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$\varepsilon-$Fréchet Differentiability of Lipschitz Functions and Applications

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Dedicated to the memory of Simon Fitzpatrick.

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We study the $\varepsilon-$Fréchet differentiability of Lipschitz functions on Asplund generated Banach spaces. We prove a mean valued theorem and its equivalent, a formula for Clarke's subdifferential, in terms of this concept. We inspect proofs of several statements based on the deep Preiss's theorem on Fréchet differentiability of Lipschitz functions and we recognize that it is enough to use a simpler lemma on $\varepsilon-$Fréchet differentiability due to Fabian and Preiss. We do so for generic differentiability results of Giles and Sciffer, for the existence of nearest points of Borwein and Fitzpatrick, etc. We also show that the $\varepsilon-$Fréchet differentiability is separably reducible.

Keywords: $\varepsilon-$Fréchet differentiability, mean-value theorem, local $\varepsilon-$support, intermediate differentiability, Asplund generated space, boundedly Asplund set, separable reduction

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1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space with continuous dual X^* and duality pairing between X and X^* denoted by $\langle \cdot, \cdot \rangle$. We denote by $\mathbb{B}_X$ the closed unit ball in X, and by $\mathbb{B}_{X^*}$ the closed unit ball of X^*. Let $\varepsilon > 0$ be given, let $\emptyset \neq G \subset X$ be an open set, let $x \in G$, and consider a function $f : G \to \mathbb{R}$. We say that f is $\varepsilon-$*Fréchet differentiable* at x if there exist $x^* \in X^*$ and $\delta > 0$ such that

$$|f(x+h) - f(x) - \langle x^*, h \rangle| \leq \varepsilon \|h\| \quad \text{whenever } h \in X \text{ and } \|h\| < \delta$$

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[1, page 279]. The set of all such x^*'s with the above property is denoted by $\nabla_\varepsilon f(x)$. The ε−Fréchet differentiability at a point gives an affine approximation of f in a vicinity of x up to ε. According to [4], the *local ε−support* $S_\varepsilon f(x)$ of f at x is the set of all $x^* \in X^*$ for which there is $\delta > 0$ such that

$$f(x+h) - f(x) - \langle x^*, h\rangle \geq -\varepsilon\|h\| \quad \text{whenever } h \in X \text{ and } \|h\| < \delta.$$

If $S_\varepsilon f(x) \neq \emptyset$, then we say that f is *locally ε−supported* at x. It is well-known that $S_\varepsilon f(x)$ is closely related to the *ε−Fréchet subdifferential* set [9, pages 27, 37],

$$\hat{\partial}_\varepsilon f(x) = \left\{ x^* \in X^*;\ \liminf_{h\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\epsilon \right\}.$$

Indeed,

$$S_\varepsilon f(x) \subset \hat{\partial}_\varepsilon f(x) = \bigcap_{\nu>0} S_{\varepsilon+\nu} f(x). \tag{1}$$

Both $\nabla_\varepsilon f$ and $S_\varepsilon f$ are multifunctions from X into X^*, and

$$\nabla_\varepsilon f(x) = S_\varepsilon f(x) \cap (-S_\varepsilon(-f)(x)) \quad \text{for each } x \in X.$$

Banach spaces where every continuous convex function on an open subset is Fréchet differentiable on a dense G_δ subset of its domain are called *Asplund spaces* and are characterized by the property that every nonempty bounded subset in X^* admits weak* slices of arbitrarily small diameter [17, page 31]. A major theorem on the differentiability of Lipschitz functions, due to Preiss [19], states that *a Lipschitz function on an open subset of an Asplund space is Fréchet differentiable at the points of a dense subset of its domain.* In the meantime there has appeared a weaker, ε−Fréchet differentiability statement due to Fabian and Preiss [10] where the Fréchet differentiability was replaced by the ε−Fréchet differentiability.

While $S_\varepsilon f$ has been intensively studied [5, 8, 4], and used in constructing basic subdifferential [15], $\nabla_\varepsilon f$ is less explored. We believe that it is also interesting to investigate the concept $\nabla_\varepsilon f$ and to apply it in nonsmooth analysis.

The aim of this note is twofold: One goal is to find a mean value theorem in terms of $\nabla_\varepsilon f$ and thus to get a corresponding characterization of Clarke's subdifferential for Lipschitz functions on Asplund spaces. The other goal is to show that instead of applying the deep Preiss' Differentiability Theorem, one can use an easier lemma of Fabian and Preiss [10] and thus to prove several known results. For example, we get intermediate differentiability statements on Asplund generated spaces given by Giles and Sciffer in [12, 13] and the result on Lipschitz quotients of Asplund spaces by Benyamini and Lindenstrauss [1]. Moreover, we prove the existence of nearest points to boundedly Asplund sets in Banach spaces, which generalizes a result due to Borwein and Fitzpatrick [2]. Finally, we also show that the existence of ε−Fréchet differential on nonseparable Banach spaces can be reduced to separable spaces with the help of a separable reduction.

Of course, we present here only a representative rather than exhaustive list of statements which can be proved by the Fabian-Preiss lemma instead of the deep Preiss' theorem. Readers are encouraged to look for further statements where the use of ε−Fréchet differentiability is enough.

2. A Mean Value Theorem for $\varepsilon - T\mathbb{B}_Y-$Fréchet Differential

Let X be a Banach space, $\emptyset \neq G \subset X$ an open set, $f : G \to \mathbb{R}$ a function, and $x \in G$. The *Clarke directional derivative* and *Clarke subdifferential* of f at x are defined respectively by

$$f^\circ(x; h) = \limsup_{y \to x, t \downarrow 0} \frac{f(y + th) - f(y)}{t}, \quad h \in X,$$

$$\partial f(x) = \{\xi \in X^*;\ \langle \xi, h\rangle \leq f^\circ(x; h) \quad \text{for all } h \in X\}.$$

Further, for $e \in X$, we define the *partial derivative* of f at x along direction e as

$$f'(x; e) = \lim_{t \to 0} \frac{f(x + te) - f(x)}{t}$$

if the the two-sided limit exists.

A Banach space X is called *Asplund generated* if there exists an Asplund space Y and a continuous linear mapping $T : Y \to X$ such that $X = \overline{TY}$. Without loss of any generality, we may assume $\|T\| \leq 1$. In what follows, whenever we say that $X = \overline{TY}$ is an Asplund generated space we shall understand that Y and T are as above. Let $M \subset X^*$ be a bounded set. For $e \in X$ and $\alpha \in \mathbb{R}$ we put

$$S(M, e, \alpha) = \big\{x^* \in M;\ \big\langle x^*, e\big\rangle > \alpha\big\}$$

and we call it a *weak* slice* of M. Of course, we are usually interested in $\alpha < \sup \big\langle M, e\big\rangle$; then $S(M, e, \alpha) \neq \emptyset$. For $\emptyset \neq A \subset X$ we put

$$A - \operatorname{diam} M = \sup \big\{\big\langle x_1^* - x_2^*, x\big\rangle;\ x_1^*, x_2^* \in M, x \in A\big\}.$$

The following $\varepsilon-$Fréchet differentiability lemma will be our key tool throughout the whole note. It roughly says that the existence of a weak* slice of $\partial f(G)$ of diameter less than ε guarantees the $3\varepsilon-$Fréchet differentiability of f at some point of G. Because we want to work in Asplund generated spaces we need the following generalization of $\varepsilon-$Fréchet differentiability. Given an Asplund generated space $X = \overline{TY}$, $x \in G \subset X$, with G open, and $f : G \to \mathbb{R}$, we say that f is $\varepsilon - T\mathbb{B}_Y-$*Fréchet differentiable* at x if there exist $x^* \in X^*$ and $\delta > 0$ such that

$$|f(x + tTy) - f(x) - \langle x^*, tTy\rangle| \leq \varepsilon t \quad \text{whenever } 0 < t < \delta \text{ and } y \in \mathbb{B}_Y.$$

The set of all such x^*'s with the above property is denoted by $\nabla_{\varepsilon - T\mathbb{B}_Y} f(x)$. Likewise, we consider the following generalization of the $\varepsilon-$Fréchet subdifferential. By $\partial_{\varepsilon - T\mathbb{B}_Y} f(x)$ we understand the set of all $x^* \in X^*$ for which there is $\delta > 0$ such that

$$f(x + tTy) - f(x) - \langle x^*, tTy\rangle \geq -\varepsilon t \quad \text{whenever } 0 < t < \delta \text{ and } y \in \mathbb{B}_Y.$$

Proposition 2.1 (Fabian & Preiss [10]). *Consider a Lipschitz function f on an open subset G of an Asplund generated space $X = \overline{T(Y)}$ and let $\varepsilon > 0$ be given. If $e \in X$ and $\alpha \in \mathbb{R}$ are such that $S(\partial f(G), e, \alpha) \neq \emptyset$ and*

$$T\mathbb{B}_Y - \operatorname{diam} S(\partial f(G), e, \alpha) < \varepsilon,$$

then there exist $x \in G$ and $x^* \in \partial f(x) \cap S(\partial f(G), e, \alpha)$ such that

(a) $f'(x;e) = \langle x^*, e\rangle$ *and*
(b) $x^* \in \nabla_{3\varepsilon - T\mathbb{B}_Y} f(x) \neq \emptyset$, *that is, for some* $\delta > 0$ *we have*

$$|f(x+tTy) - f(x) - \langle x^*, tTy\rangle| \leq 3\varepsilon t \quad \textit{whenever } 0 < t < \delta \textit{ and } y \in \mathbb{B}_Y.$$

Now we are ready to formulate a mean value theorem in terms of $\varepsilon - T\mathbb{B}_Y-$Fréchet differentials on Asplund generated spaces.

Theorem 2.2. *Let G be a nonempty open subset of an Asplund generated space $X = \overline{TY}$ and consider a Lipschitz function $f : G \to \mathbb{R}$. Then for any $a, b \in G$, for any open set $\Omega \subset G$ containing the linear segment $[a, b]$, and for any $\varepsilon > 0$ there exist $c \in \Omega$ and $\xi \in \nabla_{\varepsilon - T\mathbb{B}_Y} f(c)$ such that*

$$f(b) - f(a) \ < \langle \xi, b-a\rangle + \varepsilon.$$

Proof. Let a, b, Ω, and ε be as in the statement. Without loss of generality, we may assume that f Lipschitz constant 1. By Lebourg's mean value theorem [3, page 41], there exist $u \in [a, b]$ and $x^* \in \partial f(u)$ such that

$$f(b) - f(a) \leq \langle x^*, b-a\rangle.$$

Since $\overline{TY} = X$, Y is an Asplund space, and $T^*(\partial f(\Omega))$ is a bounded set, there exist $y \in Y$, with $\|(b-a) - Ty\| < \frac{\varepsilon}{3}$, and $\alpha \in \mathbb{R}$ so that the weak* slice $S(T^*(\partial f(\Omega)), y, \alpha)$ is nonemty and has norm-diameter less than $\frac{\varepsilon}{3}$. Obviously, nothing important will be changed if α is chosen to satisfy $\sup\big\langle \partial f(\Omega), Ty\big\rangle - \frac{\varepsilon}{3} < \alpha < \sup\big\langle \partial f(\Omega), Ty\big\rangle$. Then $T^*\big(S(\partial f(\Omega), Ty, \alpha)\big) = S(T^*(\partial f(\Omega)), y, \alpha)$. Hence $S(\partial f(\Omega)), Ty, \alpha) \neq \emptyset$ and

$$T\mathbb{B}_Y - \operatorname{diam} S(\partial f(\Omega), Ty, \alpha) = \mathbb{B}_Y - \operatorname{diam} S(T^*(\partial f(\Omega)), y, \alpha) < \frac{\varepsilon}{3}.$$

By Proposition 2.1 we find $c \in \Omega$ and $\xi \in \partial f(c) \cap S(\partial f(\Omega), Ty, \alpha) \cap \nabla_{\varepsilon - T\mathbb{B}_Y} f(c)$. We can then estimate

$$\begin{aligned} & f(b) - f(a) \\ \leq\ & \langle x^*, (b-a) - Ty\rangle + \langle x^*, Ty\rangle \leq \|(b-a) - Ty\| + \langle x^*, Ty\rangle \\ <\ & \frac{\varepsilon}{3} + \langle x^*, Ty\rangle \leq \frac{\varepsilon}{3} + \sup\big\langle \partial f(\Omega), Ty\big\rangle < \frac{\varepsilon}{3} + \langle \xi, Ty\rangle + \sup\big\langle \partial f(\Omega), Ty\big\rangle - \alpha \\ <\ & \frac{\varepsilon}{3} + \langle \xi, Ty - (b-a)\rangle + \langle \xi, b-a\rangle + \frac{\varepsilon}{3} < \varepsilon + \langle \xi, b-a\rangle. \end{aligned}$$

□

We note that if X is an Asplund space, then $\nabla_{\varepsilon - T\mathbb{B}_Y} f(c)$ in the above theorem reduces to $\nabla_\varepsilon f(c)$.

As a consequence of our $\varepsilon - T\mathbb{B}_Y-$Fréchet differential mean value theorem, on Asplund generated spaces we have the following formula for Clarke's subdifferential. This is a weakening of Preiss' formula which uses Fréchet derivatives [19, pages 319, 320]. By $\overline{\text{co}}^* E$, we mean the weak* closed convex hull of $E \subset X^*$.

Theorem 2.3. *Let $X = \overline{TY}$ be an Asplund generated space, let $\emptyset \neq G \subset X$ be an open set, and $f : G \to \mathbb{R}$ be a Lipschitz function. Then for every $\varepsilon > 0$ the function f is $\varepsilon - T\mathbb{B}_Y$−Fréchet differentiable at the points of a dense subset of G, and for every $\bar{x} \in G$ we have*

$$\lim_{\varepsilon\downarrow 0}\left[\sup\left\langle \nabla_{\varepsilon-T\mathbb{B}_Y} f(\mathbb{B}(\bar{x},\varepsilon)), h\right\rangle\right] = \lim_{\varepsilon\downarrow 0}\left[\sup\left\langle S_{\varepsilon-T\mathbb{B}_Y} f(\mathbb{B}(\bar{x},\varepsilon)), h\right\rangle\right] = f^\circ(\bar{x};h)$$

for every $h \in X$, or equivalently,

$$\bigcap_{\varepsilon>0} \overline{\mathrm{co}}^* \nabla_{\varepsilon-T\mathbb{B}_Y} f(\mathbb{B}(\bar{x},\varepsilon)) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}^* S_{\varepsilon-T\mathbb{B}_Y} f(\mathbb{B}(\bar{x},\varepsilon)) = \partial f(\bar{x}).$$

Proof. The first statement follows directly from Proposition 2.1 and the weak* dentability of Asplund spaces; see the proof of Theorem 2.2.

Fix $\bar{x} \in G$. Consider any $y \in \mathbb{B}_Y$. For $k = 1, 2, \dots$ find $\varepsilon_k \downarrow 0$, $x_k \in \mathbb{B}(\bar{x}, \varepsilon_k)$, and $\xi_k \in S_{\varepsilon_k - T\mathbb{B}_Y} f(x_k)$ such that $\lim_{k\to\infty}\langle \xi_k, Ty\rangle = \lim_{\varepsilon\downarrow 0}\left[\sup\left\langle S_{\varepsilon-T\mathbb{B}_Y} f(\mathbb{B}(\bar{x},\varepsilon)), Ty\right\rangle\right]$. Find $t_k \downarrow 0$ such that $\left\langle \xi_k, Ty\right\rangle - \varepsilon_k \le \frac{1}{t_k}\big(f(x_k + t_k Ty) - f(x_k)\big)$. Then

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\left[\sup\left\langle S_{\varepsilon-T\mathbb{B}_Y} f(\mathbb{B}(\bar{x},\varepsilon)), Ty\right\rangle\right] &= \lim_{k\to\infty}\langle \xi_k, Ty\rangle \\ &\le \limsup_{k\to\infty} \frac{1}{t_k}\big(f(x_k + t_k Ty) - f(x_k)\big) \le f^\circ(\bar{x}; Ty).\end{aligned}$$

Reversely. For $k = 1, 2, \dots$ find $x_k \in X$, with $\|x_k - \bar{x}\| < \frac{1}{k}$, and $t_k \in (0, \frac{1}{k})$ such that $f^\circ(\bar{x}; Ty) = \lim_{k\to\infty} \frac{1}{t_k}\big(f(x_k + t_k Ty) - f(x_k)\big)$. For every k find, by Theorem 2.2, a $c_k \in X$, with $\|c_k - x_k\| < 2/k$, $\xi_k \in \nabla_{t_k^2 - T\mathbb{B}_Y} f(c_k)$ and

$$f(x_k + t_k Ty) - f(x_k) \le \langle \xi_k, t_k Ty\rangle + t_k^2.$$

Then $\xi_k \in \nabla_{t_k^2 - T\mathbb{B}_Y} f\big(\mathbb{B}(\bar{x}, 3/k)\big) \subset \nabla_{3/k - T\mathbb{B}_Y} f\big(\mathbb{B}(\bar{x}, 3/k)\big)$, and hence

$$\begin{aligned} f^\circ(\bar{x}; Ty) &= \lim_{k\to\infty} \frac{1}{t_k}\big(f(x_k + t_k Ty) - f(x_k)\big) \\ &\le \limsup_{k\to\infty} \big(\langle \xi_k, Ty\rangle + t_k\big) \le \lim_{\varepsilon\downarrow 0}\left[\sup\left\langle \nabla_{\varepsilon-T\mathbb{B}_Y} f\big(\mathbb{B}(\bar{x},\varepsilon)\big), Ty\right\rangle\right].\end{aligned}$$

Now, since f is Lipschitzian, and $\overline{TY} = X$, we can replace Ty in the above arguments by any $h \in X$. Finally, it remains to realize that $\nabla_{\varepsilon-T\mathbb{B}_Y} f(\cdot) \subset S_{\varepsilon-T\mathbb{B}_Y} f(\cdot)$, and thus the first chain of equalities is proved.

The equivalence of the second chain of equalities with the first one can be proved by a standard “Hahn-Banach" argument and is left to the reader. □

Remarks. 1. Assume for simplicity that X is Asplund. Let $f'(x)$ denote the Fréchet derivative of f at x, if it exists, and otherwise we put $f'(x) = \emptyset$. Then Preiss' result says that $\bigcap_{\varepsilon>0} \overline{\mathrm{co}}^* f'(B(\bar{x},\varepsilon)) = \partial f(\bar{x})$ [19]. From this, we get of course, Theorem 2.3. The point with our proof was that we got it with less effort.

2. We deduced Theorem 2.3 from Theorem 2.2. The reverse process is also possible (and easy).

3. By (1), in Theorem 2.3 one may substitute $S_{\varepsilon-T\mathbb{B}_Y}$ by $\hat{\partial}_{\varepsilon-T\mathbb{B}_Y}$.

3. Intermediate Differentiability on Asplund Generated Spaces

In this section, we show that the intermediate differentiability results on Asplund generated spaces proved by Giles and Sciffer [13, 12] can be also obtained by Proposition 2.1 instead of Preiss' differentiability theorem.

A Lipschitz function f on an open subset G of an Asplund generated space $X = \overline{TY}$ is said to be *uniformly* $T\mathbb{B}_Y-$*intermediately differentiable* at $x \in G$ if there exist $x^* \in X^*$ and a sequence $\lambda_n \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{f(x+\lambda_n Ty) - f(x)}{\lambda_n} = \langle x^*, Ty\rangle \qquad \text{for every } y \in \mathbb{B}_Y.$$

We notice here that the same sequence (λ_n) is used for all $y \in \mathbb{B}_Y$, but the rate of convergence to the limit $\langle x^*, Ty\rangle$ needs not be uniform.

Theorem 3.1. *A Lipschitz function f on a nonempty open subset G of an Asplund generated space $X = \overline{TY}$ is uniformly $T\mathbb{B}_Y-$intermediately differentiable at the points of a dense G_δ subset of G.*

Proof. We may and do assume that the Lipschitz constant of f is 1. For $n = 1, 2, \dots$ define

$$\Omega_n = \left\{ x \in G;\ \sup_{y\in\mathbb{B}_Y} \left| \frac{f(x+\lambda Ty) - f(x)}{\lambda} - \langle x^*, Ty\rangle \right| < \frac{1}{n} \right.$$
$$\left. \text{for some } \lambda \in (0, \tfrac{1}{n}) \text{ and } x^* \in \mathbb{B}_{X^*} \right\}.$$

Since f is Lipschitz, $\Omega_{\frac{1}{n}}$ is open. By Theorem 2.3, the set of points at which f is $\frac{1}{n} - T\mathbb{B}_Y-$Fréchet differentiable is dense in G. So Ω_n is dense in G. Now let $\Omega = \bigcap_{n=1}^\infty \Omega_n$. Then Ω is a dense G_δ set in G. Fix any $x \in \Omega$. For $n = 1, 2, \dots$ we find $0 < \lambda_n < \frac{1}{n}$ and $x_n^* \in \mathbb{B}_{X^*}$ such that

$$\sup_{y\in\mathbb{B}_Y} \left| \frac{f(x+\lambda_n Ty) - f(x)}{\lambda_n} - \langle x_n^*, Ty\rangle \right| < \frac{1}{n}.$$

Since X is Asplund generated, its dual ball is weak* sequentially compact [7, Theorems 1.3.3, 2.1.2]. Hence we may take a subsequence of (x_n^*), weak* converging to x^*. Without loss of any generality, we may assume that $x_n^* \xrightarrow{w^*} x^*$. Then for any fixed $y \in \mathbb{B}_Y$ we have

$$\begin{aligned} \left| \frac{f(x+\lambda_n Ty) - f(x)}{\lambda_n} - \langle x^*, Ty\rangle \right| &\le \left| \frac{f(x+\lambda_n Ty) - f(x)}{\lambda_n} - \langle x_n^*, Ty\rangle \right| + |\langle x_n^* - x^*, Ty\rangle| \\ &< \frac{1}{n} + |\langle x_n^* - x^*, Ty\rangle| \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

that is, $\lim_{n\to\infty} \frac{1}{\lambda_n}\big(f(x+\lambda_n Ty) - f(x)\big) = \langle x^*, Ty\rangle$ for every $y \in \mathbb{B}_Y$. □

Remarks. Theorem 3.1 covers that of Giles & Sciffer [12] who considered X an Asplund space.

A Lipschitz function f on a nonempty open subset G of an Asplund generated space $X = \overline{TY}$ is said to be $T\mathbb{B}_Y-$*Fréchet intermediately differentiable* at $x \in G$ if there exist $x^* \in X^*$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|\frac{f(x+tTy)-f(x)}{t} - \langle x^*, Ty\rangle\right| < \varepsilon \quad \text{whenever } \frac{\delta}{2} < t < \delta \text{ and } y \in \mathbb{B}_Y.$$

A set-valued mapping Φ from a nonempty open subset G of a Banach space X into X^* is called *quasi weak* lower semicontinuous* at $x_0 \in G$ if for every weak* open subset W in X^* such that $W \cap \Phi(x_0) \neq \emptyset$ there exists a non-empty open subset U of G such that $x_0 \in \overline{U}$ and $\Phi(x) \cap W \neq \emptyset$ for each $x \in U$.

Armed with Proposition 2.1 and [16, Lemma 6], we now can prove the following $T\mathbb{B}_Y-$Fréchet intermediate differentiability result given by Giles & Sciffer in [13].

Theorem 3.2 (Giles & Sciffer [13]). *Let f be a Lipschitz function on a non-empty open subset G of an Asplund generated space $X = \overline{TY}$, and assume that its Clarke subdifferential ∂f is quasi weak* lower semicontinuous. Then f is $T\mathbb{B}_Y-$Fréchet intermediately differentiable at the points of a residual subset of G.*

Proof. We may assume that f has Lipschitz constant 1. We shall play a Banach-Mazur game between two players $\mathcal{A}$ and $\mathcal{B}$ on G. A play is a decreasing sequence $G \supset U_1 \supset V_1 \supset U_2 \supset V_2 \supset \ldots$ of non-empty open subsets of G, chosen alternatively by players $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$'s first move U_1. We shall show that, no matter how the player $\mathcal{A}$ moves, the player $\mathcal{B}$ can play with a strategy to ensure that $\bigcap_{n=1}^{\infty} V_n$ consists only of points where f is $T\mathbb{B}_Y-$Fréchet intermediately differentiable. By the Banach-Mazur Theorem [7, Lemma 4.2.1] this will then imply that the set of points where f is $T\mathbb{B}_Y-$Fréchet intermediately differentiable is residual in G.

Assume that player $\mathcal{A}$'s first move was $U_1 \subset G$. From the weak* dentability of Y we find $y_1 \in Y$ and $\alpha_1 \in \mathbb{R}$ such that the weak* slice $S(T^*(\partial f(U_1)), y_1, \alpha_1)$ is nonempty and has (norm)-diameter < 1. Then $S(\partial f(U_1), Ty_1, \alpha_1) \neq \emptyset$ and

$$T\mathbb{B}_Y - \operatorname{diam} S(\partial f(U_1), Ty_1, \alpha_1) = \operatorname{diam} S(T^*(\partial f(U_1)), y_1, \alpha_1) < 1.$$

By Proposition 2.1 we find $x_1 \in U_1,\ x_1^* \in \partial f(x_1) \cap S(\partial f(U_1), Ty_1, \alpha_1)$, and $\delta_1 > 0$ so that

$$\left|f(x_1 + \lambda Ty) - f(x_1) - \langle x_1^*, \lambda Ty\rangle\right| \leq 3\lambda \quad \text{whenever } 0 < \lambda < \delta_1 \text{ and } y \in \mathbb{B}_Y.$$

Observing that $x_1^* \in \partial f(x_1) \cap \left\{x^* \in X^*;\ \langle x^*, Ty_1\rangle > \alpha_1\right\}$, we find, from the quasi weak* lower semicontinuity of ∂f, an open set $V_1 \subset \overline{V_1} \subset U_1$, with $\operatorname{diam} V_1 < \delta_1$, such that $x_1 \in \overline{V_1}$ and $\partial f(x) \cap \left\{x^* \in X^*;\ \langle x^*, Ty_1\rangle > \alpha_1\right\} \neq \emptyset$ for every $x \in V_1$. This V_1 will be player $\mathcal{B}$'s answer to player $\mathcal{A}$'s move U_1.

Let $n \in \{2, 3, \ldots\}$ be fixed and assume that the moves $U_1 \supset V_1 \supset U_2 \supset V_2 \supset \ldots \supset V_{n-1}$ have already been done, and the strategy of player $\mathcal{B}$ was such that $x_i \in \overline{V_i},\ \operatorname{diam} V_i < \frac{1}{i}\delta_i$, and $\partial f(x) \cap \left\{x^* \in X^*;\ \langle x^*, Ty_i\rangle > \alpha_i\right\} \neq \emptyset$ for every $x \in V_i$, where $i = 1, 2 \ldots, n-1$. Let U_n be $\mathcal{A}$'s answer to $\mathcal{B}$'s move V_{n-1}. Then for sure $S(\partial f(U_n), Ty_{n-1}, \alpha_{n-1}) \neq \emptyset$, and hence $S(T^*(\partial f(U_n)), y_{n-1}, \alpha_{n-1}) \neq \emptyset$. By [16, Lemma 6] there are $y_n \in Y$ and $\alpha_n \in \mathbb{R}$ such that

$$\emptyset \neq S(T^*(\partial f(U_n)), y_n, \alpha_n) \subset S(T^*(\partial f(U_n)), y_{n-1}, \alpha_{n-1})$$

and $\operatorname{diam} S(T^*(\partial f(U_n)), y_n, \alpha_n) < \frac{1}{n}$. Then

$$T\mathbb{B}_Y - \operatorname{diam} S(\partial f(U_n), Ty_n, \alpha_n) = \operatorname{diam} S(T^*(\partial f(U_n)), y_n, \alpha_n) < \frac{1}{n}.$$

Proposition 2.1 yields $x_n \in U_n,\ x_n^* \in \partial f(x_n) \cap S(\partial f(U_n), Ty_n, \alpha_n)$, and $\delta_n > 0$ such that

$$\left| f(x_n + \lambda Ty) - f(x_n) - \langle x_n^*, \lambda Ty\rangle \right| \le \frac{3}{n}\lambda \quad \text{whenever } 0 < \lambda < \delta_n \text{ and } y \in \mathbb{B}_Y.$$

The quasi weak* lower semicontinuity of ∂f gives an open set $V_n \subset \overline{V_n} \subset U_n$, with $\operatorname{diam} V_n < \frac{1}{n}\delta_n$, such that $x_n \in \overline{V_n}$ and $\partial f(x) \cap \{x^* \in X^*;\ \langle x^*, Ty_n\rangle > \alpha_n\} \neq \emptyset$ for every $x \in V_n$. This V_n will be player $\mathcal{B}$'s answer to player $\mathcal{A}$'s move U_n.

The strategy of player $\mathcal{B}$ is so defined in each step. It remains to show that

$$\bigcap_{n=1}^{\infty} V_n \subset \{x \in G;\ f \text{ is } T\mathbb{B}_Y - \text{Fréchet intermediately differentiable}\} =: \Omega.$$

So take any $\bar{x} \in \bigcap_{n=1}^{\infty} V_n$. (Actually, this intersection is a singleton.) Since

$$x_n^* \in S(\partial f(U_n), Ty_n, \alpha_n) \subset S(\partial f(U_n), Ty_{n-1}, \alpha_{n-1}) \subset S(\partial f(U_{n-1}, Ty_{n-1}, \alpha_{n-1}) \ni x_{n-1}^*$$

for all $n = 2, 3, \ldots$, the sequence (x_n^*) is Cauchy in the metric of uniform convergence on $T\mathbb{B}_Y$. It is also norm-bounded; let x^* be its weak* cluster point. Thus we get that $\|T^*x_n^* - T^*x^*\| \to 0$ as $n \to \infty$. Now fix any $\varepsilon > 0$. Find a positive integer n so big that $\frac{4}{n} < \frac{\varepsilon}{3}$ and $\|T^*x_n^* - T^*x^*\| < \frac{\varepsilon}{3}$. Then for all $\frac{\delta_n}{2} < \lambda < \delta_n$ and all $y \in \mathbb{B}_Y$ we have

$$\begin{aligned}
& |f(\bar{x} + \lambda Ty) - f(\bar{x}) - \langle x^*, \lambda Ty\rangle| \\
\le\ & 2\|x_n - \bar{x}\| + |f(x_n + \lambda Ty) - f(x_n) - \langle x_n^*, \lambda Ty\rangle| + |\langle x_n^* - x^*, \lambda Ty\rangle| \\
<\ & 2 \cdot \frac{\delta_n}{n} + \frac{3}{n}\lambda + \frac{\varepsilon}{3}\lambda < \varepsilon\lambda.
\end{aligned}$$

Therefore $\bar{x} \in \Omega$. □

Note that our proofs exactly follow Giles and Sciffer's ones [12, 13]. The only difference is that we use Proposition 2.1 instead of Preiss' differentiability theorem.

4. Lipschitz Quotient Mappings

Let X and Y be Banach spaces. A mapping $\Phi : X \to Y$ is called a *Lipschitz quotient mapping* if it is Lipschitz and there exists a constant $C > 0$ such that $\Phi(\mathbb{B}(x, \varepsilon)) \supset \mathbb{B}(\Phi(x), \varepsilon/C)$ for every $x \in X$ and every $\varepsilon > 0$. When there is a Lipschitz quotient mapping from X onto Y, we say that Y is a *Lipschitz quotient* of X. The following result by Benyamini and Lindenstrass [1, page 274] is proved by Preiss' differentiability theorem. It is used to prove that a Lipschitz quotient of an Asplund space is also Asplund. Here we show that the same proof works if we replace Preiss' theorem by Proposition 2.1, see also [1, page 279].

Theorem 4.1 (Benyamini & Lindenstrauss). *Let X be a Banach space with a separable dual, and assume that the Banach space Y is a Lipschitz quotient of X. Then Y^* is separable.*

Proof. Let $\Phi : X \to Y$ be a Lipschitz quotient mapping. Of course, we may assume that Φ is 1−Lipschitz. Find then $C > 0$ so that $\Phi(\mathbb{B}(x,\varepsilon)) \supset \mathbb{B}(\Phi(x),\varepsilon/C)$ for every $x \in X$ and $\varepsilon > 0$. Assume, by contradiction, that Y^* is not separable. Then there is an uncountable set $\{y^*_\gamma\}_{\gamma\in\Gamma}$ in $\mathbb{B}_{Y^*}$ such that $\|y^*_\gamma - y^*_\beta\| > 1/2$ whenever $\gamma,\beta \in \Gamma$ and $\gamma \neq \beta$. Put $\varepsilon = 1/(10C)$. For $\gamma \in \Gamma$ we define a function $f_\gamma : X \to \mathbb{R}$ by $f_\gamma(x) = \big\langle y^*_\gamma, \Phi(x)\big\rangle$, $x \in X$; it is 1-Lipschitz. Since X is Asplund, for every $\gamma \in \Gamma$ Proposition 2.1 yields $x_\gamma \in X$, $x^*_\gamma \in X^*$, and $\delta_\gamma > 0$ such that

$$\big|f_\gamma(x_\gamma + h) - f_\gamma(x_\gamma) - \langle x^*_\gamma, h\rangle\big| \leq \varepsilon\|h\| \quad \text{whenever } h \in X \text{ and } \|h\| < \delta_\gamma.$$

By passing to an uncountable subset of Γ – denoted again by Γ – we can find $\delta > 0$ such that $\delta_\gamma > \delta$ for every $\gamma \in \Gamma$. Since X, X^*, and $\mathbb{R}$ are all separable, by further diminishing Γ, we may and do assume that Γ is still uncountable and the inequalities

$$\|x_\gamma - x_\beta\| < \varepsilon\delta, \quad |f_\gamma(x_\gamma) - f_\beta(x_\beta)| < \varepsilon\delta, \quad \|x^*_\gamma - x^*_\beta\| < \varepsilon$$

hold for all $\gamma, \beta \in \Gamma$. From now on fix two distinct $\gamma, \beta \in \Gamma$. It follows that for all $h \in X$, with $\|h\| < \delta$, we can estimate

$$\begin{aligned}
& |f_\gamma(x_\gamma + h) - f_\beta(x_\gamma + h)| \\
\leq\ & |f_\gamma(x_\gamma + h) - f_\gamma(x_\gamma) - \langle x^*_\gamma, h\rangle| + |f_\beta(x_\beta + h) - f_\beta(x_\beta) - \langle x^*_\beta, h\rangle| \\
& \quad + |f_\gamma(x_\gamma) - f_\beta(x_\beta)| + |\langle x^*_\gamma - x^*_\beta, h\rangle| + |f_\beta(x_\gamma + h) - f_\beta(x_\beta + h)| < 5\varepsilon\delta = \delta/(2C).
\end{aligned}$$

On the other hand, $\Phi(\mathbb{B}(x_\gamma,\delta)) \supset \mathbb{B}(\Phi(x_\gamma),\delta/C)$ and $\|y^*_\gamma - y^*_\beta\| > 1/2$. Therefore we have

$$\begin{aligned}
& \sup\{|f_\gamma(x_\gamma + h) - f_\beta(x_\gamma + h)|;\ h \in X,\ \|h\| < \delta\} \\
=\ & \sup\{|\langle y^*_\gamma - y^*_\beta, \Phi(x_\gamma + h)\rangle|;\ h \in X,\ \|h\| < \delta\} \\
\geq\ & \sup\{|\langle y^*_\gamma - y^*_\beta, \Phi(x_\gamma) + y\rangle|;\ y \in Y,\ \|y\| < \tfrac{\delta}{C}\} \\
=\ & \big|\big\langle y^*_\gamma - y^*_\beta, \Phi(x_\gamma)\big\rangle\big| + \tfrac{\delta}{C}\|y^*_\gamma - y^*_\beta\| \geq \tfrac{\delta}{C}\|y^*_\gamma - y^*_\beta\| > \tfrac{\delta}{2C},
\end{aligned}$$

which contradicts the previous inequality. □

5. Nearest Points to Boundedly Asplund Sets

In this section, we prove that whenever X is a strictly convex Banach space and $M \subset X$ is a boundedly Asplund set, each point of a dense G_δ subset of $X\backslash M$ has at most one nearest point in M. This generalizes a result due to Borwein and Fitzpatrick on the existence of nearest points to closed boundedly weakly compact subsets of Banach spaces [2]. A subset M of a Banach space X is called *Asplund* if M is bounded in norm, and for every countable subset $A \subset M$, the dual X^* is separable in the topology of uniform convergence on A. The family of Asplund sets in X is closed under taking subsets, closure, finite unions, finite linear combinations, and continuous linear images [7, Section 1.4]. Bounded sets of Asplund spaces and weakly compact sets of any Banach space are Asplund sets. We say that $M \subset X$ is *boundedly Asplund* if for every $r > 0$, the set $M \cap \mathbb{B}(0,r)$ is an Asplund set.

The distance function d_M for a set $M \subset X$ is defined by

$$d_M(x) = \inf_{z\in M} \|x - z\| \quad \text{for } x \in X.$$

Fix $x \in X \backslash M$. A point $z \in M$ (if any) satisfying $\|x - z\| = d_M(x)$ is called a *nearest point* in M to x. A sequence (z_n) of elements in M is called a *minimizing sequence* for M and x if

$$\lim_{n\to\infty} \|x - z_n\| = d_M(x).$$

Modifying arguments given by Borwein and Fitzpatrick [2, pages 712, 713], we have:

Theorem 5.1. *Let X be a strictly convex Banach space, and let $M \subset X$ be a non-empty, boundedly Asplund, closed set, that is $r\mathbb{B}_X \cap M$ is Asplund for every $r > 0$. Then each point of a dense G_δ subset of $X \setminus M$ has at most one nearest point.*

Proof. For $\varepsilon > 0$ we put

$$\Omega_\varepsilon := \big\{x \in X \setminus M;\ \exists x^* \in \mathbb{B}_{X^*}\ \exists \delta > 0 \text{ such that}$$
$$\inf\{\langle x^*, x - z\rangle;\ z \in M \cap \mathbb{B}(x, d_M(x) + \delta)\} > d_M(x) - \varepsilon\big\}.$$

We **claim** that Ω_ε is dense in $X \setminus M$. To prove this, let $x_0 \in X \setminus M$ and $d_M(x_0) > \varepsilon > 0$. Consider the (Asplund) set

$$K = (M \cap \mathbb{B}(0, N)) \cup \{x_0\},$$

in X, with $N = d_M(x_0) + \|x_0\| + 2\varepsilon$. Let Z denote the closed linear span of K. By an interpolation result due to Stegall [7, Theorem 1.4.4], we find an Asplund space Y a linear bounded mapping $T : Y \to Z$ such that $T\mathbb{B}_Y \supset K$ and $\overline{TY} = Z$. Since the function d_M is 1−Lipschitz, by Theorem 2.3, the set of points at which d_M is $\varepsilon/2 - T\mathbb{B}_Y$−Fréchet differentiable is dense in Z. Since $T\mathbb{B}_Y \supset K$, we may select $y_0 \in \mathbb{B}_Y$ such that $z_0 = Ty_0$ satisfies $\|z_0 - x_0\| < \varepsilon$, and for some $z_0^* \in \mathbb{B}_{Z^*}$ and some $\delta > 0$ we have

$$-\frac{\varepsilon}{2} \le \frac{d_M(z_0 + tTy) - d_M(z_0)}{t} - \langle z_0^*, Ty\rangle \le \frac{\varepsilon}{2}$$

whenever $0 < t < \delta$ and $y \in \mathbb{B}_Y$. By the Hahn-Banach theorem, we extend z_0^* to $z_0^* \in \mathbb{B}_{X^*}$. Now let (z_n) be a minimizing sequence in M for z_0. Since $\|z_n - z_0\| \to d_M(z_0)$, and

$$\|z_n\| \le \|z_n - z_0\| + \|z_0 - x_0\| + \|x_0\| \le \|z_n - z_0\| + \varepsilon + \|x_0\|,$$

for n sufficiently large we have $\|z_n\| < d_M(z_0) + \|x_0\| + 2\varepsilon$, and so $z_n \in K$. Now for $0 < t < 1$ we have

$$\begin{aligned} d_M(z_0 + t(z_n - z_0)) - d_M(z_0) &\le \|z_0 + t(z_n - z_0) - z_n\| - d_M(z_0) \\ &= (1-t)\|z_n - z_0\| - d_M(z_0) \\ &= -t\|z_n - z_0\| + [\|z_n - z_0\| - d_M(z_0)]. \end{aligned}$$

Let $t_n = [\|z_n - z_0\| - d_M(z_0) + 1/n]^{1/2}$. Since $2T\mathbb{B}_Y \supset K - z_0$, for n sufficiently large we have

$$-\frac{\varepsilon}{2} \le \frac{d_M(z_0 + t_n(z_n - z_0)) - d_M(z_0)}{t_n} - \langle z_0^*, z_n - z_0\rangle \le \frac{\varepsilon}{2}.$$

Then

$$\begin{aligned} -\frac{\varepsilon}{2} &\le \frac{d_M(z_0 + t_n(z_n - z_0)) - d_M(z_0)}{t_n} - \langle z_0^*, z_n - z_0\rangle \\ &\le (-\|z_n - z_0\| - \langle z_0^*, z_n - z_0\rangle) + t_n. \end{aligned}$$

We have that

$$\liminf_{n\to\infty}\left(\langle z_0^*, z_0 - z_n\rangle - \|z_n - z_0\|\right) \geq -\frac{\varepsilon}{2}.$$

Thus

$$\liminf_{n\to\infty}\langle z_0^*, z_0 - z_n\rangle \geq \lim_{n\to\infty}\|z_n - z_0\| - \frac{\varepsilon}{2} = d_M(z_0) - \frac{\varepsilon}{2}.$$

Since this holds for every minimizing sequence (z_n) in M for z_0, we conclude that there exists $\delta > 0$ such that

$$\langle z_0^*, z_0 - z\rangle > d_M(z_0) - \varepsilon, \quad \text{whenever } z \in M \cap \mathbb{B}(z_0, d_M(z_0) + \delta).$$

Because $\|z_0 - x_0\| < \varepsilon$ this establishes our density claim.

Further, we **claim** that for every $\varepsilon > 0$ the set Ω_ε is open. To prove this, fix any $x \in \Omega_\varepsilon$. Then there exists $x^* \in \mathbb{B}_{X^*}$ and $\delta > 0$ so that

$$\beta := \inf\{\langle x^*, x - z\rangle \mid z \in M \cap \mathbb{B}(x, d_M(x) + \delta)\} - (d_M(x) - \varepsilon) > 0.$$

Choose $0 < \delta' < \min\{\delta/3, \beta/2\}$. Whenever $\|y - x\| < \delta'$, we have

$$\mathbb{B}(y, d_M(y) + \delta') \subset \mathbb{B}(x, d_M(x) + \delta).$$

Indeed, fix one $y \in Y$ satisfying $\|z - y\| \leq d_M(y) + \delta'$. Then

$$\begin{aligned}\|z - x\| &\leq \|z - y\| + \|y - x\| \leq d_M(y) + \delta' + \|y - x\| \\ &\leq d_M(x) + \delta' + 2\|y - x\| < d_M(x) + 3\delta' < d_M(x) + \delta.\end{aligned}$$

Moreover, for $z \in M$ with $\|z - y\| \leq d_M(y) + \delta'$, we have

$$\begin{aligned}\langle x^*, y - z\rangle &= \langle x^*, y - x\rangle + \langle x^*, x - z\rangle \\ &\geq -\|y - x\| + d_M(x) - \varepsilon + \beta \geq -\|y - x\| + d_M(y) - \|y - x\| - \varepsilon + \beta \\ &= d_M(y) - \varepsilon + (\beta - 2\|y - x\|) > d_M(y) - \varepsilon + (\beta - 2\delta') > d_M(y) - \varepsilon.\end{aligned}$$

Therefore $\mathbb{B}(x, \delta')$ lies in Ω_ε. We proved that Ω_ε is open.

Now, put

$$\Omega(M) = \bigcap_{n=1}^{\infty} \Omega_{1/n};$$

this is a dense G_δ subset of $X \backslash M$. Fix any $x \in \Omega(M)$. It remains to show that it does not have two different nearest points. So assume there are $u, v \in M$ with $\|u - x\| = \|v - x\| = d_M(x) > 0$. By the above claims for $n = 1, 2, \ldots$ we find $x_n^* \in \mathbb{B}_{X^*}$ such that

$$\langle x_n^*, x - u\rangle \geq d_M(x) - 1/n, \quad \text{and} \quad \langle x_n^*, x - v\rangle \geq d_M(x) - 1/n,$$

and $\langle x_n^*, (x - u) + (x - v)\rangle \geq 2d_M(x) - 2/n$. When $n \to \infty$, we have

$$\|(x - u) + (x - v)\| \geq 2d_M(x) = \|x - u\| + \|x - v\|.$$

By strict convexity, we then get $u = v$. □

Remarks. 1. It should be noted that, in the above argument, we actually needed less than Theorem 2.3: It was enough to know that $S_{\frac{\varepsilon}{2}-T\mathbb{B}_Y} d_M(z_0) \neq \emptyset$.

2. Theorem 5.1 covers that of Fabian & Zhivkov [11] who considered X an Asplund space and also that of Borwein & Fitzpatrick [2] who had M a boundedly relatively weakly compact set, that is $r\mathbb{B}_X \cap M$ has a weakly compact closure for each $r > 0$.

Let C be a closed bounded convex subset of a Banach space X with $0 \in \operatorname{int} C$. The *Minkowski functional* $p_C : X \to [0, +\infty)$ of the set C is defined by

$$p_C(x) = \inf\{\alpha > 0;\ x \in \alpha C\} \quad \text{for } x \in X.$$

For a closed subset M of X, the distance function is defined by

$$d_M(x) = \inf_{z \in M} p_C(x - z) \quad \text{for } x \in X.$$

In [14], Li considered the minimization problem, denoted by $\min_C(x, M)$, which consists in finding a point $\bar{z} \in M$ such that

$$p_C(x - \bar{z}) = d_M(x).$$

This $\bar{z}$ is called a *solution* to the problem. Any sequence (z_n) from M satisfying $\lim_{n\to\infty} p_C(x-z_n) = d_M(x)$ is called a *minimizing sequence* of the problem $\min_C(x, M)$. The problem $\min_C(x, M)$ is said to be *well-posed* if it has a unique solution $\bar{z} \in M$, and every minimizing sequence (z_n) converges to $\bar{z}$ in norm.

Put

$$\mathbb{B}_C(x, r) = \{z \in X;\ p_C(x - z) < r\} \quad \text{for } x \in X \text{ and } r > 0,$$
$$q_C(x^*) = \sup_{x \in C} \langle x^*, x\rangle \quad \text{for } x^* \in X^*,$$

and for $\varepsilon > 0$

$$\Omega_\varepsilon = \big\{x \in X \setminus M;\ \exists x^* \in X^* \text{ with } q_C(x^*) \le 1,\ \delta > 0 \text{ such that}$$
$$\inf\{\langle x^*, x - z\rangle;\ z \in M \cap \mathbb{B}_C(x, d_M(x) + \delta)\} > d_M(x) - \varepsilon\big\}.$$

By repeating exactly the proof of Theorem 5.1 we get

Theorem 5.2. *Let C be a strictly convex set in a Banach space X, with $0 \in \operatorname{int} C$, and let M be a nonempty, boundedly Asplund, closed subset of X. Then there exists a dense G_δ subset $\Omega(M)$ of $X \setminus M$ such that $\min_C(x, M)$ has at most one solution for every $x \in \Omega(M)$.*

We say that $C \subset X$ is *sequentially Kadec* provided that for each sequence (x_n) in X which converges weakly to x_0 and satisfies $\lim_{n\to\infty} p_C(x_n) = p_C(x_0)$ we have

$$\lim_{n\to\infty} \|x_n - x_0\| = 0.$$

Theorem 5.2 allows us to recover:

Corollary 5.3 (Li [14]). *Suppose that $C \subset X$ is both strictly convex and sequentially Kadec, and $0 \in \operatorname{int} C$. Let M be a closed, boundedly relatively weakly compact, non-empty subset of X. Then the set of points $x \in X \setminus M$ such that $\min_C(x, M)$ is well-posed contains a dense G_δ subset of $X \setminus M$.*

Proof. By Theorem 5.2, it suffices to show that $\min_C(x, M)$ is well-posed for each $x \in \Omega(M)$. If $x \in \Omega(M)$, then for every m there exists $\delta_m > 0$ and $x_m^* \in X^*$ with $q_C(x_m^*) \leq 1$ such that

$$\inf\{\langle x_m^*, x - z\rangle;\ z \in M \cap \mathbb{B}_C(x, d_M(x) + \delta_m)\} > d_M(x) - \frac{1}{m}.$$

Let (z_n) be any minimizing sequence in M for x. By extracting a subsequence if necessary, we may assume that $\lim_{n\to\infty} z_n = z$ weakly, and that

$$p_C(x - z_n) \leq d_M(x) + \delta_m \quad \text{if } n \geq m.$$

Then for every $m = 1, 2, \ldots$

$$\langle x_m^*, x - z_n\rangle > d_M(x) - \frac{1}{m} \quad \text{if } n \geq m,$$

so that

$$\langle x_m^*, x - z\rangle \geq d_M(x) - \frac{1}{m}.$$

Hence we have

$$p_C(x - z) \geq \limsup_{m\to\infty} \langle x_m^*, x - z\rangle \geq d_M(x).$$

By the weak lower semicontinuity of p_C,

$$\lim_{n\to\infty} p_C(x - z_n) \geq p_C(x - z).$$

It follows that

$$p_C(x - z) = \lim_{n\to\infty} p_C(x - z_n) = d_M(x).$$

Since $x - z_n$ converges weakly to $x - z$, we may deduce from the sequential Kadec property of C that z_n converges in norm to z; which must then lie in M since M is norm closed. Thus z is a solution to $\min_C(x, M)$, and the only solution by Theorem 5.2. Since (z_n) was any minimizing sequence of $\min_C(x, M)$, we conclude that $\min_C(x, M)$ is well-posed. $\square$

It should be noted that in [14] Li proved Corollary 5.3 by Preiss' differentiability theorem.

6. Separable Reduction for $\varepsilon-$Fréchet Differentiability

Preiss showed that the Fréchet differentiability is separably determined [18]. In [11], it was showed that the local $\varepsilon-$Fréchet subdifferentiability ($\varepsilon > 0$) is separably determined. In this section, we shall show that the $\varepsilon-$Fréchet differentiability is separably determined. We use $f|_M$ to denote the restriction of a function f to a set M. We can organize the above fact into the following statement.

Theorem 6.1. *Assume that X is a Banach space and $f, g : X \to \mathbb{R}$ are locally bounded from below functions, defined on an open set $\emptyset \neq G \subset X$. Then for every separable subspace $V \subset X$, there exists a separable subspace $W \subset X$ containing V such that*

(i) *For every $\varepsilon > 0$, whenever both $f|_{G\cap W}$, $g|_{G\cap W}$ are locally $\varepsilon-$supported at $x \in G \cap W$, then both f, g are locally $\varepsilon-$supported at $x \in G$.*

(ii) *For every $\varepsilon > 0$, whenever $f|_{G\cap W}$ is $\varepsilon-$Fréchet differentiable at $x \in G \cap W$, then f is $3\varepsilon-$Fréchet differentiable at $x \in G$.*

(*iii*) *Whenever $f|_{G\cap W}$ is Fréchet differentiable at $x \in G\cap W$, then f is Fréchet differentiable at $x \in G$.*

Proof. (*i*). can be proved by a slight modification of the argument from [11]. Indeed, it is enough to take into account one more parameter – a rational $\varepsilon > 0$. Thus we get a subspace W serving for this case.

(*ii*). We keep the subspace W from (*i*). Consider $x \in G\cap W$ such that $f|_{G\cap W}$ is $\varepsilon-$Fréchet differentiable at x. Then both $f|_{G\cap W}$ and $(-f)|_{G\cap W}$ are locally $\varepsilon-$supported at x. And (*i*) guarantees that both f and $-f$ are locally $\varepsilon-$supported at x. Thus there exist $x_1^*, x_2^* \in X^*$ and $\delta > 0$ such that

$$f(x+h) - f(x) \geq \langle x_1^*, h\rangle - \varepsilon\|h\| \quad \text{and} \quad -f(x+h) + f(x) \geq \langle x_2^*, h\rangle - \varepsilon\|h\|$$

whenever $h \in X$ and $\|h\| < \delta$. Adding these two inequalities we obtain $\langle x_1^* + x_2^*, h\rangle \leq 2\varepsilon\|h\|$ for all $h \in X$ and so $\|x_1^* + x_2^*\| \leq 2\varepsilon$. Then for $h \in X$, with $\|h\| < \delta$, we have

$$\begin{aligned} -f(x+h) + f(x) &\geq \langle x_1^* + x_2^*, h\rangle + \langle -x_1^*, h\rangle - \varepsilon\|h\| \\ &\geq -\langle x_1^*, h\rangle - 3\varepsilon\|h\|. \end{aligned}$$

Thus

$$|f(x+h) - f(x) - \langle x_1^*, h\rangle| \leq 3\varepsilon\|h\|$$

whenever $h \in X$ and $\|h\| < \delta$. Therefore, $x_1^* \in \nabla_{3\varepsilon} f(x)$.

(*iii*). We again keep the subspace W from (*i*). Consider $x \in G\cap W$ such that $f|_{G\cap W}$ is Fréchet differentiable at x. Then for every $n = 1, 2, \ldots$ the function $f_{|G\cap W}$ is $\frac{1}{3n}-$Fréchet differentiable at x, and hence, by (*ii*), the whole f is $\frac{1}{n}-$Fréchet differentiable at x, that is, there are $x_n^* \in X^*$ and $\delta_n > 0$ such that

$$\big|f(x+h) - f(x) - \big\langle x_n^*, h\big\rangle\big| \leq \frac{1}{n}\|h\| \quad \text{whenever } h \in X \text{ and } 0 < \|h\| < \delta_n.$$

Hence, for all natural numbers m, n, if $h \in X$ and $0 < \|h\| < \min\{\delta_m, \delta_n\}$, we get

$$\big|\langle x_n^* - x_m^*, h\rangle\big| \leq \Big(\frac{1}{m} + \frac{1}{n}\Big)\|h\|, \quad \text{and so} \quad \|x_n^* - x_m^*\| \leq \frac{1}{n} + \frac{1}{m}.$$

This shows that $(x_n^*)_{n=1}^\infty$ is a Cauchy sequence in norm. Put $x^* = \lim_{n\to\infty} x_n^*$. Fix any $\varepsilon > 0$. Choose n such that $\frac{1}{n} < \frac{\varepsilon}{2}$ and $\|x_n^* - x^*\| < \frac{\varepsilon}{2}$. Then for all $h \in X$, with $0 < \|h\| < \delta_n$,

$$\begin{aligned} \big|f(x+h) - f(x) - \langle x^*, h\rangle\big| &\leq \big|f(x+h) - f(x) - \langle x_n^*, h\rangle\big| + \big|\langle x_n^* - x^*, h\rangle\big| \\ &< \Big(\frac{1}{n} + \frac{\varepsilon}{2}\Big)\|h\| < \varepsilon\|h\|. \end{aligned}$$

Therefore f is Fréchet differentiable at x. □

The statement (*i*) above does not cover the case $\varepsilon = 0$; this needs some extra care [6].

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