Journal of Convex Analysis
Volume 13 (2006), No. 3+4, 623–632

There Are Many Totally Convex Functions

Dan Butnariu
Department of Mathematics,
University of Haifa, 31905 Haifa, Israel
dbutnaru@math.haifa.ac.il

Simeon Reich
Department of Mathematics,
Technion - Israel Institute of Technology, 32000 Haifa, Israel
sreich@tx.technion.ac.il

Alexander J. Zaslavski
Department of Mathematics,
Technion - Israel Institute of Technology, 32000 Haifa, Israel
ajzasl@tx.technion.ac.il

Dedicated to the memory of Simon Fitzpatrick.

Received: Februrary 4, 2005

Let K be a convex subset of a normed linear space and let R^1 denote the real line. We show that there are many (in the sense of Baire category) strictly convex and totally convex functions $f : K \to R^1$. It is known that the existence of such functions is crucial in numerous optimization algorithms.

Keywords: Complete metric space, essentially strictly convex function, generic property, strictly convex function, totally convex function

2000 Mathematics Subject Classification: 46N10, 52A41, 54E50, 54E52

Introduction

Convergence analysis of many iterative algorithms for convex optimization in Banach spaces shows that they produce bounded sequences of vectors the weak accumulation points of which are optimal solutions of the problems these algorithms are supposed to solve. Obviously, the identification of a convergent subsequence of a given sequence is difficult, if not even impossible. Thus, such algorithms can be used to compute approximate solutions of the given problem only to the extent to which either the objective function of the optimization problem is strictly convex (in which case the sequences those algorithms generate converge weakly to the necessarily unique optimal solution of the problem), or one can regularize the problem by replacing the objective function with a strictly convex approximation of it in such a way that the optimal solution of the regularized problem exist and be close to the optimal solution set of the original problem (see, for instance, [1], [2], [4] [8], [9], [21], [26], [27]).

Keeping the optimal solution of the regularized problem close to the optimal solution set of the original problem usually demands that the strictly convex approximation of the objective function should be uniform on bounded sets. Also, the regularization process

ISSN 0944-6532 / $ 2.50

often requires the use of functions satisfying, among other conditions, a stronger form of strict convexity, namely, total convexity. Total convexity of a function f, first studied in [6], is a weaker form of local uniform convexity (cf. [16]) which is equivalent (see [24]) to the requirement that the conditions

$$x \in Dom\, f,\ \{x^k\}_{k \in N} \subset Dom\, f,\ \lim_{k \to \infty} D_f(x^k, x) = 0$$

imply that $\lim_{k \to \infty} ||x^k - x|| = 0$, where $D_f : X \times Domf \to [0, \infty]$ is the Bregman distance defined by

$$D_f(y, x) = f(y) - f(x) - f^0(x, y - x)$$

and $f^0(x, \cdot) :\ X \to \bar{R}$ is the right-hand directional derivative of f at x (see also (12)).

Because of this equivalence, for a large number of optimization algorithms, regularization of the optimization problem using totally convex and sufficiently uniform (see Section 2 below) approximations of the objective function, which preserve some, if not all, of its continuity properties, is an implicit guarantee of a better convergence behavior of the computational procedure (see, for example, [4], [5], [9], [12], [22]). Thus, the abundance of totally convex functions and the possibility of using them as good approximations of given convex functions are crucial in numerous optimization algorithms. These facts naturally lead to the following questions: (i) Given a convex real valued function f (the objective function of an optimization problem) defined on a nonempty, closed and convex subset K of a Banach space X (the feasible solution set), can we always find uniform (on bounded sets) approximations of f by strictly convex functions? (ii) Given the convex function f and the set K as above, are there totally convex approximations g of f on K which preserve some desirable continuity features (like lower semicontinuity or continuity) of f and such that the Bregman distances associated with f and g will be uniformly (on bounded sets) close to each other? In the present paper we not only answer both questions in the affirmative, but also show that such approximations are the typical elements in the relevant function spaces. In particular, most perturbations of the given function f have the required properties.

When we say that a certain property holds for most elements of a complete metric space Y, we mean that the set of points which have this property contains a G_δ everywhere dense subset of Y. Such an approach, when a certain property is investigated for the whole space Y and not just for a single point in Y, has already been successfully applied in many areas of Analysis. See, for example, [13-15, 20, 28, 29] and the references mentioned there.

In Section 1 we show (see Theorem 1.2) that the strictly convex functions form a G_δ dense subset of the topological space of the real-valued convex functions on K endowed with the topology of uniform convergence on bounded sets. Moreover, this is also true for the space of lower semicontinuous convex functions on K and for the space of continuous functions on K provided with the same topology. This certainly means that any convex (or lower semicontinuous and convex, or continuous and convex) function can be approximated uniformly (on bounded sets) by strictly convex functions (respectively, by lower semicontinuous strictly convex functions or continuous strictly convex functions). Moreover, it shows that most elements (in the sense of Baire category) of the relevant topological spaces are strictly convex.

In Section 2 we focus on the existence of totally convex approximations of convex functions. We define two metric topologies on the set of real-valued convex functions on K, one stronger than the other. Two convex functions f and g are close to each other in the strong metric when they and their variations are uniformly close on bounded subsets of K. The convex functions f and g are close in the weak metric when they and the variations of their right-hand derivatives are close uniformly on bounded sets. We prove that, whenever a function which is totally convex at each point of K and Lipschitz continuous on bounded subsets of K exists, then the set of totally convex functions on K, the set of lower semicontinuous totally convex functions on K, the set of continuous totally convex functions on K, as well as the set of Lipschitz continuous totally convex functions on K are large in the sense that they contain countable intersections of open (in the weak topology) and everywhere dense (in the strong topology) subsets. This result is meaningful because it implies the existence of large pools of totally convex functions. At the same time, it guarantees that given a convex function f with some continuity features, one can find uniform (on bounded sets) totally convex approximations of it which not only preserve the continuity features of f, but also have corresponding Bregman distances which are uniformly (on bounded sets) close to the Bregman distance corresponding to f itself. More information on Bregman distances, totally convex functions and their applications can be found, for instance, in [10, 11, 17-19, 23].

1. Strictly convex functions

Let K be a nonempty convex subset of a normed linear space $(X, ||\cdot||)$. We denote by $B(r)$ the closed ball of center zero and radius $r > 0$ in $(X, ||\cdot||)$. Let $\mathfrak{M}$ be the set of all convex functions $f : K \to R^1$. Denote by $\mathfrak{M}_l$ the subset of all lower semicontinuous functions $f \in \mathfrak{M}$ and by $\mathfrak{M}_c$ the subset of all continuous functions $f \in \mathfrak{M}$. We equip the set $\mathfrak{M}$ with the uniformity determined by the following base:

$$E(n) = \{(f, g) \in \mathfrak{M} \times \mathfrak{M} : \ |f(x) - g(x)| \le n^{-1}, \quad \forall x \in K \cap B(n)\}, \tag{1}$$

where n is a natural number for which $K \cap B(n) \neq \emptyset$. It is not difficult to see that this uniform space is metrizable and complete. Clearly, $\mathfrak{M}_l$ and $\mathfrak{M}_c$ are closed subsets of $\mathfrak{M}$. We provide the topological subspaces $\mathfrak{M}_l$ and $\mathfrak{M}_c$ with the relative topologies inherited from $\mathfrak{M}$.

Recall that a function $f \in \mathfrak{M}$ is called *strictly convex* if for each $x, y \in K$ such that $x \neq y$ and each $\beta \in (0, 1)$,

$$f(\beta x + (1 - \beta)y) < \beta f(x) + (1 - \beta)f(y). \tag{2}$$

Fix $\theta \in K$. Suppose that

$$\mathcal{K} = \{K_{m,n} : \ m, n \ge 1 \text{ are integers}\}$$

is a family of nonempty, bounded subsets of $K \times K$ which do not intersect the diagonal $\Delta(K) := \{(x, x) : \ x \in K\}$ of the set $K \times K$ such that, for each natural number n, we have

$$(K \cap B(n + ||\theta||)) \times (K \cap B(n + ||\theta||)) \setminus \Delta(K) \subset \cup_{m=1}^{\infty} K_{m,n}. \tag{3}$$

A function $f \in \mathfrak{M}$ is called *strictly convex with respect to the family* $\mathcal{K}$ if for each pair of natural numbers m, n, there exists $\delta := \delta(m, n) > 0$ such that the following property holds:

(P1) For each $(x, y) \in K_{m,n}$ and each $\beta \in [(2n)^{-1}, 1-(2n)^{-1}]$,

$$\beta f(x) + (1-\beta)f(y) - f(\beta x + (1-\beta)y) \geq \delta.$$

It is easy to verify that strictly convex functions with respect to $\mathcal{K}$ are strictly convex functions on K.

Denote by $\mathcal{F}_{\mathcal{K}}$ the set of all functions $f \in \mathfrak{M}$ which are strictly convex with respet to $\mathcal{K}$ and let $\mathcal{F}$ be the set of all strictly convex functions $f : K \to R^1$.

Theorem 1.1. *Suppose that the set $\mathcal{F}_{\mathcal{K}}$ contains a function f_* which is continuous and bounded on bounded subsets of K. Then $\mathcal{F}_{\mathcal{K}}$ (respectively, $\mathcal{F}_{\mathcal{K}} \cap \mathfrak{M}_l$, $\mathcal{F}_{\mathcal{K}} \cap \mathfrak{M}_c$) is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ (respectively, $\mathfrak{M}_l$, $\mathfrak{M}_c$).*

Proof. For each pair of natural numbers m, n, denote by $\mathcal{F}_{m,n}$ the set of all $f \in \mathfrak{M}$ which have property (P1) with some $\delta > 0$. Clearly,

$$\mathcal{F}_{\mathcal{K}} = \cap_{m=1}^{\infty} \cap_{n=1}^{\infty} \mathcal{F}_{m,n}. \tag{4}$$

Let $m, n \geq 1$ be integers. Since the set $K_{m,n}$ is bounded and the space $\mathfrak{M}$ is equipped with the topology of uniform convergence on bounded subsets of K, we can easily see that $\mathcal{F}_{m,n}$ is an open subset of $\mathfrak{M}$. In order to complete the proof of the theorem it is sufficient to show that $\mathcal{F}_{m,n}$ (respectively, $\mathcal{F}_{m,n} \cap \mathfrak{M}_l$, $\mathcal{F}_{m,n} \cap \mathfrak{M}_c$) is an everywhere dense subset of $\mathfrak{M}$ (respectively, $\mathfrak{M}_l$, $\mathfrak{M}_c$).

Assume that $f \in \mathfrak{M}$. For each $\gamma \in (0, 1)$, define

$$f_\gamma(x) = f(x) + \gamma f_*(x), \ x \in K. \tag{5}$$

It is clear that for each $\gamma \in (0, 1)$, $f_\gamma \in \mathfrak{M}$ and

$$f \in \mathfrak{M}_l \text{ (respectively, } f \in \mathfrak{M}_c) \Longrightarrow f_\gamma \in \mathfrak{M}_l \text{ (respectively, } f_\gamma \in \mathfrak{M}_c). \tag{6}$$

Since f_* is bounded on bounded subsets of K, we deduce that

$$f_\gamma \to f \text{ in } \mathfrak{M} \text{ as } \gamma \to 0^+. \tag{7}$$

Since f_* is strictly convex with respect to $\mathcal{K}$, there is $\delta_* > 0$ such that

$$\beta f_*(x) + (1-\beta)f_*(y) - f_*(\beta x + (1-\beta)y) \geq \delta_* \tag{8}$$

for each $(x, y) \in K_{m,n}$ and each $\beta \in [(2n)^{-1}, 1-(2n)^{-1}]$. We claim that whenever $\gamma \in (0, 1)$ it follows that $f_\gamma \in \mathcal{F}_{m,n}$. Indeed, assume that $(x, y) \in K_{mn}$ and $\beta \in [(2n)^{-1}, (1-(2n)^{-1}]$. Using (8), (5), and the convexity of f, we deduce that

$$\begin{aligned}
& \beta f_\gamma(x) + (1-\beta)f_\gamma(y) - f_\gamma(\beta x + (1-\beta)y) \\
= \ & \beta f(x) + (1-\beta)f(y) - f(\beta x + (1-\beta)y) \\
& \quad + \gamma[\beta f_*(x) + (1-\beta)f_*(y) - f_*(\beta x + (1-\beta)y)] \\
\geq \ & \gamma[\beta f_*(x) + (1-\beta)f_*(y) - f_*(\beta x + (1-\beta)y)] \geq \gamma\delta_*.
\end{aligned}$$

This shows that property (P1) holds for $f = f_\gamma$ with $\delta = \gamma\delta_*$. Hence $f_\gamma \in \mathcal{F}_{m,n}$, as claimed. When combined with (7) and (6), this implies that $\mathcal{F}_{m,n}$ (respectively, $\mathcal{F}_{m,n} \cap \mathfrak{M}_l$ and $\mathcal{F}_{m,n} \cap \mathfrak{M}_c$) is an everywhere dense subset of $\mathfrak{M}$ (respectively, $\mathfrak{M}_l$ and $\mathfrak{M}_c$). Theorem 1.1 is proved.

Remark. We assumed that each one of the sets $K_{m,n}$ was nonempty. This condition is not necessary. We may allow $K_{m,n}$ to be empty for some pairs of integers (m, n). It is clear that for such pairs $\mathcal{F}_{m,n} = \mathfrak{M}$.

We now use Theorem 1.1 to prove the main result of this section.

Theorem 1.2. *Suppose that there exists a strictly convex continuous function f_* which is bounded on bounded subsets of K. Then the set $\mathcal{F}$ (respectively, $\mathcal{F} \cap \mathfrak{M}_l$, $\mathcal{F} \cap \mathfrak{M}_c$) contains a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ (respectively, $\mathfrak{M}_l$, $\mathfrak{M}_c$).*

Proof. For each $x, y \in K$, define a function $\psi_{x,y} : [0, 1] \to R^1$ by

$$\psi_{x,y}(\beta) = \beta f_*(x) + (1 - \beta)f_*(y) - f_*(\beta x + (1 - \beta)y),\ \beta \in [0, 1].$$

Let $x, y \in K$ satisfy $x \neq y$ and let n be a natural number. Since f_* is strictly convex, $\psi_{x,y}(\beta) > 0$ for all $\beta \in [(2n)^{-1}, 1 - (2n)^{-1}]$. It follows from the continuity of f_* that the function $\psi_{x,y}$ is also continuous. Therefore

$$\min\{\psi_{x,y}(\beta) :\ \beta \in [(2n)^{-1}, 1 - (2n)^{-1}]\} > 0. \tag{9}$$

For each pair of natural numbers m and n, set

$$\begin{aligned} K_{m,n} \ = \ \{(x, y) \in (K \cap B(n + ||\theta||)) \times (K \cap B(n + ||\theta||)) \setminus \Delta(K) : \\ \psi_{x,y}(\beta) \geq m^{-1}, \quad \forall \beta \in [(2n)^{-1}, 1 - (2n)^{-1}]\}. \end{aligned} \tag{10}$$

In view of (9) and (10), for each natural number n,

$$\cup_{m=1}^{\infty} K_{m,n} = \{(x, y) :\ x, y \in K \cap B(n + ||\theta||)\} \setminus \Delta(K).$$

Denote by $\mathcal{K}$ the family $\{K_{m,n} :\ m, n \geq 1 \text{ are integers}\}$. It is easy to see that f_* is strictly convex with respect to the family $\mathcal{K}$. By Theorem 1.1, the set $\mathcal{F}_{\mathcal{K}}$ (respectively, $\mathcal{F}_{\mathcal{K}} \cap \mathfrak{M}_l$, $\mathcal{F}_{\mathcal{K}} \cap \mathfrak{M}_c$) is a countable intersection of open everywhere dense subsets of $\mathfrak{M}$ (respectively, $\mathfrak{M}_l$, $\mathfrak{M}_c$). In order to complete the proof of Theorem 1.2, it is now sufficient to note that $\mathcal{F}_{\mathcal{K}} \subset \mathcal{F}$.

2. Totally convex functions

Let $(X, ||\cdot||)$ be a Banach space and let K be a nonempty, closed and convex subset of X. Denote by $\mathcal{M}$ the set of all convex functions $f : X \to (-\infty, +\infty]$ such that $K \subset \text{Int}(\text{Dom } f)$.

Let $\mathcal{M}_v$ be the subset of $\mathcal{M}$ consisting of all those functions in $\mathcal{M}$ which are everywhere finite, i.e., with Dom $f = X$. Denote by $\mathcal{M}_l$ the subset of $\mathcal{M}$ consisting of all the lower semicontinuous functions in $\mathcal{M}$. Put $\mathcal{M}_{vl} = \mathcal{M}_v \cap \mathcal{M}_l$ and let $\mathcal{M}_c$ be the subset of $\mathcal{M}_v$ consisting of all the continuous functions in $\mathcal{M}$. We denote the collection of all those $f \in \mathcal{M}_v$ which are Lipschitz on bounded subsets of X by $\mathcal{M}_L$.

With any function $f \in \mathcal{M}$ we associate the function $D_f : X \times \text{Dom} f \to [0, \infty]$ defined by

$$D_f(y, x) = f(y) - f(x) - f^0(x, y - x), \tag{11}$$

where $f^0(x,\cdot):\ X\to\bar{R}$ is the directional derivative of f at x defined by

$$f^0(x,h)=\lim_{t\to 0^+} t^{-1}[f(x+th)-f(x)] \tag{12}$$

whenever $x\in \mathrm{Dom}f$. The function $\nu_f:(\mathrm{Dom}f)\times[0,\infty)\to[0,\infty]$ given by

$$\nu_f(x,t)=\inf\{D_f(y,x):\ y\in X,\ ||y-x||=t\} \tag{13}$$

was called in [6] the *modulus of total convexity* of $f\in\mathcal{M}$. The function $f\in\mathcal{M}$ is called *totally convex* at $x\in\mathrm{Dom}f$ if $\nu_f(x,t)>0$ for all $t>0$.

Fix $\theta\in K$. For each natural number i and each pair of functions $f,g\in\mathcal{M}$, first set

$$\begin{aligned}&\tilde{d}_i(f,g)\\ =\ &\sup\{|f(x)-g(x)|:\ x\in B(i+||\theta||)\}\\ &\quad+\sup\{|f(x)-g(x)-(f(y)-g(y))|/||x-y||: x,y\in\Omega(f,g,i)\text{ and }x\neq y\},\end{aligned}$$

where

$$\Omega(f,g,i)=\ \mathrm{Dom}(f)\cap\ \mathrm{Dom}(g)\cap B(i+||\theta||),$$

and then let

$$d_i(f,g)=\tilde{d}_i(f,g)(1+\tilde{d}_i(f,g))^{-1}. \tag{14}$$

(Here we use the conventions that $\infty-\infty=0$ and $\infty/\infty=1$.) Finally, set

$$d(f,g)=\sum_{i=1}^{\infty}2^{-i}d_i(f,g),\quad f,g\in\mathcal{M}. \tag{15}$$

Clearly, d is a metric on $\mathcal{M}$. For each natural number n, let $E(n)$ be the subset of $\mathcal{M}\times\mathcal{M}$ consisting of all pairs (f,g) such that

$$\mathrm{Dom}(f)\cap B(n+||\theta||)=\mathrm{Dom}(g)\cap B(n+||\theta||), \tag{16}$$

$$|f(x)-g(x)|\le 1/n,\quad \forall x\in B(n+||\theta||) \tag{17}$$

and

$$|(f(x)-g(x))-(f(y)-g(y))|\le n^{-1}||x-y||,\quad \forall x,y\in\Omega(f,g,n).$$

The sets $\{E(n):\ n=1,2,\dots\}$ form a base of the uniformity induced by the metric d. It is not difficult to see that the metric space $(\mathcal{M},d)$ is complete, and that $\mathcal{M}_v$, $\mathcal{M}_l$, $\mathcal{M}_{vl}$, $\mathcal{M}_c$ and $\mathcal{M}_L$ are all closed subsets of it. We call the metric topology of $(\mathcal{M},d)$ *the strong topology* of $\mathcal{M}$. For each natural number i and each $f,g\in\mathcal{M}$, we also define

$$\begin{aligned}&\tilde{d}_{wi}(f,g)\\ =\ &\sup\{|f(x)-g(x)|:\ x\in B(i+||\theta||)\}+\sup\{|f^0(x,y-x)-g^0(x,y-x)/|||y-x||:\\ &\quad x\in K,\ y\in\mathcal{D}(f)\cap\mathcal{D}(g),\ x\neq y,\ x,y\in B(i+||\theta||)\},\end{aligned} \tag{18}$$

and

$$d_{wi}(f,g)=\tilde{d}_{wi}(f,g)(1+\tilde{d}_{wi}(f,g))^{-1}.$$

For each pair of functions $f, g \in \mathcal{M}$, we put

$$d_w(f,g) = \sum_{i=1}^{\infty} 2^{-i} d_{wi}(f,g). \tag{19}$$

It is clear that d_w is a metric on $\mathcal{M}$ and $d_w(f,g) \leq d(f,g)$ whenever $f, g \in \mathcal{M}$.

For each natural number n, denote by $E_w(n)$ the subset of $\mathcal{M} \times \mathcal{M}$ consisting of those pairs (f,g) satisfying (16), (17), and the following condition:

$$|f^0(x, y-x) - g^0(x, y-x)| \leq n^{-1}||y-x|| \tag{20}$$
$$\text{for all } x \in K \cap B(n+||\theta||) \text{ and all } y \in \text{Dom}(f) \cap \text{Dom}(g) \cap B(n+||\theta||).$$

The family of sets $\{E_w(n) : \ n = 1, 2, \dots\}$ forms a base of the uniformity induced on $\mathcal{M}$ by the metric d_w. The metric d_w provides $\mathcal{M}$ with a topology which we will call *the weak topology*. We consider the topological subspaces $\mathcal{M}_v$, $\mathcal{M}_l$, $\mathcal{M}_{vl}$, $\mathcal{M}_c$ and $\mathcal{M}_L$ with the relative weak and strong topologies inherited from $\mathcal{M}$. Let $\mathcal{A}$ be one of the spaces $\mathcal{M}_v$, $\mathcal{M}_l$, $\mathcal{M}_{vl}$, $\mathcal{M}_c$ and $\mathcal{M}_L$ equipped with the weak and strong relative topologies.

We note that when the functions f and g are close in the metric d_w, then not only are they close uniformly on bounded subsets of K, but the Bregman distances determined by them are also close in the same sense.

Theorem 2.1. *Suppose that $\mathcal{M}_L$ contains a function f_* which is totally convex at each point $x \in K$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that each $f \in \mathcal{F}$ is totally convex at each point of K.*

Proof. Fix $t > 0$. We first show that there exists a set $\mathcal{F}_t \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that, for each $f \in \mathcal{F}_t$ and each $x \in K$, we have $\nu_f(x,t) > 0$. To this end, let $f \in \mathcal{A}$. For each $\gamma \in (0,1)$, set

$$f_\gamma(x) = f(x) + \gamma f_*(x), \quad \forall x \in X. \tag{21}$$

It is clear that $f_\gamma \in \mathcal{A}$. Since $f_* \in \mathcal{M}_L$, it follows that $f_\gamma \to f$ as $\gamma \to 0^+$ in the strong topology. For each natural number n, set

$$K_n = \{x \in K : \ ||x|| \leq n, \ \nu_{f_*}(x,t) \geq n^{-1}\}. \tag{22}$$

Clearly,

$$\cup_{n=1}^{\infty} K_n = K. \tag{23}$$

Let $f \in \mathcal{A}$, $\gamma \in (0,1)$, and let $n \geq 1$ be an integer. By (13), (11), (12), (21) and (22), for each $x \in K_n$, we have

$$\begin{aligned}
\nu_{f_\gamma}(x,t) &= \inf\{D_{f_\gamma}(y,x) : \ y \in X, \ ||y-x|| = t\} \\
&= \inf\{f_\gamma(y) - f_\gamma(x) - f_\gamma^0(x, y-x) : \ y \in X, \ ||y-x|| = t\} \\
&= \inf\{f(y) - f(x) - f^0(x, y-x) + \gamma[f_*(y) - f_*(x) - f_*^0(x, y-x)] : \\
&\qquad y \in X, \ ||y-x|| = t\} \\
&\geq \gamma \inf\{f_*(y) - f_*(x) - f_*^0(x, y-x) : y \in X, \ ||y-x|| = t\} \\
&= \gamma \nu_{f_*}(x,t) \geq \gamma/n.
\end{aligned}$$

Thus $\nu_{f_\gamma}(x,t) \geq \gamma/n$ for each $x \in K_n$. Choose a natural number q such that

$$q > n + t + (t+2)2n/\gamma. \tag{24}$$

There exists an open neighborhood $\mathcal{U}(f,\gamma,n)$ of f_γ in $\mathcal{A}$ equipped with the weak topology such that

$$\mathcal{U}(f,\gamma,n) \subset \{g \in \mathcal{A} : \ (f_\gamma, g) \in E_w(q)\}. \tag{25}$$

Assume that

$$g \in \mathcal{U}(f,\gamma,n), \ x \in K_n, \ y \in X, \ ||y-x|| = t. \tag{26}$$

By (26), (25), (20), (24) and (22), we deduce that

$$\max\{|f_\gamma(x) - g(x)|, \ |f_\gamma(y) - g(y)|\} \leq q^{-1} \tag{27}$$

and

$$|f_\gamma^0(x, y-x) - g^0(x, y-x)| \leq q^{-1}||y-x||. \tag{28}$$

It follows from (27), (28), (26), (11), (13) and (24) that

$$\begin{aligned} D_g(y,x) &= g(y) - g(x) - g^0(x, y-x) \\ &\geq f_\gamma(y) - f_\gamma(x) - f_\gamma^0(x, y-x) - 2/q - q^{-1}||y-x|| \\ &= D_{f_\gamma}(y,x) - 2/q - t/q \geq \nu_{f_\gamma}(x,t) - (t+2)/q \\ &\geq \gamma/n - (t+2)/q \geq \gamma/(2n). \end{aligned}$$

Together with (13) this implies that

$$\nu_g(x,t) \geq \gamma(2n)^{-1}, \quad \forall x \in K_n \text{ and } \forall g \in \mathcal{U}(f,\gamma,n). \tag{29}$$

Set

$$\mathcal{F}_t = \cap_{n=1}^{\infty} \cup \{\mathcal{U}(f,\gamma,n) : \ f \in \mathcal{A}, \ \gamma \in (0,1)\}. \tag{30}$$

It is easy to see that $\mathcal{F}_t$ is a countable intersection of open (in the weak topology) eveywhere dense (in the strong topology) subsets of $\mathcal{A}$. Let $g \in \mathcal{F}_t$ and $x \in K$. By (23), there is a natural number n such that $x \in K_n$. By (30), there exist $f \in \mathcal{A}$ and $\gamma \in (0,1)$ such that $g \in \mathcal{U}(f,\gamma,n)$. When combined with (29), this implies that $\nu_g(x,t) \geq \gamma/(2n)$. Thus, for each $g \in \mathcal{F}_t$ and for each $x \in K$, we have $\nu_g(x,t) > 0$, as claimed.

Taking now $t = 1/n$, where n is a positive integer, we deduce that for each integer $n \geq 1$, there exists a set $\mathcal{F}_{1/n} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that $\nu_g(x, n^{-1}) > 0$ for each $g \in \mathcal{F}_{1/n}$ and each $x \in K$. Set

$$\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{F}_{1/n}.$$

Clearly, $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $g \in \mathcal{F}$, each $x \in K$ and each $t > 0$, we have $\nu_g(x,t) > 0$. In other words, each $g \in \mathcal{F}$ is totally convex at each point of K. This completes the proof of Theorem 2.1.

Remarks. (i) According to Proposition 3.2 in [16], the square of the norm in any reflexive Banach space with the Kadec-Klee property is an example of the function f_* the existence of which is the hypothesis of Theorem 2.1.

(ii) According to Proposition 2.13 in [16], any totally convex function is essentially strictly convex in the sense of [25] and [3]. Thus Theorem 2.1 also shows that there are many (in the sense of Baire category) essentially strictly convex functions.

Acknowledgements. The work of the first and second author was partially supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities (Grant 592/00). The second author was also supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund.

References

[1] Y. Alber, R. Burachik, A. N. Iusem: A proximal point method for nonsmooth optimization in Banach spaces, Abstr. Appl. Anal. 2 (1997) 97–120.

[2] Y. Alber, D. Butnariu, I. Ryazantseva: Regularization and resolution of monotone variational inequalities with operators given by hypomonotone approximations, J. Nonlinear Convex Anal. 6 (2005) 23–53.

[3] H. Bauschke, J. M. Borwein, P. L. Combettes: Essential smoothness, essential strict convexity, and Legendre functions in Banach spaces, Commun. Contemp. Math. 3 (2001) 615–647.

[4] H. Bauschke, J. M. Borwein, P. L. Combettes: Bregman monotone optimization algorithms, SIAM J. Control Optim. 42 (2003) 596–636.

[5] H. Bauschke, P. L. Combettes: Construction of best Bregman approximations in reflexive Banach spaces, Proc. Amer. Math. Soc. 131 (2003) 3757–3766.

[6] D. Butnariu, Y. Censor, S. Reich: Iterative averaging of entropic projections for solving stochastic convex feasibility problems, Comput. Optim. Appl. 8 (1997) 21–39.

[7] D. Butnariu, A. N. Iusem: Local moduli of convexity and their application to finding almost common fixed points of measurable families of operators, in: Recent Developments in Optimization Theory and Nonlinear Analysis, Y. Censor, S. Reich (eds.), Contemporary Mathematics 204, American Mathematical Society, Providence (1997) 61–91.

[8] D. Butnariu, A. N. Iusem: On a proximal point method for convex optimization in Banach spaces, Numer. Funct. Anal. Optim. 18 (1998) 723–744.

[9] D. Butnariu, A. N. Iusem: Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, Kluwer, Dordrecht (2000).

[10] D. Butnariu, A. N. Iusem, R. S. Burachik: Iterative methods of solving stochastic convex feasibility problems and applications, Comput. Optim. Appl. 15 (2000) 269–307.

[11] D. Butnariu, A. N. Iusem, E. Resmerita: Total convexity of the powers of the norm in uniformly convex Banach spaces, J. Convex Analysis 7 (2000) 319–334.

[12] D. Butnariu, A. N. Iusem, C. Zalinescu: Uniform convexity, total convexity and the convergence of the outer Bregman projection and proximal point algorithms in Banach spaces, J. Convex Analysis 10 (2003) 35–61.

[13] D. Butnariu, S. Reich, A. J. Zaslavski: Generic power convergence of operators in Banach spaces, Numer. Funct. Anal. Optim. 20 (1999) 629–650.

[14] D. Butnariu, S. Reich, A. J. Zaslavski: Asymptotic behavior of relatively nonexpansive operators in Banach spaces, J. Appl. Anal. 7 (2001) 151–174.

[15] D. Butnariu, S. Reich, A. J. Zaslavski: Asymptotic behavior of quasi-nonexpansive mappings, in: Inherently Parallel Algorithms in Feasibility and Optimization and their Applications, D. Butnariu, Y. Censor, S. Reich (eds.), Elsevier, Amsterdam (2001) 49–68.

[16] D. Butnariu, E. Resmerita: Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, Abstr. Appl. Anal. 2006.

[17] Y. Censor, A. Lent: An iterative row-action method for interval convex programming, J. Optim. Theory Appl. 34 (1981) 321–353.

[18] Y. Censor, S. Reich: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization, Optimization 37 (1996) 323–339.

[19] Y. Censor, S. A. Zenios: Parallel Optimization, Oxford, New York (1997).

[20] F. S. De Blasi, J. Myjak: Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach, C. R. Acad. Sci., Paris, Sér. A 283 (1976) 185–187.

[21] A. N. Iusem, R. G. Otero: Augmented Lagrangian methods for cone constrained convex optimization in Banach spaces, J. Nonlinear Convex Anal. 3 (2002) 155–176.

[22] R. G. Otero, B. F. Svaiter: A strongly convergent hybrid proximal method in Banach spaces, J. Math. Anal. Appl. 289 (2004) 700–711.

[23] S. Reich: A weak convergence theorem for the alternating method with Bregman distances, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, A. G. Kartsatos (ed.), Marcel Dekker, New York (1996) 313–318.

[24] E. Resmerita: On total convexity, Bregman projections and stability in Banach spaces, J. Convex Analysis 11 (2004) 1–16.

[25] R. T. Rockafellar: Convex Analysis, Princeton Univ. Press, Princeton (1970).

[26] R. T. Rockafellar: Augmented Lagrangeans and applications of the proximal point algorithm in convex programming, Math. Oper. Res. 1 (1976) 97–116.

[27] A. N. Tikhonov: Regularization of incorrectly posed problems, Soviet Math. Dokl. 4 (1963) 1035–1038.

[28] G. Vidossich: Existence, uniqueness and approximation of fixed points as a generic property, Bol. Soc. Bras. Mat. 5 (1974) 17–29.

[29] A. J. Zaslavski: Well-posedness and porosity in optimal control without convexity assumptions, Calc. Var. Partial Differ. Equ. 13 (2001) 265–293.