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Variational Methods in Classical Open Mapping Theorems

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Dedicated to the memory of Simon Fitzpatrick.

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We describe some basic facts from the theory of linear error bounds for lower semicontinuous functions on complete metric spaces, relying upon Ekeland's variational principle and on the notion of strong slope. We then show how this variational method yields the classical Banach-Schauder and Lusternik-Graves open mapping theorems.

1. Introduction

The purpose of this paper is to show how the variational approach yields open mapping theorems. We illustrate this fact through two classical results of linear and smooth non-linear analysis, namely the Banach-Schauder open mapping theorem, and the so-called Lusternik-Graves theorem. Generally speaking, this is of course a known fact, and indeed such approach can be seen as the basis for the development, over the last thirty years or so, of the branch of analysis which is now called *nonsmooth analysis*, that was motivated primarily by its applications in optimization theory (see Ioffe [18]).

It is by now widely (if not unanimously) acknowledged, among nonsmooth analysts, that the key tool in the said approach, is the celebrated variational principle of Ekeland [12]. This was probably first put into light by Ioffe in his paper [15], dealing with locally Lipschitz functions defined on a Banach space. This variational approach was also pioneered by Aubin in [1, 2] and by Borwein in [7]. In that line, we showed in our recent paper with Lucchetti [5], that the theory of *metric regularity* (a notion intimately related to that of openness of (set-valued) mappings) can be developed in the framework of (lower semi)continuous functions defined on a complete metric space — the framework of the variational principle itself. This is possible thanks to the notion of *strong slope* introduced by De Giorgi, Marino, and Tosques [9], which indeed provides for the best estimates in such matters, as was then also stressed (and developed) independently by Ioffe in [17, 18]. Thus, this framework appears appropriate for the abstract theory, leaving as corollaries the results involving notions related to (sub)differential calculus in Banach spaces, which of course remain important from the point of view of specific applications.

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In [3, 4], refining the approach of [5], we presented our main abstract results as characterizations, in terms of the strong slope, of so-called *(linear) error bounds* for lower semicontinuous functions on metric spaces. Here, we wish to present a self-contained, clear and succinct overview of the methods and principles involved in this variational approach. For that purpose, in Section 2, we discuss the variational principle, its relationship with the strong slope, and two abstract results that are particular cases of the afore-mentioned ones in [3, 4] (while we still slightly refine the proofs). In Section 3, we show how these two results naturally and directly yield the equality of some "openness constants" associated with linear and nonlinear operators between Banach spaces, and how these results of a *quantitative* type in turn readily yield the Banach-Schauder and Lusternik-Graves open mapping theorems.

2. The method

Let X be a metric space endowed with the metric d, and $f : X \to \mathbb{R}$ be a function. For $x \in X$ and $r > 0$ (resp., $r \geq 0$), we denote by $B_r(x)$ (resp., $\bar{B}_r(x)$) the open (resp., closed) ball of center x and radius r. For $\alpha \in \mathbb{R}$, we set:

$$[f{>}\alpha] := \{x \in X : \ f(x) > \alpha\}, \quad [f{\leq}\alpha] := \{x \in X : \ f(x) \leq \alpha\},$$

and, whenever $[f{\leq}\alpha] \neq \emptyset$, we let $d(x, [f{\leq}\alpha]) := \inf\{d(x, y) : \ y \in [f{\leq}\alpha]\}$, $x \in X$. For $\sigma > 0$, we denote by d_σ the metric on X defined by $(x, y) \mapsto \sigma d(x, y)$ (in particular, $d_1 = d$), and we say that $x \in X$ is a *d_σ-point* of f if

$$f(x) < f(z) + \sigma d(z, x) \quad \text{for all } z \in X\,, z \neq x\,.$$

Defining, for $x \in X$, the set $M_\sigma(x) := \{z \in X : \ f(z) + \sigma d(z, x) \leq f(x)\}$, we thus have:

$$x \text{ is a } d_\sigma\text{-point of } f \iff M_\sigma(x) = \{x\}\,. \tag{1}$$

Moreover, it follows from the triangle inequality that for any $x, y \in X$:

$$y \in M_\sigma(x) \implies M_\sigma(y) \subset M_\sigma(x)\,. \tag{2}$$

If $\sigma = 1$, we simply write $M(x) := M_1(x)$. We recall the following basic result (see Remark 2.3 below for a brief comment).

Theorem 2.1. *The following are equivalent:*

(a) *(X, d) is complete;*

(b) *every continuous and bounded from below function $f : X \to \mathbb{R}$ has a d-point.*

Proof. $(a) \Rightarrow (b)$: Let $x_0 \in X$, and define recursively a sequence $(x_n)_{n\in\mathbb{N}} \in X$ by:

$$x_n \in M(x_{n-1})\,, \qquad f(x_n) \leq \inf_{M(x_{n-1})} f + \frac{1}{n}\,.$$

Then, $M(x_n) \subset M(x_{n-1})$ for each n, so that for all $n, p \in \mathbb{N}$ and all $y \in M(x_{n+p-1}) \subset M(x_{n-1})$, we have:

$$d(y, x_n) \leq f(x_n) - f(y) \leq \frac{1}{n}\,. \tag{3}$$

Letting $y := x_{n+p}$ in these inequalities we see, first, that (x_n) is a Cauchy sequence in (X, d), and then, letting $p \to +\infty$ for fixed n, and since f is continuous, that $x := \lim x_n \in \bigcap_{n\in\mathbb{N}} M(x_n)$. Then, $M(x) \subset \bigcap_{n\in\mathbb{N}} M(x_n)$ according to (2), so that if $y \in M(x)$, (3) again shows that $y = x$. Thus, $M(x) = \{x\}$, that is, x is a d-point of f according to (1).

$(b) \Rightarrow (a)$: Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d). Then, setting $f(x) := 2 \lim d(x, x_n)$ defines a continuous function $f : X \to \mathbb{R}$ such that $f(x_n) \to 0 = \inf_X f$. Let $x \in X$ be a d-point of f, we have:

$$f(x) \leq f(x_n) + d(x, x_n) \quad \text{for every } n \in \mathbb{N}\,,$$

so that, letting $n \to \infty$ yields $2f(x) \leq f(x)$. Thus, $f(x) = 0$, that is, $x = \lim x_n$. □

Corollary 2.2. *The following are equivalent:*

(a) (X, d) *is complete;*

(b) *if a continuous* $f : X \to \mathbb{R}$, $y \in X$, *and* $\sigma, r > 0$ *are such that* $f(y) < \inf_X f + \sigma r$, *then* f *has a* d_σ*-point* $x \in B_r(y) \cap [f \leq f(y)]$.

Proof. $(a) \Rightarrow (b)$: Applying Theorem 2.1 to the restriction of f to the complete metric space $(M_\sigma(y), d_\sigma)$, we find $x \in M_\sigma(y)$ such that

$$z \in M_\sigma(y) \setminus \{x\} \implies z \notin M_\sigma(x)\,.$$

Since $M_\sigma(x) \subset M_\sigma(y)$ (recall (2)), we thus have $M_\sigma(x) = \{x\}$, that is (recall (1)), x is a d_σ-point of f. Moreover, $f(x) + \sigma d(x, y) \leq f(y) < f(x) + \sigma r$, so that $d(x, y) < r$ and $f(x) \leq f(y)$.

$(b) \Rightarrow (a)$ is obvious, since (b) is clearly stronger than assertion (b) of Theorem 2.1. □

Remark 2.3. Clearly, Theorem 2.1 and Corollary 2.2 still hold (with the same proofs) if we replace "continuous" by "lower semicontinuous". Indeed, this observation applies to all the results of this section. The implication $(a) \Rightarrow (b)$ in Theorem 2.1 (or in Corollary 2.2), is Ekeland's variational principle [12, 13]. The reverse implication, yielding a characterization of metric completeness, is due to Weston [23]. □

Definition 2.4. For $f : X \to \mathbb{R}$ and $x \in X$, set:

$$|\nabla f|(x) := \begin{cases} 0 & \text{if } x \text{ is a local minimum point of } f, \\ \displaystyle\limsup_{\substack{y\to x \\ (y\neq x)}} \frac{f(x) - f(y)}{d(x, y)} & \text{otherwise.} \end{cases}$$

The extended real number $|\nabla f|(x) \in [0, +\infty]$ is called the *strong slope* of f at x.

This notion was introduced by De Giorgi, Marino, and Tosques in [9]. Before exhibiting its connection with the variational principle, let us recall the following:

Example 2.5. Let X be a normed space with norm $\|\cdot\|$, U be an open subset of X, and let $f : U \to \mathbb{R}$. For $x, u \in X$, define:

$$f'(x;u) = \liminf_{t \searrow 0} \frac{f(x+tu) - f(x)}{t}$$

(which is called the *lower Dini derivative* of f at x in the direction u). We have:

$$-f'(x;u) \leq \|u\| \cdot |\nabla f|(x) . \tag{4}$$

Indeed, $-f'(x;u) \leq 0$ if either x is a local minimum of f, or if $u = 0$, while, otherwise:

$$-f'(x;u) = \|u\| \limsup_{t \searrow 0} \frac{f(x) - f(x+tu)}{t\|u\|} \leq \|u\| \cdot |\nabla f|(x) .$$

(Observe, for later use, that if $f := \|\cdot\|$, then for every $x \in X$ and every $\alpha \in \mathbb{R}$, we have: $\|\cdot\|'(x; \alpha x) = \alpha\|x\|$ and $\|\cdot\|'(x; \cdot)$ is continuous.)

If f is (Fréchet-)differentiable at $x \in U$, then:

$$|\nabla f|(x) = \|Df(x)\|_* := \sup_{\|u\|=1} |Df(x)(u)| .$$

Indeed, since $Df(x)(u) = f'(x;u)$ for any $u \in X$, inequality (4) yields $\|Df(x)\|_* \leq |\nabla f|(x)$. Conversely, if $|\nabla f|(x) > \sigma > 0$, there exists a sequence (x_n) in U converging to x and such that $f(x) - f(x_n) > \sigma\|x - x_n\|$, so that

$$Df(x)\left(\frac{x - x_n}{\|x - x_n\|}\right) \geq \sigma ,$$

showing that $\|Df(x)\|_* \geq \sigma$, whence $\|Df(x)\|_* \geq |\nabla f|(x)$. □

If $f : X \to \mathbb{R}$ (X an arbitrary metric space), $x \in X$, and $\sigma > 0$, it is readily seen, from the definitions, that:

$$x \text{ is a } d_\sigma\text{-point of } f \implies |\nabla f|(x) \leq \sigma . \tag{5}$$

Thus, we get from Corollary 2.2:

Corollary 2.6. *Let X be complete, $f : X \to \mathbb{R}$ be continuous and bounded from below. If $y \in X$ and $\sigma, r > 0$ are such that $f(y) < \inf_X f + \sigma r$, then, there exists $x \in B_r(y)$ with $f(x) \leq f(y)$ and $|\nabla f|(x) \leq \sigma$. (In particular, $\inf\limits_X |\nabla f| = 0$.)*

Let us also mention the following:

Corollary 2.7. *Let X be complete and $f : X \to \mathbb{R}$ be continuous. Then, the set $\{x \in X : |\nabla f|(x) < +\infty\}$ is dense in X.*

Proof. Let $y \in X$ and $\varepsilon > 0$, and let $0 < \varepsilon' \leq \varepsilon$ be such that f is bounded from below on $\bar{B}_{\varepsilon'}(y)$. Considering any

$$\sigma > \frac{1}{\varepsilon'}\left(f(y) - \inf_{\bar{B}_{\varepsilon'}(y)} f\right),$$

and applying Corollary 2.2 to the restriction $\tilde{f}$ of f to $\bar{B}_{\varepsilon'}(y)$, we find a d_σ-point x of $\tilde{f}$ with $x \in B_{\varepsilon'}(y)$, so that $|\nabla f|(x) = |\nabla \tilde{f}|(x) \leq \sigma$, according to (5). □

We are now ready to state our main abstract results. The first one is of "global" type, the second one is a "local" result.

Theorem 2.8. *Let X be complete, and $f : X \to \mathbb{R}$ be a continuous function. Assume that $[f{>}0] \neq \emptyset$ and that:*

$$\tau := \inf_{[f>0]} |\nabla f| > 0\,. \tag{6}$$

Then, $[f{\le}0] \neq \emptyset$ and

$$f(x) \geq \tau d(x, [f{\le}0]) \quad \text{for all } x \in [f{>}0]\,.$$

Proof. Observe first that $\tau < +\infty$, according to Corollary 2.7. Also, $[f{\le}0] \neq \emptyset$ — for, otherwise, $\inf_{[f>0]} |\nabla f| = \inf_X |\nabla f| = 0$, according to Corollary 2.6. Let now $y \in [f{>}0]$, set $r := d(y, [f{\le}0]) > 0$, and let $\sigma > \dfrac{f(y)}{r}$: we need to show that $\sigma \geq \tau$. For $x \in X$, let $g(x) := \max\{f(x), 0\}$, so that $g(y) < \inf_X g + \sigma r$. Applying Corollary 2.6, we find $x \in B_r(y)$ such that $|\nabla g|(x) \leq \sigma$. Thus, $f(x) > 0$ (since $d(x, y) < r$), so that $|\nabla f|(x) = |\nabla g|(x) \leq \sigma$, whence the conclusion. □

Theorem 2.9. *Let X be complete, $f : X \to \mathbb{R}$ be a continuous function, $x_0 \in X$, and $\tau, \rho > 0$ be such that $f(x_0) < \tau\rho$ and:*

$$[x \in B_{2\rho}(x_0)\,,\ 0 < f(x) < \tau\rho] \implies |\nabla f|(x) \geq \tau\,.$$

Then, $[f{\le}0] \neq \emptyset$, and:

$$f(x) \geq \tau d(x, [f{\le}0]) \quad \text{for all } x \in B_\rho(x_0) \text{ with } 0 < f(x) < \tau\rho\,.$$

Proof. We may assume that there exists some $y \in B_\rho(x_0)$ with $0 < f(y) < \tau\rho$ (otherwise, there is nothing to prove). Let such y be fixed, and set:

$$r := \begin{cases} \rho & \text{if } [f{\le}0] = \emptyset, \\ \min\{\rho, d(y, [f{\le}0])\} & \text{if } [f{\le}0] \neq \emptyset. \end{cases}$$

Let further $\sigma > \dfrac{f(y)}{r}$ be fixed, and set $g := \max\{f, 0\}$, so that $g(y) < \inf_X g + \sigma r$. Applying Corollary 2.6, we find $x \in B_r(y)$ such that $g(x) \leq g(y) < \tau\rho$ and $|\nabla g|(x) \leq \sigma$. By the choice of r, we have $x \in B_{2\rho}(x_0)$ and $f(x) > 0$, whence $f(x) = g(x)$ and $|\nabla f|(x) = |\nabla g|(x)$. Thus, we have $\sigma \geq \tau$, and we deduce that $f(y) \geq \tau r$. We conclude that $r < \rho$, that is, $[f{\le}0] \neq \emptyset$ and $r = d(y, [f{\le}0])$. □

Remark 2.10. According to [3], the following result, sharpening Theorem 2.8, holds: if $f : X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous (f not identically equal to $+\infty$), then:

$$\inf_{[f>0]} |\nabla f| = \inf_{\gamma \geq 0} \inf_{x \in [f>\gamma]} \frac{f(x) - \gamma}{d(x, [f{\le}\gamma])}\,,$$

where the right-hand side is 0 if $[f{\le}0] = \emptyset$, and both sides are $+\infty$ if $[f{>}0] = \emptyset$. See also [4] for extensions, and more on "local" results of the type of Theorem 2.9. □

3. Open mapping theorems

Let X and Y be two Banach spaces: we shall denote either norm by $\|\cdot\|$, and by $\bar{B}_X$ and $\bar{B}_Y$ the respective closed unit balls. Let $A : X \to Y$ be a continuous linear map, and set:

$$\tau_A := \sup\{\tau \geq 0 :\ \tau \bar{B}_Y \subset A(\bar{B}_X)\}\,, \tag{7}$$

$$\bar{\tau}_A := \sup\{\tau \geq 0 :\ \tau \bar{B}_Y \subset \overline{A(\bar{B}_X)}\} \geq \tau_A\,. \tag{8}$$

Clearly, $\tau_A > 0$ if and only if A is *open* (i.e., $A(U)$ is open in Y whenever U is open in X), in which case A is *onto* (i.e., $A(X) = Y$). We may observe that:

$$\tau_A = \inf_{\|y^*\|_*=1} \|A^* y^*\|_*\,, \tag{9}$$

where $A^* : Y^* \to X^*$ is the adjoint operator of A.

We shall show, using Theorem 2.8, that the constants τ_A and $\bar{\tau}_A$ are indeed equal. For this, we introduce another constant, which is precisely connected with the conclusion of Theorem 2.8. We set:

$$\tau'_A := \begin{cases} \sup\{\tau \geq 0 :\ \|Ax - y\| \geq \tau d(x, A^{-1}(y))\ \ \forall x \in X,\ \forall y \in Y\} & \text{if } A \text{ is onto,} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.1. $\tau_A = \tau'_A$.

Proof. Assume (first) that $\tau'_A > 0$. Let $0 < \tau < \tau'_A$, let then $\varepsilon > 0$ be such that $\tau + \varepsilon < \tau'_A$, and let $y \in \tau \bar{B}_Y$. Then:

$$d(0, A^{-1}(y)) \leq \frac{\|y\|}{\tau + \varepsilon} < 1\,,$$

so that $A^{-1}(y) \cap \bar{B}_X \neq \emptyset$. Thus, $\tau_A \geq \tau$, and we conclude that $\tau_A \geq \tau'_A$.

Conversely, assume that $\tau_A > 0$. Let $0 < \tau < \tau_A$, and let $(x, y) \in X \times Y$ be fixed. Then, there exists $x' \in X$ with:

$$\|x'\| \leq \frac{\|Ax - y\|}{\tau} \quad \text{and} \quad Ax' = Ax - y\,,$$

so that $x - x' \in A^{-1}(y)$ and $\tau d(x, A^{-1}(y)) \leq \tau\|x'\| \leq \|Ax - y\|$. Thus, $\tau'_A \geq \tau$, and we conclude that $\tau'_A \geq \tau_A$. □

Theorem 3.2. $\bar{\tau}_A = \tau_A$.

Proof. Taking Proposition 3.1 and (8) into account, we need only show that:

$$\bar{\tau}_A > 0 \implies \tau'_A \geq \bar{\tau}_A\,.$$

Let $0 < \tau < \bar{\tau}_A$, let $y \in Y$ be fixed, and define $f : X \to \mathbb{R}$ by $f(x) := \|Ax - y\|$. Let $x \in X$ with $f(x) > 0$ (that is, $Ax \neq y$), and let (x_n) be a sequence in $\bar{B}_X$ such that

$$z := -\tau \frac{Ax - y}{\|Ax - y\|} = \lim_{n\to\infty} Ax_n\,. \tag{10}$$

Then:

$$f'(x;x_n) = \|\cdot\|'(Ax-y;Ax_n) \to \|\cdot\|'(Ax-y;z) = -\tau\,,$$

while:

$$|\nabla f|(x) \geq \|x_n\| |\nabla f|(x) \geq -f'(x;x_n)$$

for each n, according to (4). Thus, $|\nabla f|(x) \geq \tau$, and it follows from Theorem 2.8 that $[f{\leq}0] = A^{-1}(y) \neq \emptyset$ and

$$\|Ax-y\| \geq \tau d(x, A^{-1}(y)) \quad \text{for all } x \in X$$

(since $f \geq 0$). Thus, since y is arbitrary in Y, $\tau'_A \geq \tau$, and the conclusion follows. □

This result directly yields the Banach-Schauder open mapping theorem (see [6, 22]) through the following additional, standard argument: if A is onto, so that $Y = \overline{\bigcup\{nA(\bar{B}_X) : n \in \mathbb{N}\}}$, it follows from the Baire category theorem that $\bar{\tau}_A > 0$. Thus, Theorem 3.2 allows to conclude that A is open whenever it is onto.

Remark 3.3. Observe that in the preceding proof, we do not actually need to explicitely invoke (4). Namely, from (10) and for $0 < t \leq \frac{\|Ax-y\|}{\tau}$, we directly compute:

$$\frac{f(x+tx_n) - f(x)}{t} \leq \frac{\|Ax-y+tz\| - \|Ax-y\|}{t} + \|Ax_n - z\| = -\tau + \|Ax_n - z\|$$

(which is negative for large n), so that:

$$|\nabla f|(x) \geq \limsup_{t \searrow 0} \frac{f(x) - f(x+tx_n)}{t} \geq \tau - \|Ax_n - z\|$$

for large n, whence $|\nabla f|(x) \geq \tau$. □

Let now U be an open subset of X, and $F : U \to Y$ be a continuous mapping. For a given $y \in Y$, we define a (continuous) function $f_y : U \to \mathbb{R}_+$ by:

$$f_y(x) := \|F(x) - y\|\,. \tag{11}$$

For $x_0 \in U$, we denote by τ_{F,x_0} the supremum of the nonnegative reals τ such that for some neighborhood V_0 of x_0 we have:

$$\bar{B}_{\tau\rho}(F(x)) \subset F(\bar{B}_\rho(x)) \quad \text{for all } x \in V_0 \text{ and all } \rho > 0 \text{ with } \bar{B}_\rho(x) \subset V_0\,. \tag{12}$$

Of course, this is satisfied with $\tau = 0$ and arbitrary V_0 (so that τ_{F,x_0} is well defined), and it is readily seen that for any $x_0 \in U$, we have $\tau_{F,x} \geq \tau_{F,x_0}$ for every $x \in U$ sufficiently close to x_0. If $\tau_{F,x_0} > 0$, F is said to be *open at a linear rate* near the point x_0.

Proposition 3.4. *Assume that F is differentiable at $x_0 \in U$, and let $y \in Y$ with $y \neq F(x_0)$. Then:*

$$|\nabla f_y|(x_0) \geq \tau_{DF(x_0)} \geq \tau_{F,x_0}\,.$$

Proof. Given $\varepsilon > 0$, we let $\rho_\varepsilon > 0$ be such that:

$$\|F(x) - F(x_0) - DF(x_0)(x - x_0)\| \le \varepsilon \|x - x_0\| \quad \text{for all } x \in \bar{B}_{\rho_\varepsilon}(x_0) \subset U\,. \tag{13}$$

We establish the first inequality, where $\tau_{DF(x_0)}$ is defined as in (7), and for which we may assume that $\tau_{DF(x_0)} > 0$. Let $0 < \tau < \tau_{DF(x_0)}$, let then $\varepsilon > 0$ be such that $\tau + \varepsilon < \tau_{DF(x_0)}$, and let $u \in \bar{B}_X$ be such that

$$DF(x_0)(u) = -(\tau + \varepsilon)\frac{F(x_0) - y}{\|F(x_0) - y\|} \in (\tau + \varepsilon)\bar{B}_Y\,,$$

according to the definition of $\tau_{DF(x_0)}$. Then, for $0 < t \le \min\left\{\rho_\varepsilon, \frac{\|F(x_0)-y\|}{\tau+\varepsilon}\right\}$, we have:

$$\begin{aligned}\frac{f_y(x_0 + tu) - f_y(x_0)}{t} &= \frac{\|F(x_0 + tu) - y\| - \|F(x_0) - y\|}{t} \\ &\le \frac{\|F(x_0) - y + tDF(x_0)(u)\| - \|F(x_0) - y\|}{t} + \varepsilon = -\tau\,,\end{aligned}$$

so that:

$$|\nabla f_y|(x_0) \ge \limsup_{t \searrow 0} \frac{f_y(x_0) - f_y(x_0 + tu)}{t} \ge \tau\,,$$

and the conclusion follows.

For the second inequality, we may assume that $\tau_{F,x_0} > 0$, so let $0 < \tau < \tau_{F,x_0}$, and let $\rho_0 > 0$ be such that (12) is satisfied with $V_0 := \bar{B}_{\rho_0}(x_0)$. Let $z \in \tau\bar{B}_Y$, and $\varepsilon > 0$ be fixed. Let then $0 < \rho \le \min\{\rho_0, \rho_\varepsilon\}$. Since $F(x_0) + \rho z \in \bar{B}_{\tau\rho}(F(x_0))$, we find $x \in \bar{B}_\rho(x_0)$ such that $F(x_0) + \rho z = F(x)$, and (13) yields:

$$\|z - DF(x_0)(\tilde{x})\| \le \varepsilon\,,$$

where $\tilde{x} := \dfrac{x - x_0}{\rho} \in \bar{B}_X$. Since $\varepsilon > 0$ is arbitrary, we obtain that $z \in \overline{DF(x_0)(\bar{B}_X)}$, and since z is arbitrary in $\tau\bar{B}_Y$, that $\bar{\tau}_{DF(x_0)} \ge \tau$ (where, of course, $\bar{\tau}_{DF(x_0)}$ is defined as in (8)). Thus, $\bar{\tau}_{DF(x_0)} \ge \tau_{F,x_0}$, and the conclusion follows from Theorem 3.2. □

For $x_0 \in U$ such that $F(U)$ is a neighborhood of $F(x_0)$, we denote by τ'_{F,x_0} the supremum of the nonnegative reals τ such that for some neighborhoods V of x_0 and W of $F(x_0)$, we have:

$$\|F(x) - y\| \ge \tau d(x, F^{-1}(y)) \quad \text{for all } (x, y) \in V \times W \tag{14}$$

(since $F^{-1}(y) \ne \emptyset$ for all y in some neighborhood of $F(x_0)$, τ'_{F,x_0} is well-defined). On the other hand, if $F(U)$ is *not* a neighborhood of $F(x_0)$, we set $\tau'_{F,x_0} := 0$. If $\tau'_{F,x_0} > 0$, F is said to be *metrically regular* near the point x_0. The following proposition is just a nonlinear, local (and, in fact, purely *metric*) version of the afore-mentioned equality $\tau_A = \tau'_A$ (see Remark 3.9 (c) for a comment).

Proposition 3.5. $\tau_{F,x_0} = \tau'_{F,x_0}$.

Proof. Assume (first) that $\tau'_{F,x_0} > 0$. Let $0 < \tau < \tau'_{F,x_0}$, let then $\varepsilon > 0$ be such that $\tau + \varepsilon < \tau'_{F,x_0}$, and let V and W be neighborhoods of x_0 and $F(x_0)$, respectively, such that:

$$\|F(x) - y\| \geq (\tau + \varepsilon) d(x, F^{-1}(y)) \quad \text{for all } (x, y) \in V \times W \,.$$

Let $r > 0$ be such that $B_{(\tau+1)r}(F(x_0)) \subset W$, and let $0 < \rho_0 \leq r$ be such that $B_{\rho_0}(x_0) \subset V$ and $F(B_{\rho_0}(x_0)) \subset \bar{B}_r(F(x_0))$. Then, if $x \in V_0 := B_{\rho_0}(x_0)$, if $0 < \rho < \rho_0 - \|x - x_0\|$, and if $y \in \bar{B}_{\tau\rho}(F(x))$, it is readily verified that $(x, y) \in V \times W$, so that:

$$d(x, F^{-1}(y)) \leq \frac{\|F(x) - y\|}{\tau + \varepsilon} < \rho \,,$$

which shows that $y \in F(\bar{B}_\rho(x))$. Thus, $\tau_{F,x_0} \geq \tau$, and we conclude that $\tau_{F,x_0} \geq \tau'_{F,x_0}$.

Conversely, assume that $\tau_{F,x_0} > 0$. Let $0 < \tau < \tau_{F,x_0}$, and let $\rho_0 > 0$ be such that (12) holds with $V_0 := \bar{B}_{2\rho_0}(x_0)$. Let then V and W be neighborhoods of x_0 and $F(x_0)$, respectively, such that:

$$V \subset \bar{B}_{\rho_0}(x_0) \quad \text{and} \quad \|F(x) - y\| \leq \tau\rho_0 \quad \text{for all } (x, y) \in V \times W \,.$$

Let $(x, y) \in V \times W$ be such that $F(x) \neq y$ and set $\rho := \dfrac{\|F(x) - y\|}{\tau} \in]0, \rho_0]$. Then, $\bar{B}_\rho(x) \subset \bar{B}_{2\rho_0}(x_0)$ and $y \in \bar{B}_{\tau\rho}(F(x))$, so that we find $x' \in \bar{B}_\rho(x)$ such that $y = F(x')$. Thus:

$$\|F(x) - y\| = \tau\rho \geq \tau\|x - x'\| \geq d(x, F^{-1}(y)) \,,$$

which shows that $\tau'_{F,x_0} \geq \tau$, and we conclude that $\tau'_{F,x_0} \geq \tau_{F,x_0}$. □

Recall now that F is said to be *strictly differentiable* at $x_0 \in U$ if F is differentiable at x_0 and if, for every $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ such that $F - DF(x_0)$ is ε-Lipschitzian on $\bar{B}_{\rho_\varepsilon}(x_0)$, that is:

$$\|F(x) - F(x') - DF(x_0)(x - x')\| \leq \varepsilon\|x - x'\| \quad \text{for all } x, x' \in \bar{B}_{\rho_\varepsilon}(x_0) \,. \tag{15}$$

(This is the case if F is differentiable near x_0 and DF is continuous at x_0.)

Theorem 3.6. *Assume that F is strictly differentiable at x_0. Then, $\tau_{DF(x_0)} = \tau_{F,x_0} = \tau'_{F,x_0}$.*

Proof. We need only show, taking Propositions 3.4 and 3.5 into account, that:

$$\tau_{DF(x_0)} > 0 \implies \tau'_{F,x_0} \geq \tau_{DF(x_0)} \,.$$

Let $0 < \tau < \tau_{DF(x_0)}$, let $\varepsilon > 0$ be such that $\tau + \varepsilon < \tau_{DF(x_0)}$, and let $\rho_\varepsilon > 0$ be such that (15) holds. Let $y \in Y$ be fixed, and let $x \in B_{\rho_\varepsilon}(x_0)$ be such that $F(x) \neq y$. Then, arguing exactly as in the first part of the proof of Proposition 3.4, but with x_0 replaced by x, and with $0 < t \leq \min\left\{\rho_\varepsilon - \|x - x_0\|, \frac{F(x) - y}{\tau + \varepsilon}\right\}$, we obtain that $|\nabla f_y|(x) \geq \tau$. Letting $2\rho := \rho_\varepsilon$ thus yields:

$$\inf_{B_{2\rho}(x_0) \cap [f_y > 0]} |\nabla f_y| \geq \tau \,. \tag{16}$$

Consider now a neighborhood V of x_0, and a neighborhood W of $F(x_0)$, such that $V \subset B_\rho(x_0)$ and $f_y(x) < \tau\rho$ for all $(x,y) \in V \times W$. Thanks to (16), we may apply Theorem 2.9 for each $y \in W$ to obtain that $[f_y \le 0] = F^{-1}(y) \neq \emptyset$ and:

$$\|F(x) - y\| \ge \tau d(x, F^{-1}(y)) \quad \text{for all } (x,y) \in V \times W\,.$$

Thus, $\tau'_{F,x_0} \ge \tau$, and the conclusion follows. □

Theorem 3.6 readily yields the so-called Lusternik-Graves theorem: if F is strictly differentiable at x_0 and if $DF(x_0)$ is onto, so that $\tau_{DF(x_0)} > 0$ according to the Banach-Schauder theorem, then F is open at a linear rate near x_0, or, equivalently, F is metrically regular near x_0. (This is indeed a "modern" version of the original results of Lusternik and Graves [19, 14], see Remark 3.9 (a) below.) Observe that, conversely, if F is only assumed to be differentiable at x_0, and is open at a linear rate near x_0, then $DF(x_0)$ is onto, as follows from the second inequality in Proposition 3.4. Of course, Theorem 3.6 shows that if F is strictly differentiable at x_0, then F is open at a linear rate near x_0 if and only if $DF(x_0)$ is onto, which provides a nonlinear (hence local) analogue of the Banach-Schauder theorem. Such a statement was already derived by Dontchev in [10] through a different approach, namely, as a consequence of a stability result of the openness at a linear rate property, for locally closed set-valued maps between Banach spaces.

Remark 3.7. Under the assumption of Theorem 3.6, and as a corollary, it can be established that $DF(x_0)$ is an isomorphism if and only if there exists a neighborhood V of x_0 such that $F(V)$ is a neighborhood of $F(x_0)$, F is injective on V, and both $F : V \to F(V)$ and $F^{-1} : F(V) \to V$ are Lipschitz continuous. □

Before proceeding with a few concluding remarks, let us mention the following local estimate for the openness (or metric regularity) constant of F:

Theorem 3.8. *Assume that F is differentiable in a neighborhood of $x_0 \in U$. Then:*

$$\tau_{F,x_0} = \tau_0 := \liminf_{x \to x_0} \tau_{DF(x)} = \liminf_{x \to x_0} \left(\inf_{\|y^*\|_* = 1} \|DF^*(x)(y^*)\|_* \right).$$

(where $DF^*(x)$ is the adjoint of $DF(x)$).

Proof. As already observed, we have $\tau_{F,x} \ge \tau_{F,x_0}$ for x close enough to x_0, while $\tau_{DF(x)} \ge \tau_{F,x}$ if F is differentiable at x, according to Proposition 3.4. Thus, $\tau_0 \ge \tau_{F,x_0}$. Conversely, we may assume that $\tau_0 > \tau > 0$. Let $\rho > 0$ be such that F is differentiable on $B_{2\rho}(x_0) \subset U$ and $\tau_{DF(x)} \ge \tau$ for every $x \in B_{2\rho}(x_0)$. Thanks to Proposition 3.4, we have $|\nabla f_y|(x) \ge \tau$ for every $x \in B_{2\rho}(x_0)$ and every $y \in Y$ such that $F(x) \neq y$, that is: (16) holds. Then, like in the proof of Theorem 3.6, we obtain that $\tau'_{F,x_0} \ge \tau$, so that $\tau_{F,x_0} = \tau'_{F,x_0} \ge \tau_0$. For the last equality, just recall (9). □

Remark 3.9. (a) Let X and Y be Banach spaces, $A : X \to Y$ be linear and continuous, U be a neighborhood of $x_0 \in X$, $F : U \to Y$ be continuous, and $\delta > 0$ be such that:

$$\|F(x') - F(x) - A(x' - x)\| \le \delta \|x' - x\| \quad \text{for every } x, x' \in U\,.$$

Then, arguing as in Theorem 3.6 shows that:

$$\tau'_{F,x_0} = \tau_{F,x_0} \ge \tau_A - \delta\,.$$

Indeed, we may assume that $\tau_A > \delta$, and considering $\tau > 0$ with $\tau_A > \tau + \delta$, the same argument with $DF(x_0)$ replaced by A and ε replaced by δ, shows that (16) holds for any $\rho > 0$ such that $\bar{B}_{2\rho}(x_0) \subset U$. So, if $\tau_A > \delta$, we obtain in particular that given $0 < \tau < \tau_A - \delta$, there exists $\rho_0 > 0$ such that for $0 < \rho \leq \rho_0$ and $y \in \bar{B}_{\rho\tau}(x_0)$, there exists $x \in \bar{B}_\rho(x_0)$ such that $F(x) = y$. This statement corresponds to Graves' formulation of his result, as was recalled by Dontchev in [10], and involving no differentiability assumption on F.

(b) It is clear (from the proof) that the first inequality in Proposition 3.4 holds, assuming only that F is Gâteaux-differentiable at x_0. Consequently, we also get, for example, $\tau_{F,x_0} \geq \tau_0$ in Theorem 3.8, assuming only Gâteaux-differentiability of F (which is [16, Corollary 1.6]). Similarly, most if not all sufficient conditions for local metric regularity that can be found in the literature follow from the method presented here, based on the use of the strong slope. Indeed, as mentioned in the introduction, Theorem 3.6 affords considerable generalization, involving closed set-valued maps F between complete metric spaces, see, e.g., [4], especially Theorem 5.3 therein, dealing with the *characterization* of local metric regularity for such mappings, in terms of estimates of the strong slope for functions of the type of the f_y's. Such estimates in turn translate, for example and in a Banach space context, into estimates for the so-called *coderivative* of F (see [20, 4]), of which Theorem 3.8 is a special case.

(c) The fact that openness at a linear rate — also called *covering (near a given point)* — and metric regularity are equivalent notions, was established in a fairly general setting (set-valued maps between normed spaces) in the papers of Borwein and Zhuang [8] and Penot [21], but the underlying properties had been in use long before (see, e.g., [18]). As a matter of fact (see [8, 21]), these notions are indeed equivalent to a third one, namely, J.-P. Aubin's *pseudo-Lipschitz property* for the inverse (set-valued) map [2], called the *Aubin property* by Dontchev and Rockafellar in [11] and known as such since then. We observe that traditionally, the constant τ appearing in (14) is written *on the other side* of the inequality, so that the "best openness and metric regularity constants" were often considered as mutually inverse. We believe it is preferable to adopt our formulation (and proceed in order to get *equality* of the best constants, as in Proposition 3.5), because it is what comes out naturally from the abstract results, starting with the variational principle. This also suggests that the notion, yielding the *theory*, of metric regularity (we note that the terminology is due to Penot [21]), should prevail over that of openness at a linear rate. □

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References

[1] J.-P. Aubin: Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions, Adv. Math., Suppl. Stud. 7A (1981) 159–229.

[2] J.-P. Aubin: Lipschitz behavior of solutions to convex minimization problems, Math. Oper. Res. 9 (1984) 87–111.

[3] D. Azé, J.-N. Corvellec: On the sensitivity analysis of Hoffman constants for systems of linear inequalities, SIAM J. Optim. 12 (2002) 913–927.

[4] D. Azé, J.-N. Corvellec: Characterizations of error bounds for lower semicontinuous functions on metric spaces, ESAIM, Control Optim. Calc. Var. 10 (2004) 409–425.

[5] D. Azé, J.-N. Corvellec, R. E. Lucchetti: Variational pairs and applications to stability in nonsmooth analysis, Nonlinear Anal., Theory Methods Appl. 49 (2002) 643–670.

[6] S. Banach: Théorie des Opérations Linéaires, Monografie Matematyczne, Warsaw (1932).

[7] J. M. Borwein: Stability and regular points of inequality systems, J. Optim. Theory Appl. 48 (1986) 9–52.

[8] J. M. Borwein, D. M. Zhuang: Verifiable necessary and sufficient conditions for openness and regularity of set-valued maps, J. Math. Anal. Appl. 134 (1988), 441–459.

[9] E. De Giorgi, A. Marino, M. Tosques: Problemi di evoluzione in spazi metrici e curve di massima pendenza, Atti Accad. Naz. Lincei, VIII. Ser., Rend., Cl. Sci. Fis. Mat. Nat. 68 (1980) 180–187.

[10] A. L. Dontchev: The Graves theorem revisited, J. Convex Analysis 3 (1996) 45–53.

[11] A. L. Dontchev, R. T. Rockafellar: Characterizations of strong regularity for variational inequalities over polyhedral convex sets, SIAM J. Optim. 6 (1996) 1087–1105.

[12] I. Ekeland: On the variational principle, J. Math. Anal. Appl. 47 (1974) 324–353.

[13] I. Ekeland: Nonconvex minimization problems, Bull. Am. Math. Soc., New Ser. 1 (1979) 443–474.

[14] L. M. Graves: Some mapping theorems, Duke Math. J. 17 (1950) 111–114.

[15] A. Ioffe: Regular points of Lipschitz functions, Trans. Amer. Math. Soc. 251 (1979) 61–69.

[16] A. Ioffe: On the local surjection property, Nonlinear Anal., Theory Methods Appl. 11 (1987) 565–592.

[17] A. Ioffe: Towards metric theory of metric regularity, in: Approximation, Optimization and Mathematical Economics (Guadeloupe, 1999), M. Lassonde (ed.), Physica, Heidelberg (2001) 165–176.

[18] A. Ioffe: Metric regularity and subdifferential calculus, Russ. Math. Surv. 55 (2000) 501–558.

[19] L. A. Lusternik: Conditional extrema of functionals, Mat. Sb. 41 (1934) 390–401.

[20] B. S. Mordukhovich, Y. Shao: Differential characterizations of covering, metric regularity and Lipschitzian properties of multifunctions, Nonlinear Anal., Theory Methods Appl. 25 (1995) 1401–1428.

[21] J.-P. Penot: Metric regularity, openness and Lipschitzian behavior of multifunctions, Nonlinear Anal., Theory Methods Appl. 13 (1989) 629–643.

[22] J. Schauder: Über die Umkehrung linearer, stetiger Funktionaloperationen, Studia Math. 2 (1930) 1–6.

[23] J. D. Weston: A characterization of metric completeness, Proc. Amer. Math. Soc. 64 (1977) 186–188.