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A New Convexity Property for Monotone Operators

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Dedicated to the memory of Simon Fitzpatrick.

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It is well known that the interior of the domain of a maximal monotone operator is convex, and coincides with the core of its convex hull. In fact this is also true for the interior of the difference of the domains of two maximal monotone operators S and T. Recently, Borwein [1, Th. 19] used the condition that the origin is in the core of the convex hull of the difference of the graphs of S and $-T$ in order to obtain that the origin is in the image of $S+T$. In this short note we show that $\operatorname{gph} S - \operatorname{gph}(-T)$ has similar properties to $\operatorname{dom} S - \operatorname{dom} T$.

Throughout this note X is a reflexive Banach space and X^* is its topological dual. The coupling (pairing) function on $X \times X^*$ is denoted by c; so $c(x, x^*) := \langle x, x^*\rangle := x^*(x)$. We identify the dual of $X \times X^*$ with $X^* \times X$ by the pairing

$$\langle (x, x^*), (y^*, y)\rangle := \langle x, y^*\rangle + \langle y, x^*\rangle .$$

The class of maximal monotone operators $S : X \rightrightarrows X^*$ is denoted by $\mathfrak{M}(X)$. The domain, image and graph of $S \in \mathfrak{M}(X)$ are defined as usual and are denoted by $\operatorname{dom} S$, $\operatorname{Im} S$ and $\operatorname{gph} S$, respectively. We also use the usual notation from convex analysis for functions f, like $\operatorname{dom} f$ (domain), f^* (conjugate), ∂f (subdifferential), $\overline{\operatorname{co}} f$ (convex lsc envelope); ι_A denotes the indicator function of A (that is ι_A is 0 on A and $+\infty$ outside A). To S we associate the functions $c_S := c + \iota_{\operatorname{gph} S}$, the Fitzpatrick function $\varphi_S : X \times X^* \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, that is,

$$\varphi_S(x, x^*) := \sup\{\langle x, u^*\rangle + \langle u, x^*\rangle - \langle u, u^*\rangle \mid (u, u^*) \in \operatorname{gph} S\} \quad (x, x^*) \in X \times X^*,$$

and $\psi_S := \overline{\operatorname{co}}\, c_S$; the function φ_S was introduced in [2]. In fact φ_S is the transpose (denoted by $^\top$) of the conjugate of c_S and ψ_S is the transpose of the conjugate of φ_S, that is, the biconjugate of c_S. It is known (see [3], [4]) that $c \le \varphi_S \le \psi_S$ for any $S \in \mathfrak{M}(X)$. We call a representative of $S \in \mathfrak{M}(X)$ a lsc convex function $f : X \times X^* \to \overline{\mathbb{R}}$ such that $f \ge c$, $f^* \ge c^\top$ and

$$\operatorname{gph} S = \{(x, x^*) \in X \times X^* \mid f(x, x^*) = \langle x, x^*\rangle\}. \tag{1}$$

By [4, Prop. 2.6], the lsc convex function $f : X \times X^* \to \overline{\mathbb{R}}$ is a representative of $S \in \mathfrak{M}(X)$ if and only if $\varphi_S \le f \le \psi_S$; in particular, φ_S and ψ_S are representatives of S. It follows that $f^{*\top}$ is a representative of S when f is so.

In the sequel, for a subset A of a normed vector space Z, we use the notation $\operatorname{cl} A$, $\operatorname{int} A$, $\operatorname{aff} A$, $\operatorname{co} A$, $\operatorname{core} A$, $\operatorname{icr} A$ for the closure, interior, affine hull, convex hull, core

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(or algebraic interior), intrinsic core (or relative algebraic interior) of A, respectively; moreover, $\overline{\mathrm{co}}\, A := \mathrm{cl}(\mathrm{co}\, A)$ and $\overline{\mathrm{aff}} A = \mathrm{cl}(\mathrm{aff}\, A)$. By ${}^{ic}A$ we mean $\mathrm{icr}\, A$ when $\mathrm{aff}\, A$ is closed and $\emptyset$ (the empty set) otherwise. So, when we write $a \in {}^{ic}A$ we mean that $\mathrm{aff}\, A$ is closed and $a \in \mathrm{icr}\, A$. Note that for a convex set A, $a \in {}^{ic}A$ if and only if $\mathbb{R}_+(A - a)$ is a closed linear space.

The next result extends slightly (with a very similar proof) Borwein's Theorem 19 in [1]. Here, for $g : X \times X^* \to \overline{\mathbb{R}}$ we denote by $\widehat{g}$ the function on $X \times X^*$ defined by $\widehat{g}(x, x^*) := g(x, -x^*)$. Note that

$$\widehat{g}^*(x^*, x) = g^*(x^*, -x) \quad \forall (x, x^*) \in X \times X^*. \tag{2}$$

Lemma 1. *Let f, g be representatives for $S, T \in \mathfrak{M}(X)$, respectively. If $(0, 0) \in {}^{ic}(\mathrm{dom}\, f - \mathrm{dom}\, \widehat{g})$ then $0 \in \mathrm{Im}(S + T)$.*

Proof. We have $f, g \ge c$ and $f^*, g^* \ge c^\top$; moreover,

$$\mathrm{gph}\, S = \{(x, x^*) \mid f(x, x^*) = \langle x, x^* \rangle\} = \{(x, x^*) \mid f^*(x^*, x) = \langle x, x^* \rangle\},$$

and similarly for T. Of course,

$$f(x, x^*) + \widehat{g}(x, x^*) = f(x, x^*) + g(x, -x^*) \ge \langle x, x^* \rangle + \langle x, -x^* \rangle = 0$$

for every $(x, x^*) \in X \times X^*$, that is $\inf(f + \widehat{g}) \ge 0$. Because $(0, 0) \in {}^{ic}(\mathrm{dom}\, f - \mathrm{dom}\, \widehat{g})$, the Fenchel duality theorem (see f.i. [5, Cor. 2.8.5]) yields some $(u, u^*) \in X \times X^*$ such that $f^*(u^*, u) + \widehat{g}^*(-u^*, -u) \le 0$, that is, by (2), $f^*(u^*, u) + g^*(-u^*, u) \le 0$. Since $f^* \ge c^\top$ and $g^* \ge c^\top$,

$$0 \le \langle u, u^* \rangle + \langle u, -u^* \rangle \le f^*(u^*, u) + g^*(-u^*, u) \le 0,$$

which yields $f^*(u^*, u) = \langle u, u^* \rangle$ and $g^*(-u^*, u) = \langle u, -u^* \rangle$. Therefore $u^* \in S(u)$ and $-u^* \in T(u)$, and so $0 \in \mathrm{Im}(S + T)$. □

Note that

$$x^* \in \mathrm{Im}(S + T) \quad \Longleftrightarrow \quad (0, x^*) \in \mathrm{gph}\, S - \mathrm{gph}(-T). \tag{3}$$

Hence the conclusion of Lemma 1 can be written as $(0, 0) \in \mathrm{gph}\, S - \mathrm{gph}(-T)$. In fact, as we shall see in Theorem 3 below, the conclusion of Lemma 1 can be replaced by $(0, 0) \in {}^{ic}(\mathrm{gph}\, S - \mathrm{gph}(-T))$.

Let us establish the following lemma.

Lemma 2. *Let f, g be representatives for $S, T \in \mathfrak{M}(X)$, respectively. Then*

$${}^{ic}(\mathrm{dom}\, f - \mathrm{dom}\, \widehat{g}) \subset \mathrm{gph}\, S - \mathrm{gph}(-T) \subset \mathrm{co}\,(\mathrm{gph}\, S - \mathrm{gph}(-T)) \subset \mathrm{dom}\, f - \mathrm{dom}\, \widehat{g}. \tag{4}$$

Proof. The second inclusion is obvious. From (1) we obtain $\mathrm{gph}\, S \subset \mathrm{dom}\, f$ and $\mathrm{gph}(-T) \subset \mathrm{dom}\, \widehat{g}$. Since $\mathrm{dom}\, f - \mathrm{dom}\, \widehat{g}$ is convex, the third inclusion follows.

Let $(u, u^*) \in {}^{ic}(\mathrm{dom}\, f - \mathrm{dom}\, \widehat{g})$ and take $\mathrm{gph}\, S_0 := \mathrm{gph}\, S - (u, u^*)$. Note that $S_0 \in \mathfrak{M}(X)$ and the function h defined by

$$h(x, x^*) := f(x + u, x^* + u^*) - \langle x, u^* \rangle - \langle u, x^* \rangle - \langle u, u^* \rangle$$

for $(x,x^*) \in X \times X^*$ is a representative for S_0 (as a simple computation shows). Since $\operatorname{dom} h = \operatorname{dom} f - (u,u^*)$, we have $(0,0) \in {}^{ic}(\operatorname{dom} h - \operatorname{dom} \widehat{g})$. From Lemma 1 and the equivalence (3) we obtain that $(0,0) \in \operatorname{gph} S_0 - \operatorname{gph}(-T)$, and so, $(u,u^*) \in \operatorname{gph} S - \operatorname{gph}(-T)$. □

Theorem 3. *Let $S,T \in \mathfrak{M}(X)$. Then*

$$ {}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T)) = {}^{ic}\big(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T))\big) = {}^{ic}\big(\operatorname{dom} \psi_S - \operatorname{dom} \widehat{\psi_T}\big). \tag{5} $$

Moreover, if f and g are representatives for S and T respectively, and ${}^{ic}(\operatorname{dom} f - \operatorname{dom} \widehat{g})$ is nonempty then

$$ {}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T)) = {}^{ic}(\operatorname{dom} f - \operatorname{dom} \widehat{g}). \tag{6} $$

Proof. Let first f and g be representatives for S and T, respectively, with ${}^{ic}(\operatorname{dom} f - \operatorname{dom} \widehat{g})$ nonempty. Taking into account that for a convex set A with ${}^{ic}A \neq \emptyset$ we have $\operatorname{aff}({}^{ic}A) = \operatorname{aff} A$, from (4) we obtain

$$ {}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T)) = {}^{ic}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T))) = {}^{ic}(\operatorname{dom} f - \operatorname{dom} \widehat{g}). $$

In particular (6) holds. Since ψ_S and ψ_T are representatives for S and T, from the preceding result we get (5) when ${}^{ic}(\operatorname{dom} \psi_S - \operatorname{dom} \widehat{\psi_T})$ is nonempty; in particular ${}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T))$ and ${}^{ic}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)))$ are nonempty in this case. So, in order to have (5) it is sufficient to prove that ${}^{ic}(\operatorname{dom} \psi_S - \operatorname{dom} \widehat{\psi_T})$ is nonempty when ${}^{ic}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)))$ is so, because it is clear that ${}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T)) \subset {}^{ic}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)))$. For this purpose just note that from the definition of ψ_S we have $\operatorname{gph} S \subset \operatorname{dom} \psi_S \subset \overline{\operatorname{co}}(\operatorname{gph} S)$. Hence

$$ \operatorname{gph} S - \operatorname{gph}(-T) \subset \operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)) \subset \operatorname{dom} \psi_S - \operatorname{dom} \widehat{\psi_T} \subset \overline{\operatorname{co}}(\operatorname{gph} S - \operatorname{gph}(-T)), \tag{7} $$

and so

$$ \begin{aligned} \operatorname{aff}(\operatorname{gph} S - \operatorname{gph}(-T)) = \operatorname{aff}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T))) &\subset \operatorname{aff}\big(\operatorname{dom} \psi_S - \operatorname{dom} \widehat{\psi_T}\big) \\ &\subset \overline{\operatorname{aff}}(\operatorname{gph} S - \operatorname{gph}(-T)). \end{aligned} \tag{8} $$

Thus, if ${}^{ic}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)))$ is nonempty then $\operatorname{aff}(\operatorname{gph} S - \operatorname{gph}(-T))$ is closed, and so the inclusions in (8) become equalities; the claim follows immediately from (7). □

Corollary 4. *Let f, g be representatives for $S,T \in \mathfrak{M}(X)$. If $(0,x^*) \in {}^{ic}(\operatorname{dom} f - \operatorname{dom} \widehat{g})$ then $x^* \in \operatorname{Im}(S+T)$. Hence $S+T$ is surjective whenever $\{0\} \times X^* \subset {}^{ic}(\operatorname{dom} f - \operatorname{dom} \widehat{g})$.*

Proof. The conclusion follows immediately from Theorem 3 and (3). □

From Theorem 3 we obtain that ${}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T))$ is a convex set for any maximal monotone operators S and T.

Corollary 5. *Let $S,T \in \mathfrak{M}(X)$ be such that ${}^{ic}(\operatorname{gph} S - \operatorname{gph}(-T))$ is nonempty. Then*

$$ \operatorname{cl}(\operatorname{gph} S - \operatorname{gph}(-T)) = \overline{\operatorname{co}}(\operatorname{gph} S - \operatorname{gph}(-T)) = \operatorname{cl}\big(\operatorname{dom} \psi_S - \operatorname{dom} \widehat{\psi_T}\big). $$

Proof. From the preceding theorem we get

$$^{ic}(\operatorname{dom}\psi_S - \operatorname{dom}\widehat{\psi_T}) \subset \operatorname{gph} S - \operatorname{gph}(-T) \subset \operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)) \subset \operatorname{dom}\psi_S - \operatorname{dom}\widehat{\psi_T}.$$

Because $\operatorname{cl}(^{ic}A) = \operatorname{cl} A$ for a convex set A with $^{ic}A \neq \emptyset$, the conclusion follows. □

Taking $\operatorname{gph} T := \{0\} \times X^*$ in Theorem 3 and Corollary 5 we recover the following known result.

Corollary 6. *Let $S \in \mathfrak{M}(X)$. Then $^{ic}(\operatorname{dom} S) = {}^{ic}(\operatorname{co}(\operatorname{dom} S))$. Moreover, if these sets are nonempty then $\operatorname{cl}(\operatorname{dom} S) = \overline{\operatorname{co}}(\operatorname{dom} S)$.*

When we deal with the core instead of the intrinsic core, in Theorem 3 we can use arbitrary representatives of S and T.

Corollary 7. *Let $S, T \in \mathfrak{M}(X)$ have the representatives $f, g \in \Gamma(X \times X^*)$. Then*

$$\operatorname{core}(\operatorname{gph} S - \operatorname{gph}(-T)) = \operatorname{core}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T))) = \operatorname{core}(\operatorname{dom} f - \operatorname{dom}\widehat{g}). \quad (9)$$

In particular $\operatorname{core}(\operatorname{gph} S - \operatorname{gph}(-T)) = \operatorname{core}(\operatorname{dom}\varphi_S - \operatorname{dom}\widehat{\varphi_T})$.

Proof. Of course, the inclusions

$$\operatorname{gph} S - \operatorname{gph}(-T) \subset \operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T)) \subset \operatorname{dom} f - \operatorname{dom}\widehat{g}$$

provide the corresponding inclusions in (9). If $\operatorname{core}(\operatorname{dom} f - \operatorname{dom}\widehat{g})$ is empty then (9) holds. Because $^{ic}A = \operatorname{core} A$ when $\operatorname{core} A \neq \emptyset$, if $\operatorname{core}(\operatorname{dom} f - \operatorname{dom}\widehat{g})$ is nonempty, then $^{ic}(\operatorname{dom} f - \operatorname{dom}\widehat{g}) = \operatorname{core}(\operatorname{dom} f - \operatorname{dom}\widehat{g}) \neq \emptyset$, and so, by Theorem 3, (5) and (6) hold. Moreover $\operatorname{aff}(\operatorname{dom} f - \operatorname{dom}\widehat{g}) = X \times X^*$, and so (9) holds, too. □

The preceding result shows that the interiority conditions in the hypothesis of [1, Th. 19] are equivalent.

Open problem. Let $S, T \in \mathfrak{M}(X)$. Is the implication

$$^{ic}(\operatorname{co}(\operatorname{gph} S - \operatorname{gph}(-T))) \neq \emptyset \quad \Longrightarrow \quad {}^{ic}(\operatorname{dom}\varphi_S - \operatorname{dom}\widehat{\varphi_T}) \neq \emptyset$$

true?

Of course, as seen in Corollary 7, the answer is affirmative when $\operatorname{aff}(\operatorname{gph} S - \operatorname{gph}(-T)) = X \times X^*$. It is also positive when $\dim X < \infty$. In fact the answer is positive if and only if $\operatorname{dom}\varphi_S - \operatorname{dom}\widehat{\varphi_T} \subset \overline{\operatorname{aff}}(\operatorname{gph} S - \operatorname{gph}(-T))$.

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