in C$ *such that*

$$\sum_{t \in T} \lambda_t f_t(x_\lambda) \leq 0.$$

Proof. $[(i) \Longrightarrow (iii)]$ This implication is obvious.

$[(iii) \Longrightarrow (ii)]$ We shall prove this implication without using any regularity condition. Suppose that (iii) holds and assume, on the contrary, that (ii) does not hold, i.e., $(0,-1) \in K$. Since

$$\text{cone}\left\{\bigcup_{t \in T} \text{gph} f_t^* \cup \text{gph}\delta_C^*\right\} \subset \text{cone}\left\{\bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\},$$

we can suppose that

$$(0,-1) \in \text{cone}\left\{\bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\} = \text{cone}\left\{\bigcup_{t \in T} \text{epi} f_t^*\right\} + \text{epi}\delta_C^*,$$

so that, by (1), there exist $\lambda \in \mathbb{R}_+^{(T)}$, $u_t \in \text{dom} f_t^*$ and $\alpha_t \geq 0$, for each $t \in T$, $v \in \text{dom}\delta_C^*$, and $\beta \geq 0$, such that only finitely many λ_t are positive, and the following equation holds

$$(0,-1) = \sum_{t \in T} \lambda_t (u_t, f_t^*(u_t) + \alpha_t) + (v, \delta_C^*(v) + \beta).$$

Hence, $-1 = \sum_{t \in T} \lambda_t (f_t^*(u_t) + \alpha_t) + \delta_C^*(v) + \beta$ and $0 = \sum_{t \in T} \lambda_t u_t(x) + v(x)$, for all $x \in X$, so that

$$\begin{aligned} 1 &= \sum_{t \in T} \lambda_t (u_t(x) - f_t^*(u_t) - \alpha_t) + v(x) - \delta_C^*(v) - \beta \\ &\leq \sum_{t \in T} \lambda_t f_t(x) + \delta_C(x) - \sum_{t \in T} \lambda_t \alpha_t - \beta. \end{aligned}$$

Thus,

$$1 \leq 1 + \sum_{t \in T} \lambda_t \alpha_t + \beta \leq \sum_{t \in T} \lambda_t f_t(x)$$

for all $x \in C$, which contradicts (iii).

$[(ii) \Longrightarrow (i)]$ Assume that (ii) holds. If (**C**) is satisfied (i.e., K is weak*-closed), (i) and (ii) are equivalent by Theorem 3.1.

Now assume that (**D**) holds. Consider, first, that

$$(0,-1) \notin K = \text{cone}\left\{\bigcup_{t \in T} \text{gph} f_t^* \cup \text{gph}\delta_C^*\right\}.$$

We can apply the weak separation theorem ([10, 11E] if X is infinite dimensional) to conclude the existence of $z \in X$ and $\alpha \in \mathbb{R}$, not both simultaneously equal to zero, such that

$$0(z) + (-1)\alpha = -\alpha \geq 0,$$

at the same time that

$$\begin{array}{ll} u_t(z) + f_t^*(u_t)\alpha \leq 0, & \forall u_t \in \mathrm{dom} f_t^*,\ \forall t \in T, \\ v(z) + \delta_C^*(v)\alpha \leq 0, & \forall v \in \mathrm{dom}\delta_C^*. \end{array}$$

If $\alpha = 0$, we get

$$\begin{array}{ll} u_t(z) \leq 0, & \forall u_t \in \mathrm{dom} f_t^*,\ \forall t \in T, \\ v(z) \leq 0, & \forall v \in \mathrm{dom}\delta_C^*. \end{array}$$

(5) yields

$$f_t^\infty(z) = \delta^*_{\mathrm{cl}(\mathrm{dom} f_t^*)}(z) = \delta^*_{\mathrm{dom} f_t^*}(z) \leq 0,\ \forall t \in T,$$

and

$$\delta_C^\infty(z) = \delta^*_{\mathrm{cl}(\mathrm{dom}\delta_C^*)}(z) = \delta^*_{\mathrm{dom}\delta_C^*}(z) \leq 0.$$

Due to (4) one has

$$\{z \in X \mid \delta_C^\infty(z) \leq 0\} = \{x \in X \mid \delta_C(x) \leq 0\}^\infty = C^\infty,$$

and we obtain a contradiction because $z \neq 0$ and

$$z \in C^\infty \cap \{u \in X \mid f_t^\infty(u) \leq 0,\ t \in T\}.$$

Thus, we have proved that $\alpha < 0$ and $\widehat{z} := \frac{z}{|\alpha|}$ satisfies

$$\begin{array}{ll} u_t(\widehat{z}) - f_t^*(u_t) \leq 0, & \forall u_t \in \mathrm{dom} f_t^*,\ \forall t \in T, \\ v(\widehat{z}) - \delta_C^*(v) \leq 0, & \forall v \in \mathrm{dom}\delta_C^*. \end{array}$$

Taking suprema in the left-hand sides we get $f_t(\widehat{z}) \leq 0$, for all $t \in T$, and $\delta_C(\widehat{z}) \leq 0$ (i.e., $\widehat{z} \in C$). Hence $\widehat{z}$ is a solution of σ and (i) holds.

The proof of this implication is the same, under (**D**), when

$$(0,-1) \notin K = \mathrm{cone}\left\{ \bigcup_{t \in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^* \right\}.$$

□

Obviously, if (**C**) holds for $K := \mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* + \mathrm{epi}\delta_C^*\right\}$, then (i), (ii) and (iii) are also equivalent by (7), (8) and the own Theorem 3.5.

The equivalence $[(i) \Leftrightarrow (iii)]$ was proved by the first time in [1] under the assumption that $X = \mathbb{R}^n$ and C is compact (so that (9) trivially holds). The compactness was replaced by the weaker recession condition (9), which is equivalent to (**D**) in this context (as far as $(0,-1)$ can be weakly separated from K even though K is nonsolid), in [18, Theorem 21.3]. The simpler proof of this extension in [15, Theorem 3.1] has been adapted to an arbitrary X in Theorem 3.5.

The first infinite dimensional version of $[(i) \Leftrightarrow (iii)]$ was proved in [5, Theorem 1], assuming that all the functions f_t, $t \in T$, are real-valued and C is compact. Since then, Fan's theorem has been extended to more general situations under different types of assumptions. For instance, the extension to functions $f_t : X \to \mathbb{R} \cup \{+\infty\}$, maintaining the compactness assumption, is [19, Theorem 2], where different applications can be found.

Now, assume that the cone-convex system $\sigma^* := \{g(x) \in -S;\ x \in C\}$ satisfies the following condition:

(**B**) C is a nonempty closed convex subset of X, the convex cone S is closed (not necessarily with nonempty interior), and the mapping g is continuous and S-convex.

Then, for each v belonging to the dual cone S^+, $v \circ g : X \to \mathbb{R}$ defined by $(v \circ g)(x) := v(g(x))$, is a continuous convex function. Moreover, it is clear that

$$g(x) \in -S \iff (v \circ g)(x) \leq 0, \text{ for all } v \in S^+.$$

Therefore the cone-convex system σ^* has the same solutions as the convex system

$$\sigma := \{(v \circ g)(x) \leq 0,\ v \in S^+;\ x \in C\},$$

with σ satisfying condition (**A**). Consider the constraint qualifications (**C**) and (**D**) as in Theorem 3.5, with (9) reformulated as

$$C^\infty \cap \left\{u \in X \mid (v \circ g)^\infty (u) \leq 0,\ v \in S^+\right\} = \{0\}. \tag{10}$$

Corollary 3.6. *Let* $\sigma^* := \{g(x) \in -S;\ x \in C\}$ *satisfying* (**B**) *and let* K *be either* $\operatorname{cone}\left\{\bigcup_{v\in S^+} \operatorname{gph}(v \circ g)^* \cup \operatorname{gph}\delta_C^*\right\}$ *or* $\operatorname{cone}\left\{\bigcup_{v\in S^+} \operatorname{epi}(v \circ g)^* \cup \operatorname{epi}\delta_C^*\right\}$. *If either* (**C**) *or* (**D**) *holds for* K*, then the following statements are equivalent:*

(*i*) σ^* *is consistent*;

(*ii*) $(0, -1) \notin K$;

(*iii*) *For any* $\lambda \in \mathbb{R}_+^{(S^+)}$, *there exists* $x_\lambda \in C$ *such that* $\sum_{v\in S^+} \lambda_v (v \circ g)(x_\lambda) \leq 0$;

(*iv*) *For any* $v \in S^+$, *there exists* $x_v \in C$ *such that* $(v \circ g)(x_v) \leq 0$.

Proof. The proof of the equivalence between (*i*), (*ii*) and (*iii*) is a straightforward consequence of Theorem 3.5. Since [(*iii*) $\Rightarrow$ (*iv*)] holds trivially, it will be enough to prove that [(*iv*) $\Rightarrow$ (*ii*)] is true.

Assume that (*iv*) holds but (*ii*) fails; more precisely, that

$$(0, -1) \in \operatorname{cone}\left\{\bigcup_{v\in S^+} \operatorname{epi}(v \circ g)^* \cup \operatorname{epi}\delta_C^*\right\}.$$

Applying the fact that $\bigcup_{v\in S^+} \operatorname{epi}(v \circ g)^*$ is a convex cone (see the proof of [13, Lemma 2.1]), and repeating the argument in the proof of the implication [(*iii*)$\Longrightarrow$(*ii*)] in Theorem 3.5, it is easy to prove the existence of $v \in S^+$ such that $1 \leq (v \circ g)(x)$ for all $x \in C$. Thus (*iv*) also fails.

Finally, it is obvious that if

$$(0, -1) \notin K = \operatorname{cone}\left\{\bigcup_{v\in S^+} \operatorname{epi}(v \circ g)^* \cup \operatorname{epi}\delta_C^*\right\},$$

one also has

$$(0, -1) \notin K = \operatorname{cone}\left\{\bigcup_{v\in S^+} \operatorname{gph}(v \circ g)^* \cup \operatorname{gph}\delta_C^*\right\}.$$

□

If the closed cone constraint qualification (**C**) holds for $K := \bigcup_{v \in S^+} \text{epi}(v \circ g)^* + \text{epi}\delta_C^*$ (which is a convex cone because it is the sum of two convex cones), then (i), (ii), (iii), and (iv) are equivalent. In this case, the equivalence between (i) and (ii) was established recently in [13, Lemma 2.1], by means of a direct proof using the separation theorem. In [13], it is shown that this closed cone constraint qualification is strictly weaker than other known interior-type regularity conditions; e.g., the generalized Slater condition, requiring the existence of $x_0 \in C$ such that $g(x_0) \in -\text{int}S$, or conditions of the form $0 \in \text{core}(g(C) + S)$ or $0 \in \text{sqri}(g(C) + S)$ (sqri$B$ stands for the strong quasi-relative interior of the set B).

Note also that without any regularity condition, (i) is equivalent to $(0, -1) \notin \text{cl}(K)$, where K is any of the three mentioned cones (by Theorem 3.1).

4. Generalized Farkas' Lemma

We are now in a position to establish some generalized Farkas' lemmas in both asymptotic and non-asymptotic forms.

Theorem 4.1 (Asymptotic Farkas' lemma). *Let $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ be a consistent convex system satisfying condition* (**A**)*, $v \in X^*$ and $\alpha \in \mathbb{R}$. Then, the following statements are equivalent:*

(i) $f_t(x) \leq 0$ *for all* $t \in T$ *and* $x \in C \implies v(x) \leq \alpha$;
(ii) $(v, \alpha) \in \text{cl cone}\left(\bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right)$.

Proof. Let

$$\sigma_2 := \left\{ \begin{array}{l} u_t(x) \leq f_t^*(u_t) + \alpha, \;\; u_t \in \text{dom} f_t^*, \;\; t \in T, \;\; \alpha \in \mathbb{R}_+ \\ u(x) \leq \delta_C^*(u) + \beta, \;\; u \in \text{dom}\delta_C^*, \;\; \beta \in \mathbb{R}_+ \end{array} \right\}.$$

Recalling that σ and σ_2 have the same solutions, it follows from Lemma 2.5 that (i) is equivalent to

$$(v, \alpha) \in \text{cl cone}\left\{B \cup (0,1)\right\},$$

where B denotes the set of coefficient vectors of σ_2, i.e., $B = \bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*$. Since $(0,1) \in \text{cl cone}B$, (i) is in fact equivalent to

$$(v, \alpha) \in \text{cl cone}\left(\bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right).$$

□

As a consequence of Theorem 4.1, $K = \text{cl cone}\left(\bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right)$ is the greatest convex cone K such that $\{v(x) \leq \alpha, (v, \alpha) \in K\}$ is a linearization of σ.

An asymptotic Farkas' lemma similar to Theorem 4.1 can be found in [11, Corollary 2.1], where the right-hand side of the inclusion in (ii) is expressed in terms of ϵ-subdifferentials of the functions f_t, for all $t \in T$. The next result was established in [11, Theorem 2.1].

Corollary 4.2. *Let $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ be a consistent convex system satisfying condition* (**A**) *and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper l.s.c. convex function. Then the following statements are equivalent:*

(i) $f_t(x) \leq 0$ *for all* $t \in T$ *and* $x \in C \implies f(x) \leq 0$;

(ii) $\mathrm{epi} f^* \subset \mathrm{cl\ cone}\left(\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right)$.

Proof. Since $\{v(x) \leq \alpha, (v,\alpha) \in \mathrm{epi} f^*\}$ is a linearization of $\{f(x) \leq 0\}$, (i) is equivalent to the fact that, for each $(v,\alpha) \in \mathrm{epi} f^*$,

$$f_t(x) \leq 0 \text{ for all } \ t \in T \text{ and } x \in C \implies v(x) \leq \alpha.$$

By Theorem 4.1 the last implication is equivalent to

$$(v,\alpha) \in \mathrm{cl\ cone}\left(\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right)$$

for each $(v,\alpha) \in \mathrm{epi} f_t^*$. This is (ii). □

The next straightforward consequence of Corollary 4.2 extends the dual characterization of set containments of convex sets in [12, Theorem 3.2].

Corollary 4.3. *Let* $\{f_t(x) \leq 0, t \in T; x \in C\}$ *and* $\{h_w(x) \leq 0, w \in W; x \in D\}$ *be consistent convex systems on* X *satisfying condition* $(\mathbf{A})$, *and let* A *and* B *be the respective solution sets. Then,* $A \subset B$ *if and only if*

$$\bigcup_{w\in W} \mathrm{epi} h_w^* \cup \mathrm{epi}\delta_D^* \subset \mathrm{cl\ cone}\left(\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right).$$

Consequently, $A = B$ *if and only if*

$$\mathrm{cl\ cone}\left(\bigcup_{w\in W} \mathrm{epi} h_w^* \cup \mathrm{epi}\delta_D^*\right) = \mathrm{cl\ cone}\left(\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right).$$

Now we prove that a nonasymptotic version of Farkas' lemma can be obtained under both regularity conditions.

Theorem 4.4 (Nonasymptotic Farkas' lemma). *Let* $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ *be a consistent convex system satisfying condition* $(\mathbf{A})$, $v \in X^* \setminus \{0\}$, *and* $\alpha \in \mathbb{R}$.

If $(\mathbf{D})$ *holds for* $K := \mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\}$, *then the following statements are equivalent to each other:*

(i) $f_t(x) \leq 0$, $\forall t \in T$, *and* $x \in C \Longrightarrow v(x) \geq \alpha$;

(ii) $-(v, \alpha - \rho) \in K$, $\forall \rho > 0$;

(iii)

$$\sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x\in C} \left\{ v(x) + \sum_{t\in T} \lambda_t f_t(x) \right\} \geq \alpha. \tag{11}$$

If $(\mathbf{C})$ *holds for* $K := \mathrm{cone}\left\{\bigcup_{t\in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_C^*\right\}$, *the following condition can be added to the list of equivalent statements:*

(iv) $-(v,\alpha) \in K$.

Moreover, the supremum in (11) *is attained, and* (iii) *can be replaced by*

(iii') *There exists* $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ *such that*

$$v(x) + \sum_{t \in T} \overline{\lambda}_t f_t(x) \geq \alpha, \ \forall x \in C. \tag{12}$$

Remark 4.5. Observe that $[(iv) \Longrightarrow (ii)]$. In fact, (iv) entails the existence of $\lambda \in \mathbb{R}_+^{(T)}$, $u_t \in \text{dom } f_t^*$, $\alpha_t \geq 0$, for each $t \in T$, $u \in \text{dom } \delta_C^*$, and $\beta \geq 0$, such that

$$-(v, \alpha) = \sum_{t \in T} \lambda_t (u_t, f_t^*(u_t) + \alpha_t) + (u, \delta_C^*(u) + \beta). \tag{13}$$

Consequently,

$$-(v, \alpha - \rho) = \sum_{t \in T} \lambda_t (u_t, f_t^*(u_t) + \alpha_t) + (u, \delta_C^*(u) + \beta + \rho) \in K.$$

Proof. Assume that (**D**) holds for $K := \text{cone}\left\{\bigcup_{t \in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right\}$.

$[(i) \Longrightarrow (ii)]$ The statement (i) is equivalent to the inconsistency of the system

$$\sigma(\rho) := \left\{ \begin{array}{l} f_t(x) \leq 0, \ t \in T, \\ v(x) - \alpha + \rho \leq 0, \\ x \in C \end{array} \right\},$$

whichever $\rho > 0$ we take. Observe that $\sigma(\rho)$ satisfies (**D**) for the cone

$$K + \text{cone}\, \text{epi}(v(.) - \alpha + \rho)^* = K + \text{cone}\left\{(v, \alpha - \rho + \mu) \mid \mu \geq 0\right\}.$$

The equivalence between (i) and (ii) in Theorem 3.5 entails that $\sigma(\rho)$ is inconsistent if and only if

$$(0, -1) \in K + \text{cone}\left\{(v, \alpha - \rho + \mu) \mid \mu \geq 0\right\}.$$

Since $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ is consistent, Theorem 3.5 precludes $(0, -1) \in K$, and there must exist $\lambda > 0$ and $\mu \geq 0$ such that

$$(0, -1) \in K + \lambda\left(v, \alpha - \rho + \mu\right),$$

and so,

$$-(v, \alpha - \rho) \in \frac{1}{\lambda}\left\{K + (0, 1 + \lambda\mu)\right\} \subset \frac{1}{\lambda} K = K.$$

$[(ii) \Longrightarrow (iii)]$ If we apply now the equivalence between (i) and (iii) in Theorem 3.5, $\sigma(\rho)$ will be inconsistent, for every $\rho > 0$, if and only if there exist $\lambda^\rho \in \mathbb{R}_+^{(T)}$ and $\mu^\rho \geq 0$ such that

$$\mu^\rho (v(x) - \alpha + \rho) + \sum_{t \in T} \lambda_t^\rho f_t(x) > 0, \ \text{for all } x \in C. \tag{14}$$

It must be $\mu^\rho > 0$ (otherwise, the system $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ will be inconsistent (once again by Theorem 3.5). Defining $\gamma^p := \lambda^\rho/\mu^\rho$, (14) yields

$$\inf_{x\in C}\left\{v(x) + \sum_{t\in T}\gamma_t^\rho f_t(x)\right\} \geq \alpha - \rho.$$

Now, letting $\rho \downarrow 0$ we obtain (11).

$[(iii) \implies (i)]$ Now we assume (11). Given $\rho > 0$, there exists $\lambda^\rho \in \mathbb{R}_+^{(T)}$ such that

$$\inf_{x\in C}\left\{v(x) + \sum_{t\in T}\lambda_t^\rho f_t(x)\right\} \geq \alpha - \frac{\rho}{2}.$$

Then

$$\inf_{x\in C}\left\{(v(x) - \alpha + \rho) + \sum_{t\in T}\lambda_t^\rho f_t(x)\right\} \geq \frac{\rho}{2} > 0,$$

so that $\sigma(\rho)$ is inconsistent (again by Theorem 3.5), i.e., (i) holds.

In the second part of the proof, we assume that (**C**) is satisfied for the cone $K = \operatorname{cone}\left\{\bigcup_{t\in T}\operatorname{epi} f_t^* \cup \operatorname{epi}\delta_C^*\right\}$. Under this assumption, (ii) implies (iv) as far as K is closed, and so, (ii) and (iv) are equivalent according to the remark previous to the proof.

Now the equivalence of (i) and (iv) follows immediately from Theorem 4.1 and the closed cone constraint qualification (**C**). It suffices to prove that (iv) implies (iii') since the implication $[(iii') \implies (iii)]$ is obvious.

$[(iv) \implies (iii')]$ We have already seen that (iv) entails the existence of $\lambda \in \mathbb{R}_+^{(T)}$, $u_t \in \operatorname{dom} f_t^*$, $\alpha_t \geq 0$, for each $t \in T$, $u \in \operatorname{dom} \delta_C^*$, and $\beta \geq 0$, such that (13) holds, which is equivalent to

$$-v = \sum_{t\in T}\lambda_t u_t + u, \tag{15}$$

and

$$\begin{aligned} -\alpha &= \sum_{t\in T}\lambda_t f_t^*(u_t) + \delta_C^*(u) + \sum_{t\in T}\lambda_t\alpha_t + \beta \\ &\geq \sum_{t\in T}\lambda_t f_t^*(u_t) + \delta_C^*(u). \end{aligned} \tag{16}$$

Note that for each $x \in X$, $\delta_C^*(u) \geq u(x) - \delta_C(x)$ and $f_t^*(u_t) \geq u_t(x) - f_t(x)$, for each $t \in T$. It follows from (16) that

$$-\alpha \geq \sum_{t\in T}\lambda_t(u_t(x) - f_t(x)) + u(x) - \delta_C(x).$$

Taking (15) into account, the last inequality implies

$$-\alpha \geq -v(x) - \sum_{t\in T}\lambda_t f_t(x) - \delta_C(x)$$

for all $x \in X$. Thus for all $x \in C$, we have

$$v(x) + \sum_{t \in T} \lambda_t f_t(x) \geq \alpha,$$

which proves (iii'). The proof is complete. $\square$

Observe that, if $\operatorname{cone}\left(\bigcup_{t\in T} \operatorname{epi} f_t^* + \operatorname{epi}\delta_C^*\right)$ is closed, then Lemma 2.1 allows us to replace "$\cup$" with "$+$" in Theorem 4.4.

The next corollary is a straightforward consequence of Theorem 4.4 (see also Corollary 3.6).

Corollary 4.6. *Let $\sigma^* := \{g(x) \in -S;\ x \in C\}$ satisfying* (**B**) *and let $u \in X^*$, $\alpha \in \mathbb{R}$. If $\operatorname{cone}(\bigcup_{v\in S^+} \operatorname{epi}(v \circ g)^* \cup \operatorname{epi}\delta_C^*)$ is weak*-closed, then the following statements are equivalent to each other:*

(i) *$g(x) \in -S$ and $x \in C \Longrightarrow\ u(x) \geq \alpha$;*
(ii) *$-(u, \alpha) \in \operatorname{cone}(\bigcup_{v\in S^+} \operatorname{epi}(v \circ g)^* \cup \operatorname{epi}\delta_C^*)$;*
(iii) *There exists $v \in S^+$ such that $u(x) + (v \circ g)(x) \geq \alpha$, for all $x \in C$.*

5. Optimality Conditions for Convex Programs

Let $\sigma = \{f_t(x) \leq 0, t \in T; x \in C\}$ be a consistent convex system satisfying condition (**A**). Consider the convex optimization problem

$$\text{(CP)} \qquad \begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & f_t(x) \leq 0,\ t \in T, \\ & x \in C, \end{array}$$

where f is a proper l.s.c. convex funtion.

Let $A := \{x \in C \mid\ f_t(x) \leq 0,\ t \in T\}$ be the feasible set of (CP), and assume that $A \cap \operatorname{dom} f \neq \emptyset$.

A first question to be addressed is the existence of points in A minimizing the value of the objective function. These points are called minimizers of the problem (CP).

Proposition 5.1. *If X is the Euclidean space and* (9) *holds, the set of minimizers of* (CP) *is non-empty.*

Proof. The function

$$h := f + \delta_A$$

is a proper l.s.c. convex function (as a consequence of the fact that $A \cap \operatorname{dom} f \neq \emptyset$) such that

$$\{z \in X \mid h^\infty(z) \leq 0\} = \{z \in X \mid f^\infty(z) \leq 0\} \cap A^\infty = \{0\},$$

since, according with [18, Theorem 9.4],

$$A^\infty = C^\infty \cap \{z \in X \mid f_t^\infty(z) \leq 0,\ t \in T\} = \{0\}.$$

By [18, Theorem 27.2] h attains its infimum at points which are, obviously, minimizers of (CP). $\square$

Remark 5.2. The statement in Proposition 5.1 can be false even in the case that X is a reflexive Banach space, as the following example shows:

Example 5.3. Let us consider the convex optimization problem in the Hilbert space ℓ^2

$$\text{(CP)} \qquad \text{Minimize } \{f(x) \mid x \in C\},$$

where $x = (\xi_n)_{n\geq 1} \in \ell^2$,

$$C := \left\{ x \in \ell^2 \,||\xi_n| \leq n, \ \forall n \in \mathbb{N} \right\},$$

and

$$f(x) := \sum_{n=1}^{\infty} \frac{\xi_n}{n}.$$

If $a := (\alpha_n)_{n\geq 1}$ with $\alpha_n = 1/n$, $n = 1, 2, ...,$ we have $a \in \ell^2$ and f is a continuous linear (and, so, convex) functional on ℓ^2.

In [21, Example 1.1.1] it is proved that C is a closed convex set which is not bounded (because $ne_n \in C$, for every $n \in \mathbb{N}$) and such that $C^\infty = \{0\}$. Therefore, the recession condition (9) trivially holds. Moreover if we define $c^k := (\gamma_n^k)_{n\geq 1}$, $k = 1, 2, ...,$

$$\gamma_n^k := \begin{cases} -n, & \text{if } n \leq k, \\ 0, & \text{if } n > k, \end{cases}$$

it is also evident that $c^k \in C$, for all $k \in \mathbb{N}$, and $f(c^k) = -k$. Thus, we conclude that f is not bounded from below on C and no minimizer exists.

[21, Exercise 2.41] provides different characterizations of the coerciveness of a proper l.s.c. convex function defined on a normed space, but none of them directly involves the notion of recession direction. Theorem 2.5.1 in [21] establishes that if X is a reflexive Banach space and the function $f + \delta_A$ is coercive, then the set of minimizers is non-empty.

Theorem 5.4. *Suppose that the set* $\text{epi} f^* + \text{epi}\delta_A^*$ *is weak*-closed. Then* $a \in A$ *is a minimizer of* (CP) *if and only if there exists* $v \in \partial f(a)$ *such that*

$$-(v, v(a)) \in \text{cl cone}\left(\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^* \right).$$

Proof. Since (CP) can be written as $\inf\{f(x) \mid x \in A\}$, we have that $a \in A$ is a minimizer if and only if

$$0 \in \partial(f + \delta_A)(a).$$

Since $\text{epi} f_t^* + \text{epi}\delta_A^*$ is weak*-closed, by Lemma 2.3 the last inclusion is equivalent to

$$0 \in \partial f(a) + N_A(a).$$

In fact this condition is also equivalent to the existence of $v \in \partial f(a)$ such that $v(x) \geq v(a)$ for each $x \in A$; i.e., there exists $v \in \partial f(a)$ such that

$$f_t(x) \leq 0, \ \forall t \in T, \text{ and } x \in C \implies v(x) \geq v(a).$$

By Theorem 4.1, the last implication is equivalent to

$$-(v, v(a)) \in \text{cl cone}\left(\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right).$$

The theorem is proved. $\square$

Theorem 5.5. *Suppose that* $\text{epi} f^* + \text{epi}\delta_A^*$ *and* $\text{cone}\left(\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right)$ *are weak***-closed sets. Then* $a \in A$ *is a minimizer of* (CP) *if and only if there exist* $v \in \partial f(a)$ *and* $\lambda \in \mathbb{R}_+^{(T)}$ *such that*

$$-v \in \partial(\sum_{t\in T} \lambda_t f_t + \delta_C)(a) \tag{17}$$

and

$$\lambda_t f_t(a) = 0, \ \forall t \in T. \tag{18}$$

Moreover, if the functions f_t*,* $t \in T$*, are continuous at* a *then* (17) *can be replaced by*

$$0 \in \partial f(a) + \sum_{t\in T} \lambda_t \partial f_t(a) + N_C(a).$$

Proof. It follows from Theorem 5.4 that $a \in A$ is a minimizer of (CP) if and only if there exist $v \in \partial f(a)$ such that

$$-(v, v(a)) \in \text{cone}\left(\bigcup_{t\in T} \text{epi} f_t^* \cup \text{epi}\delta_C^*\right). \tag{19}$$

(Note that the cone in the right-hand side of (19) is weak*-closed by the assumption.)

By Theorem 4.4, (19) is equivalent to the existence of $\lambda \in \mathbb{R}_+^{(T)}$ satisfying

$$v(x) + \sum_{t\in T} \lambda_t f_t(x) \geq v(a), \ \forall x \in C.$$

Taking $x = a$ in the last inequality, we get $\lambda_t f_t(a) = 0 \ \forall t \in T$. It is also clear that a is a minimizer of the problem

$$\text{(P1)} \qquad \inf_{x\in C} \left\{ v(x) + \sum_{t\in T} \lambda_t f_t(x) \right\},$$

which implies that

$$0 \in \partial(v + \sum_{t\in T} \lambda_t f_t + \delta_C)(a),$$

or equivalently,

$$-v \in \partial(\sum_{t\in T} \lambda_t f_t + \delta_C)(a).$$

The necessity is proved.

Conversely, if (17) is satisfied then a is a solution of Problem (P1) and hence,

$$v(x) + \sum_{t \in T} \lambda_t f_t(x) \geq v(a) + \sum_{t \in T} \lambda_t f_t(a), \ \forall x \in C.$$

Due to (18) one actually has

$$v(x) + \sum_{t \in T} \lambda_t f_t(x) \geq v(a), \ \forall x \in C.$$

Now if $x \in C$ and $f_t(x) \leq 0$, $\forall t \in T$, then $v(x) \geq v(a)$. This means that $v(x) \geq v(a)$ for all $x \in A$. This is, in turn, equivalent to $0 \in \partial f(a) + N_A(a)$, which implies that a is a minimizer of (CP).

Moreover, if all the functions f_t, $t \in T$, are continuous at a then

$$\partial(\sum_{t \in T} \lambda_t f_t + \delta_C)(a) = \sum_{t \in T} \lambda_t \partial f_t(a) + N_C(a).$$

(See, for instance, [16, Theorem 5.3.32]). In this way, the last assertion of the theorem follows. □

Remark 5.6. The set $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ is weak*-closed when f is linear. In fact, $\mathrm{epi} f^*$ is a vertical halfline (locally compact, as far it is finite-dimensional), δ_A^* is proper and so,

$$(-\mathrm{epi} f^*)^\infty \cap (\mathrm{epi}\delta_A^*)^\infty = \{0\}.$$

Then the Dieudonné theorem applies ([21, Theorem 1.1.8]).

Even in the simple linear case, the assumptions of Theorem 5.5 do not entail the existence of minimizers. This fact is illustrated in the following example.

Example 5.7. Consider the linear optimization problem, in $\mathbb{R}^2$,

$$\text{(CP)} \qquad \begin{aligned} &\text{Minimize} && x_1 \\ &\text{subject to} && -\frac{1}{t}x_1 + x_2 \leq \log(t) - 1, \ t \in]0,1], \\ & && x_1 \geq 0 \text{ and } x_2 \leq 0. \end{aligned}$$

Let $f(x) = x_1$, $f_t(x) = -\frac{1}{t}x_1 + x_2 - \log(t) + 1$, $t \in]0,1]$, $f_0(x) = -x_1$, and $f_2(x) = x_2$, with $x = (x_1, x_2) \in R^2$. Let also $T =]0,1] \cup \{0,2\}$. Note that in this case, $C = R^2$ and all the constraints are linear inequalities. In [7, Exercise 8.8] it is stated that $\mathrm{cone}\left(\bigcup_{t \in T} \mathrm{epi} f_t^* \cup \mathrm{epi}\delta_{\mathbb{R}^n}^*\right)$ is weak*-closed. It follows from Remark 5.2 that $\mathrm{epi} f^* + \mathrm{epi}\delta_A^*$ is also a weak*-closed set. However, the set of minimizers is empty.

We consider finally the cone-convex problem

$$\text{(CCP)} \qquad \begin{aligned} &\text{Minimize} && f(x) \\ &\text{subject to} && g(x) \in -S, \\ & && x \in C, \end{aligned}$$

where the constraint system σ^* satisfies condition (**B**). The following optimality condition for (CCP) was established in [3, Theorem 4.1] (see also [13]) for the case where X and Y are Banach spaces and under the conditions that the sets $\bigcup_{v\in S^+} \mathrm{epi}(v\circ g)^* + \mathrm{epi}\delta_C^*$ and $\mathrm{epi}f^* + \bigcup_{v\in S^+} \mathrm{epi}(v\circ g)^* + \mathrm{epi}\delta_C^*$ are weak*-closed. The next result relaxes these conditions.

Corollary 5.8. *Suppose that* $\mathrm{cone}\left(\bigcup_{v\in S^+} \mathrm{epi}(v\circ g)^* \cup \mathrm{epi}\delta_C^*\right)$ *and* $\mathrm{epi}f^* + \mathrm{cone}(\bigcup_{v\in S^+} \mathrm{epi}(v\circ g)^* \cup \mathrm{epi}\delta_C^*)$ *are weak*-closed. Then* $a\in A$ *is a minimizer of* (CCP) *if and only if there exist* $v\in\partial f(a)$ *and* $v\in S^+$ *such that*

$$0\in\partial f(a)+\partial(v\circ g)(a)+N_C(a) \text{ and } (v\circ g)(a)=0.$$

Proof. Note that the constraint $g(x)\in -S$ is equivalent to $g_v(x):=(v\circ g)(x)\le 0$ for all $v\in S^+$. Moreover, by the assumption on the map g, for each $v\in S^+$, g_v is continuous. On the other hand, by [13, Lemma 2.1], we have

$$\mathrm{epi}\delta_A^* = cl\left(\bigcup_{v\in S^+} \mathrm{epi}(v\circ g)^* + \mathrm{epi}\delta_C^*\right).$$

It follows from Lemma 2.1 and the assumptions of the corollary that

$$\mathrm{epi}\delta_A^* = \mathrm{cone}\left(\bigcup_{v\in S^+} \mathrm{epi}(v\circ g)^* \cup \mathrm{epi}\delta_C^*\right).$$

The conclusion follows by the same argument as in the proof of Theorem 5.5, using Corollary 4.6 instead of Theorem 4.4. □

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