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Convex Along Lines Functions and Abstract Convexity. Part I.

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The present paper investigates the property of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ with $f(0) < +\infty$ to be $\mathcal{L}_n$-subdifferentiable or $\mathcal{H}_n$-convex. The $\mathcal{L}_n$-subdifferentiability and $\mathcal{H}_n$-convexity are introduced as in Rubinov [9]. Some refinements of these properties lead to the notions of $\mathcal{L}_n^0$-subdifferentiability and $\mathcal{H}_n^0$-convexity. Their relation to the convex-along (CAL) functions is underlined in the following theorem proved in the paper (Theorem 5.6): Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be such that $f(0) < +\infty$ and f is $\mathcal{H}_n$-convex at the points at which it is infinite. Then if f is $\mathcal{L}_n^0$-subdifferentiable, it is CAL and globally calm at each $x^0 \in \operatorname{dom} f$. Here the notions of local and global calmness are introduced after Rockafellar, Wets [8] and play an important role in the considerations. The question is posed for the possible reversal of this result. In the case of a positively homogeneous (PH) and CAL function such a reversal is proved (Theorem 6.2). As an application conditions are obtained under which a CAL PH function is $\mathcal{H}_n^0$-convex (Theorems 6.3 and 6.4).

Keywords: Abstract convexity, generalized convexity, duality, $\mathcal{H}_n$-convexity, convex-along-rays functions, convex-along-lines functions, positively homogeneous functions

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1. Introduction

After the monographs of Pallaschke, Rolewicz [5], Singer [13] and Rubinov [9] abstract convex analysis has grown to a mathematical discipline with its own face and problems. Its importance is due mainly to its close relation to global optimization. In spite of the recent advances, there are still many problems in its merit, which are until so far either unsolved or insufficiently developed. Among them it is the problem to characterize the

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class of $\mathcal{H}_k$-convex functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ for k positive integer. The $\mathcal{H}_k$-convex functions are defined as in Rubinov [9]. It is known that all $\mathcal{H}_k$-convex functions are convex-along-rays (CAR). Recall that the function f is CAR if its restriction on each ray $\{\lambda x \mid \lambda \geq 0\}$ is convex. We use also the notions of convex-along-lines (CAL) functions and positively homogeneous of order one (PH) functions defined as follows. The function f is CAL if its restriction on each line $\{\lambda x \mid \lambda \in \mathbb{R}\}$ is convex. The function f is PH if $f(\lambda x) = \lambda f(x)$ for $\lambda \geq 0$ and $x \in \mathbb{R}^n$. It is known [9] that all lsc CAR functions with finite value at zero belong to the class $\mathcal{H}_{n+1}$, which in consequence turns to be very broad. Therefore, the task to establish the relation of CAR and $\mathcal{H}_n$-convex functions seems quite natural. The present paper is an investigation in this direction. We study the property of a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ with $f(0) < +\infty$ to be $\mathcal{L}_n$-subdifferentiable or $\mathcal{H}_n$-convex. More precisely, we introduce the concepts of $\mathcal{L}_n^0$-subdifferentiability and $\mathcal{H}_n^0$-convexity as some refinement of $\mathcal{L}_n$-subdifferentiability and $\mathcal{H}_n$-convexity. We study the property of a CAR function to be $\mathcal{L}_n^0$-subdifferentiable or $\mathcal{H}_n^0$-convex. The CAL property appears in a natural way in this study.

The paper is structured as follows. Section 2 is an introduction to abstract convexity with respect to min-type functions. We define there also $\mathcal{L}_n^0$-subdifferentiability and $\mathcal{H}_n^0$-convexity. Section 3 gives some properties and examples of CAL functions. The notions of local and global calmness, which play an important role in the considerations, are defined in Section 4. In Section 5 the following theorem is proved: *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be such that $f(0) < +\infty$ and f is $\mathcal{H}_n$-convex at the points at which it is infinite. Then if f is $\mathcal{L}_n^0$-subdifferentiable, it is CAL and globally calm at each $x^0 \in \operatorname{dom} f$.* Here CAL functions appear in a natural way and for this reason they are exposed in the study. We pose the question for the possible reversal of this result. In Section 6 we prove a reversal in the case of CAL PH functions. As an application we derive conditions under which a CAL PH function is $\mathcal{H}_n^0$-convex (for PH functions $\mathcal{H}_n^0$-convexity is equivalent to $\mathcal{L}_n^0$-convexity). Let us note that an attempt to generalize the results from Section 6 to CAL but not necessarily PH functions brings into light new phenomena. The authors intend to study separately this more general case.

Confining in the last section to PH functions, we would like to add the following comment. Many important functions in nonsmooth analysis are PH, say different type of generalized derivatives with respect to the directions, or the Minkowski gauge of a radiant set. For other aspects of applications of PH functions in abstract convexity see [2] and [10].

The $\mathcal{H}_{n+1}$-convex functions find interesting applications in the study of radiant (star-shaped at zero) and co-radiant sets [9, 11, 12] and also in the study of a separation of star-shaped sets by means of a finite collection of linear functions. We hope that $\mathcal{H}_n^0$-convex functions can find also similar applications.

The class of CAL functions contains the class of CAR functions possessing a minimum at $x^0 = 0$. The importance of such functions, various applications and some duality schemes in convex setting has been studied by Penot [6]. The present paper deals in fact with an extension of the duality scheme of Moreau [4] generalizing the well known duality schemes for convex functions. The latter is discussed in details e.g. in Borwein, Lewis [1]. Extension of Moreau duality can be found in Rubinov [9] and Singer [13]. Obviously, the problem of finding a general characterization of $\mathcal{H}_k$-convex and in particular of $\mathcal{H}_n$-convex functions remains open.

2. Preliminaries

In this paper $\mathbb{R}_{+\infty}$ denotes the set $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$. The abbreviation lsc stands for lower semicontinuous and usc for upper semicontinuous. We shall examine some special classes of functions $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$.

2.1. Positively homogeneous functions and convex-along-rays functions

Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a function with $\operatorname{dom} f \neq \emptyset$. Recall that the domain of $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is the set $\operatorname{dom} f = \{x \in \mathbb{R}^n \mid f(x) \neq +\infty\}$. For each $x \in \mathbb{R}^n$, $x \neq 0$, consider the ray $R_x = \{\alpha x : \alpha \geq 0\}$ and the restriction f_x of f to this ray. In other words f_x is the function of one variable defined on $[0, +\infty)$ by $f_x(\alpha) = f(\alpha x)$. We say that a certain property holds for f *along rays* if f_x possesses this property for each x. In particular, f is convex-along-rays (CAR) if f_x is convex for each x. This functions is positively homogeneous of first degree (PH) if $f(\alpha x) = \alpha f(x)$ for all $x \in \mathbb{R}^n$ and $\alpha \geq 0$. We accept in the sequel that $0 \cdot (+\infty) = 0$. This definition implies that $f(0) = 0$ for each PH function f, hence $0 \in \operatorname{dom} f$.

The examination of CAR functions can be reduced to the examination of PH functions with the help of construction of lower affine approximation that is discussed in [9, Section 5.5.3]. Let f be a CAR function and $a < f(0)$. For each $x \in \mathbb{R}^n$ consider

$$b^a(x) = \inf_{\lambda > 0} \frac{f_x(\lambda) - a}{\lambda} = \sup\{c : a + c\lambda \leq f_x(\lambda),\ \lambda \geq 0\}. \tag{1}$$

It is easy to check that b^a is a PH function for each $a < f(0)$. If f is PH then $b^a = f$ for all $a < 0 = f(0)$. The function

$$g^a(x) = a + b^a(x), \quad x \in \mathbb{R}^n, \tag{2}$$

is *affine-along-rays*. It is easy to see that $g^a(x) \leq f(x)$, $\forall x \in \mathbb{R}^n$. This function is called a *lower affine approximation* of f.

Proposition 2.1 ([9], Lemma 5.5). *A* lsc-along-rays *CAR function is the supremum of its lower affine approximations.*

In [9] this result was proved for lsc CAR functions however the proof holds for lsc-along-rays CAR functions. Using terminology of abstract convexity (see next subsection) we can say that a lsc-along-rays CAR function is abstract convex with respect to the set of abstract affine function $\mathcal{H}_{\mathcal{L}}$ where $\mathcal{L}$ is the set of PH functions.

Let $0 \in \operatorname{dom} f$. Then we can also consider the lower affine approximation with $a = f(0)$. Since $f_x(0) = f(0)$ we have in such a case:

$$b^a(x) = \inf_{\lambda > 0} \frac{f_x(\lambda) - f_x(0)}{\lambda}.$$

If the function f is CAR, the function f_x is convex on $[0, +\infty)$, hence

$$b^a(x) = \lim_{\lambda \to 0^+} \frac{f_x(\lambda) - f_x(0)}{\lambda} = (f_x)'(0) = f'(0, x),$$

where $f'(0,x)$ is the directional derivative of f at zero in the direction x. We have for $a = f(0)$:

$$g^a(x) = a + b^a(x) = f(0) + \inf_{\lambda>0} \frac{f_x(\lambda) - f_x(0)}{\lambda} \le f(0) + f_x(1) - f(0) = f(x)$$

so g^a is a lower affine approximation also for $a = f(0)$.

2.2. Abstract convex functions

We recall first some definitions from abstract convexity [9]. Let X be a given set and $\mathcal{L}$ be a set of functions $\ell : X \to \mathbb{R}$. A function $f : X \to \mathbb{R}_{+\infty}$ is said to be abstract convex with respect to $\mathcal{L}$ or $\mathcal{L}$-convex at a point x^0 if

$$f(x^0) = \sup\{\ell(x^0) \mid \ell \in \mathcal{L},\, \ell \le f\}. \tag{3}$$

Here $g \le f$ means $g(x) \le f(x)$ for all $x \in X$. In such a situation $\mathcal{L}$ is called the set of *elementary functions*. If (3) holds for each $x^0 \in X$ then f is called $\mathcal{L}$-convex (abstract convex w.r.t. $\mathcal{L}$) on X. For each $\ell \in \mathcal{L}$ and $c \in \mathbb{R}$ consider the function

$$h_{\ell,c}(x) = \ell(x) - c, \quad x \in X. \tag{4}$$

The set $\mathcal{L}$ with the property that $\ell \in \mathcal{L}$, $c \ne 0$, implies $h_{\ell,c} \notin \mathcal{L}$ is called a set of abstract linear functions. If $\mathcal{L}$ is a set of abstract linear functions, then a function of the form (4) is called abstract affine function. The set of all abstract affine functions is denoted by $\mathcal{H}_{\mathcal{L}}$.

Consider a function $f : X \to \mathbb{R}_{+\infty}$. A function $\ell \in \mathcal{L}$ is called an $\mathcal{L}$-subgradient of f at a point $x^0 \in \operatorname{dom} f$ if

$$\ell(x) - \ell(x^0) \le f(x) - f(x^0) \ \text{ for all } x \in X.$$

The set $\partial_{\mathcal{L}} f(x^0)$ of all $\mathcal{L}$-subgradients of f at x^0 is called the $\mathcal{L}$-subdifferential of f at x^0. It holds that f is $\mathcal{L}$-subdifferentiable at x^0, that is $\partial_{\mathcal{L}} f(x^0) \ne \emptyset$, if and only if

$$f(x^0) = \max\{h(x^0) \mid h \in \mathcal{H}_{\mathcal{L}},\, h \le f\}.$$

Thus a $\mathcal{L}$-subdifferentiable at $x^0 \in \operatorname{dom} f$ function f is $\mathcal{H}_{\mathcal{L}}$-convex at x^0. If

$$f(x) = \max\{h(x) \mid h \in H,\, h \le f\} \ \text{ for all } x \in \operatorname{dom} f,$$

then $\partial_{\mathcal{L}} f(x) \ne \emptyset$ for every $x \in \operatorname{dom} f$ and f is called $\mathcal{L}$-subdifferentiable on $\operatorname{dom} f$.

Assume that the set $\mathcal{L}$ consists of PH functions. Then each $\mathcal{L}$-convex function is PH. Let f be a PH function. Then $\ell \in \partial_{\mathcal{L}} f(x_0)$ if and only if

$$\ell(x) \le f(x) \ \text{ for all } x \in X \ \text{ and } \ \ell(x_0) = f(x_0). \tag{5}$$

For given two sets $\mathcal{L} \subset \mathcal{L}'$ of abstract linear functions, obviously each $\mathcal{H}_{\mathcal{L}}$-convex function is $\mathcal{H}_{\mathcal{L}'}$-convex. Further each $\mathcal{H}_{\mathcal{L}}$-convex function with nonempty $\mathcal{L}$-subdifferentials is $\mathcal{H}_{\mathcal{L}'}$-convex with nonempty $\mathcal{H}_{\mathcal{L}'}$-subdifferentials.

2.3. Abstract convexity with respect to sets of min-type functions

Let n be a positive integer. Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. Let $\langle\cdot,\cdot\rangle$ be the inner product in $\mathbb{R}^n$. For a positive integer k we define the class of abstract linear functions $\mathcal{L}_k$ as the set of all functions $\ell : \mathbb{R}^n \to \mathbb{R}$ such that $\ell(x) = \min_{1\le i\le m}\langle l_i, x\rangle$ for some $m \le k$ and $l_1, \dots, l_m \in \mathbb{R}^n$ (repeating eventually some of the vectors we can simply write k instead of m in the above minimum, a convention applied in the sequel). Clearly $\mathcal{L}_k$ consists of PH functions. Denote the set $\mathcal{H}_{\mathcal{L}_k}$ of all abstract affine with respect to $\mathcal{L}_k$ functions by $\mathcal{H}_k$. Thus $h \in \mathcal{H}_k$ if and only if there exist $l_i \in \mathbb{R}^n$, $i = 1, \dots, k$, and $c \in \mathbb{R}$ such that

$$h(x) = \min_{1\le i\le k}(\langle l_i, x\rangle - c), \quad x \in \mathbb{R}^n.$$

It is clear from this definition that if $k' < k''$ then $\mathcal{H}_{k'} \subset \mathcal{H}_{k''}$ and therefore the class of $\mathcal{H}_{k'}$-convex functions is contained in the class of $\mathcal{H}_{k''}$-convex functions. In particular the class of $\mathcal{H}_n$-convex functions is contained in the class of $\mathcal{H}_{n+1}$-convex functions.

Since a function $\ell \in \mathcal{L}_k$ is continuous and PH, it follows that each $\mathcal{L}_k$-convex function is lsc and PH. The following two proposition can be found in [9, Section 5.5.2] and Proposition 2.4 is in fact Proposition 5.53 in [9]. Further, for Proposition 2.5 see [9, Theorem 5.16].

Proposition 2.2. *Each lsc PH nonnegative function f is $\mathcal{L}_n$-convex.*

Proposition 2.3. *A function f with $0 \in \operatorname{dom} f$ is $\mathcal{L}_{n+1}$-convex if and only if f is lsc and PH.*

Proposition 2.4. *For arbitrary positive integer k a $\mathcal{H}_k$-convex function is a lsc CAR function.*

Proposition 2.5. *A function f with $0 \in \operatorname{dom} f$ is $\mathcal{H}_{n+1}$-convex if and only if f is lsc and CAR.*

A description of $\mathcal{L}_n$-convex functions and $\mathcal{H}_n$-convex functions is more complicated.

The set $\mathcal{L}_k$ of all min-type functions is very large, so it is convenient to consider some special subsets of this set. Denote by $\mathcal{L}_k(x^0)$ the set of functions $\ell(x) = \min_{1\le i\le k}\langle l_i, x\rangle$ such that

$$\langle l_1, x^0\rangle = \langle l_2, x^0\rangle = \cdots = \langle l_k, x^0\rangle. \tag{6}$$

We consider $\mathcal{L}_k(x^0)$ as a set of abstract linear functions. The set of the shifts (4) with $\ell \in \mathcal{L}_k(x^0)$ is denoted by $\mathcal{H}_k(x^0)$ and can be taken as a set of abstract affine functions. We are interested in the situation when f is $\mathcal{L}_k(x^0)$-subdifferentiable, $\mathcal{L}_k(x^0)$-convex or $\mathcal{H}_k(x^0)$-convex at x^0. In such a case we say for short that f is respectively $\mathcal{L}_k^0$-subdifferentiable, $\mathcal{L}_k^0$-convex and $\mathcal{H}_k^0$-convex at x^0. The inclusion $\mathcal{L}_k(x^0) \subset \mathcal{L}_k$ implies that each $\mathcal{L}_k^0$-subdifferentiable or $\mathcal{L}_k^0$-convex at x^0 function is respectively $\mathcal{L}_k$-subdifferentiable or $\mathcal{L}_k$-convex at x^0. Similarly, $\mathcal{H}_k(x^0) \subset \mathcal{H}_k$ implies that each $\mathcal{H}_k^0$-convex at x^0 function is $\mathcal{H}_k$-convex at x^0. We will say that f is $\mathcal{L}_k^0$-subdifferentiable if f is $\mathcal{L}_k^0$-subdifferentiable at any point $x^0 \in \operatorname{dom} f$. We say that f is $\mathcal{L}_k^0$-convex or $\mathcal{H}_k^0$-convex if it is respectively $\mathcal{L}_k^0$-convex or $\mathcal{H}_k^0$-convex at any point $x^0 \in \mathbb{R}^n$. Obviously, if f is $\mathcal{L}_k^0$-subdifferentiable, $\mathcal{L}_k^0$-convex or $\mathcal{H}_k^0$-convex, then it is respectively $\mathcal{L}_k$-subdifferentiable, $\mathcal{L}_k$-convex or $\mathcal{H}_k$-convex.

Remark 2.6. Let us note that $\mathcal{L}_k^0$-convex and $\mathcal{H}_k^0$-convex functions are not abstract convex functions in the sense of the definition of abstract convexity given in Subsection 2.2, and for this reason these notions can be considered only as a convenient notation. For instance, $\mathcal{L}_k^0$-convexity means abstract convexity at each point x^0 with respect to the set of elementary functions $\mathcal{L}_k(x^0)$ that depends on this point.

Remark 2.7. The notions of $\mathcal{H}_k^0$-convexity and $\mathcal{L}_k^0$-subdifferentiability allow us to examine a *qualified* version of $\mathcal{H}_k$-convexity and $\mathcal{L}_k$-subdifferentiability. For instance, $\mathcal{L}_k^0$-subdifferentiability of f means that f is $\mathcal{L}_k$-subdifferentiable at each point $x^0 \in \operatorname{dom} f$ and moreover f has a subgradient $\ell = (l_1, \dots, l_k) \in \mathcal{L}_k$ such that (6) holds.

Some results related to $\mathcal{L}_k^0$-convex, $\mathcal{H}_k^0$-convex and $\mathcal{L}_k^0$-subdifferentiable functions with $k = n + 1$ can be found in [9]. In this paper we study $\mathcal{L}_k^0$-convex, $\mathcal{H}_k^0$-convex and $\mathcal{L}_k^0$-subdifferentiable functions with $k = n$.

3. CAL functions and their lower affine approximation

Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a function. For each $x \in \mathbb{R}^n$, $x \neq 0$, consider the line $L = \{\lambda x : \lambda \in \mathbb{R}\}$ and the restriction f^x of f to this line. In other words f^x is the function of one variable defined on $\mathbb{R}$ by $f^x(\alpha) = f(\alpha x)$. We say that f enjoys a certain property *along lines* if f^x possesses this property for each x. In particular, f is called *convex-along-lines* (CAL) if f^x is convex for each x. The function f is called lsc-along-lines if f^x is lsc for each x. It is called linear-along-lines if f^x is linear, or in other words if $f(\alpha x) = \alpha f(x)$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Let us underline that we distinguish in the sequel between the defined here function f^x having domain $\mathbb{R}$ and the defined in Section 2.1 function f_x having domain $[0, +\infty)$.

In this paper we shall study CAL functions. First we indicate some properties of the set of CAL functions. Namely, it is easy to check that the sum of a finite number of CAL functions is a CAL function. The supremum of an arbitrary family of CAL functions is a CAL function. The pointwise limit of a net of CAL functions is a CAL function.

We give now simple examples of CAL functions.

1) Each convex function is CAL.
2) Each nonnegative PH function is CAL.
3) Let f be a CAR function with $\operatorname{dom} f \subset K$, where K is a pointed closed cone (pointed means $K \cap (-K) = \emptyset$). Then f is CAL.

Proposition 3.2 below gives a condition for a CAR function to be CAL. We insert a proof based on the notion of a Dini derivative. Consider a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$. The upper Dini-directional derivative of f at $x \in \operatorname{dom} f$ in direction $u \in \mathbb{R}^n$ is defined as element of $\overline{\mathbb{R}}$ by

$$f'_+(x, u) = \limsup_{t \to 0^+} \frac{1}{t} \left(f(x + tu) - f(x) \right).$$

We introduce the second-order upper Dini-directional derivative of f at x in the direction u if (and only if) $f'_+(x, u)$ is finite by

$$f''_+(x, u) = \limsup_{t \to 0^+} \frac{2}{t^2} \left(f(x + tu) - f(x) - t\, f'_+(x, u) \right).$$

The following lemma characterizes the property of an usc function to be convex in terms of upper Dini-derivatives and is a special case of a result proved in Ginchev, Ivanov [3, Theorem 2.1].

Lemma 3.1. *Let $\varphi : \mathbb{R} \to \mathbb{R}_{+\infty}$ be usc function and* $\operatorname{dom}\varphi$ *be a convex set. Then φ is convex if and only if the following two conditions are satisfied for each $t \in \operatorname{dom}\varphi$:*

a) $\varphi'_+(t,1) + \varphi'_+(t,-1) \geq 0$ *if the expression in the left hand side has sense.*
b) $\varphi'_+(t,1) + \varphi'_+(t,-1) = 0$ *implies* $\varphi''_+(t,1) \geq 0$.

We add the following comments to the above conditions. Concerning a) we accept that sums of the type $(\pm\infty) + (\mp\infty)$ have no sense. All the other sums involving infinities have sense: $(\pm\infty) + (\pm\infty) = \pm\infty$ and for a finite $(\pm\infty) + a = \pm\infty$. Concerning b) we see that $\varphi'_+(t,1) + \varphi'_+(t,-1) = 0$ could have place only if $\varphi'_+(t,1)$ and $\varphi'_+(t,-1)$ are finite, whence $\varphi''_+(t,1)$ does exist.

Now, applying Lemma 3.1 we get the following result:

Proposition 3.2. *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a CAR function with $0 \in \operatorname{dom} f$. Then the property that f is CAL is equivalent to any of the following conditions:*

1^0) $f(x) + f(-x) \geq 2f(0)$ *for all* $x \in \mathbb{R}^n$,
2^0) $f'(0,u) + f'(0,-u) \geq 0$ *for all* $u \in \mathbb{R}^n$.

Proof. Suppose that f is CAL. Then inequality 1^0 is a straightforward consequence of the convexity of the function f^x. Putting $x = tu$ in 1^0 we get

$$\frac{1}{t}(f(tu) - f(0)) + \frac{1}{t}(f(-tu) - f(0)) \geq 0,$$

and after passing to a limit with $t \to 0^+$ we obtain condition 2^0.

Suppose now that f is CAR and satisfies condition 2^0 (if condition 1^0 is assumed, we have shown that also 2^0 has place). Fix $x \neq 0$.

The function f^x is convex on the segments $[0, +\infty)$ and $(-\infty, 0]$. If f is identical to $+\infty$ on the interior of one of these segments, then obviously f^x is convex. Assume that this is not the case. Put $t_- = \inf \operatorname{dom} f$ and $t_+ = \sup \operatorname{dom} f$. Then $t_- < 0 < t_+$. Choose $\bar{t}_-$ and $\bar{t}_+$ such that $t_- < \bar{t}_- < 0 < \bar{t}_+ < t_+$. Define the function $\varphi : \mathbb{R} \to \mathbb{R}_{+\infty}$ which coincides with f^x on $(\bar{t}_-, \bar{t}_+)$ and is equal to $+\infty$ outside this interval.

We show that φ satisfies the assumptions of Lemma 3.1: *The function φ is usc.* At a point $t \notin (\bar{t}_-, \bar{t}_+)$ it is usc, since $f(t) = +\infty$. Further φ is convex on the intervals $(\bar{t}_-, 0)$ and $(0, \bar{t}_+)$. Therefore it is continuous and hence usc at each point t from these intervals (recall that a convex function defined on an Euclidean space is continuous at the points in the interior of its domain). At $t = 0$ φ is usc in consequence of the inequalities $\varphi(0) \geq \limsup_{t \to 0^+} \varphi(t)$ and $\varphi(0) \geq \limsup_{t \to 0^-} \varphi(t)$. To show the first of them (the second follows in a similar way) we must pass to a limit with $t \to 0^+$ in the inequality

$$\varphi(t) \leq \left(1 - \frac{t}{\bar{t}_+}\right) \varphi(0) + \frac{t}{\bar{t}_+}\varphi(\bar{t}_+), \quad t \in (0, \bar{t}_+),$$

obtained from the convexity of φ on $[0, \bar{t}_+]$. *Inequality a) holds for each $t \in \operatorname{dom}\varphi$.* For $t \in \operatorname{dom}\varphi \setminus \{0\} = (\bar{t}_-, 0) \cup (0, \bar{t}_+)$ this follows from the convexity of φ restricted to each of the intervals $(\bar{t}_-,\, 0)$ and $(0,\, \bar{t}_+)$. For $t = 0$ from condition 2^0 we have

$$\varphi_+(0,\, 1) + \varphi_+(0,\, -1) = f'(0, x) + f'(0, -x) \geq 0.$$

In b) the inequality $\varphi''_+(t,\, 1) = f''_+(tx,\, x) \geq 0$*,* $t \in \operatorname{dom} \varphi$, follows from the convexity of f^x on the intervals $(-\infty,\, 0]$ and $[0, +\infty)$, and it is satisfied both if $\varphi'_+(t,\, 1) + \varphi'_+(t,\, -1) = 0$ holds and if does not. Now Lemma 3.1 gives that the function φ is convex.

We face the following situation. The function f^x is convex on each of the overlapping open intervals $(-\infty,\, 0)$, $(\bar{t}_-,\, \bar{t}_+)$, $(0, +\infty)$, that is it is locally convex. It is a known fact that then f^x is convex on $\mathbb{R}$. □

Corollary 3.3. *Let f be a PH function. Then the condition that f is CAL is equivalent to $f(x) + f(-x) \geq 0$ for all $x \in \mathbb{R}^n$.*

The following Proposition 3.4 needs to be noticed. It follows directly from the subsequent Lemma 3.5.

Proposition 3.4. *The lower affine approximation of a CAL function is a CAL function.*

Lemma 3.5. *Let f be a CAL function and let $a \leq f(0)$. Let b^a be the PH function defined by* (1)*. Then b^a is a CAL function.*

Proof. Let $x \in \mathbb{R}^n$, $x \neq 0$. The convexity of f^x implies for $\lambda, \mu > 0$ the inequality

$$\frac{1}{\lambda}\,(f(\lambda x) - a) + \frac{1}{\mu}\,(f(-\mu x) - a) \geq \frac{\lambda + \mu}{\lambda\mu}\,(f(0) - a) \geq 0.$$

Taking infimum with respect to $(\lambda, \mu) \in (0, +\infty) \times (0, +\infty)$ we get immediately $b^a(x) + b^a(-x) \geq 0$. The result follows now from Corollary 3.3. □

4. Calmness

The notion of calmness is well known (see, for example, [8] and references therein). We shall use the following definition: a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is called locally calm at a point $x^0 \in \operatorname{dom} f$ if

$$\operatorname{calm} f(x^0) := \liminf_{x \to x^0} \frac{f(x) - f(x^0)}{\|x - x^0\|} > -\infty. \tag{7}$$

The value $\operatorname{calm} f(x^0)$ is referred to as the local calmness of f at x^0.

To be more precise we need to use the term *locally calm from below*, having in mind that the inequality $\limsup_{x \to x^0} (f(x) - f(x^0))/\|x - x^0\| < +\infty$ can be considered as *locally calmness from above*. However, we consider only (7) in this paper and we omit the words "from below" for the sake of simplicity.

Obviously, a Fréchet differentiable at x^0 function is locally calm. The following assertion is also obvious and we omit its proof.

Proposition 4.1. *If a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is locally calm at a point $x^0 \in \operatorname{dom} f$ then f is lsc at x^0.*

Clearly, the reverse statement does not hold. For PH functions however the following is true.

Proposition 4.2. *A PH function, which is lsc at zero is also locally calm at zero.*

Proof. Take $\varepsilon > 0$ and assume that $\delta > 0$ is such that $\|x\| \le \delta$ implies $f(x) \ge -\varepsilon$. For $\|x\| = \delta$ we have $f(x)/\|x\| \ge -\varepsilon/\delta$ and from the PH property the same inequality is true for all x. Therefore f is calm at zero. □

Later we use also the following assertion.

Proposition 4.3 ([9], see Theorem 5.15). *The local calmness of a lsc PH function at a point $x^0 \neq 0$ is equivalent to the nonemptyness of the subdifferential $\partial_{\mathcal{L}_{n+1}} f(x^0)$.*

We now introduce the notion of global calmness. A function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is said to be globally calm at a point $x^0 \in \operatorname{dom} f$ if

$$\operatorname{Calm} f(x^0) := \inf \left\{ \frac{f(x) - f(x^0)}{\|x - x^0\|} \mid x \in \mathbb{R}^n,\ x \neq x^0 \right\} > -\infty.$$

The value $\operatorname{Calm} f(x^0)$ is referred to as the global calmness of f at x^0. Obviously it holds $\operatorname{Calm} f(x^0) \le \operatorname{calm} f(x^0)$. Therefore if f is globally calm at x^0 it is also locally calm at x^0. The next proposition shows that it can be said more in the case of a CAR function.

Proposition 4.4. *Let a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be CAR and such that $f(0) < +\infty$. Then $\operatorname{calm} f(0) = \operatorname{Calm} f(0)$. Further, if $x^0 \in \operatorname{dom} f$ and f is locally calm at both 0 and x^0, then f is globally calm at x^0.*

Proof. Let $x \in \mathbb{R}^n$. The function $t \to f(tx)$, $t \in [0, +\infty)$, is convex, whence for $t \in (0, 1]$ it holds $f(x) - f(0) \ge \frac{1}{t}(f(tx) - f(0))$ and consequently

$$\frac{1}{\|x\|}(f(x) - f(0)) \ge \liminf_{t \to 0^+} \frac{1}{\|tx\|}(f(tx) - f(0)) \ge \operatorname{calm} f(0).$$

Therefore $\operatorname{Calm} f(0) \ge \operatorname{calm} f(0)$. Since the converse inequality is obviously true, we get $\operatorname{Calm} f(0) = \operatorname{calm} f(0)$.

Let $x^0 \in \operatorname{dom} f$ be fixed. We show that if f is locally calm at both 0 and x^0 then f is globally calm at x^0. The identity of the local and global calmness at 0 proves this assertion for $x^0 = 0$. Let now $x^0 \neq 0$. Since f is locally calm at x^0, then for any $c < \operatorname{calm} f(x^0)$ there exists $\delta > 0$ such that $(f(x) - f(x^0))/\|x - x^0\| > c$ for $\|x - x^0\| < \delta$. Let now $\|x - x^0\| \ge \delta$. Then

$$\frac{f(x) - f(x^0)}{\|x - x^0\|} = \frac{f(x) - f(0)}{\|x - x^0\|} - \frac{f(x^0) - f(0)}{\|x - x^0\|}.$$

In order to check the global calmness of f at x^0 we need to show that the two terms in the right-hand side are bounded from below.

For the second term we have

$$-\frac{f(x^0) - f(0)}{\|x - x^0\|} \ge -\frac{1}{\|x - x^0\|}\,|f(x^0) - f(0)| \ge -\frac{1}{\delta}\,|f(x^0) - f(0)|.$$

For the first term we have

$$\begin{aligned}\frac{f(x)-f(0)}{\|x-x^0\|} &= \frac{\|x\|}{\|x-x^0\|}\,\frac{f(x)-f(0)}{\|x\|} \\ \geq\ & \frac{\|x\|}{\|x-x^0\|}\operatorname{calm} f(0) \geq -\frac{\|x\|}{\|x-x^0\|}\,|\operatorname{calm} f(0)| \geq -A\,|\operatorname{calm} f(0)|,\end{aligned}$$

where

$$A = \sup\left\{\frac{\|x\|}{\|x-x^0\|} \mid \|x-x^0\| \geq \delta\right\} \leq \frac{\|x^0\|+\delta}{\delta} < +\infty.$$

The above estimation for A follows from the inequalities

$$\frac{\|x\|}{\|x-x^0\|} \leq \frac{\|x^0\|+\|x-x^0\|}{\|x-x^0\|} \leq \frac{\|x^0\|}{\|x-x^0\|}+1 \leq \frac{\|x^0\|}{\delta}+1.$$

□

Proposition 4.4 gives relations between local and global calmness for CAR functions. In the case of PH functions we can say more.

Proposition 4.5. *Let a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be PH. Let $x^0 \in \operatorname{dom} f$. Then for all $t > 0$ it holds $\operatorname{calm} f(x^0) = \operatorname{calm} f(tx^0)$ and $\operatorname{Calm} f(x^0) = \operatorname{Calm} f(tx^0)$.*

Proof. We consider only the global calmness, the equality for the local calmness is derived similarly. Applying the positive homogeneity we get

$$\begin{aligned}\operatorname{Calm} f(x^0) &= \inf_{x\neq x^0}\frac{f(x)-f(x^0)}{\|x-x^0\|} = \inf_{x\neq x^0}\frac{f(tx)-f(tx^0)}{\|tx-tx^0\|} \\ &= \inf_{x\neq tx^0}\frac{f(x)-f(tx^0)}{\|x-tx^0\|} = \operatorname{Calm} f(tx^0).\end{aligned}$$

□

Proposition 4.6. *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a PH function. Let $x^0 \in \operatorname{dom} f$ and f be locally calm at both 0 and x^0. Then $\inf_{t\geq 0}\operatorname{Calm} f(tx^0) > -\infty$. If also $-x^0 \in \operatorname{dom} f$ and $\operatorname{calm} f(-x^0) > -\infty$, then $\inf_{t\in\mathbb{R}}\operatorname{Calm} f(tx^0) > -\infty$.*

Proof. According to Proposition 4.4 it holds $\operatorname{Calm} f(0) > -\infty$ and $\operatorname{Calm} f(x^0) > -\infty$. According to Proposition 4.5 $\operatorname{Calm} f(tx^0) = \operatorname{Calm} f(x^0)$ for $t > 0$, hence $\inf_{t\geq 0}\operatorname{Calm} f(tx^0) > -\infty$. In the case of $x^0 \neq 0$, $-x^0 \in \operatorname{dom} f$ and $\operatorname{calm} f(-x^0) > -\infty$ we have also $\operatorname{Calm} f(-tx^0) = \operatorname{Calm} f(-x^0) > -\infty$ for all $t > 0$ and finally $\inf_{t\in\mathbb{R}}\operatorname{Calm} f(tx^0) > -\infty$. □

We now show that lower semicontinuity at the origin of a CAR function implies globally calmness at zero of lower affine approximations of this function.

Proposition 4.7. *Let a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be lsc at 0 and CAR with $f(0) < +\infty$. Then for each $a < f(0)$ the PH function b^a defined by (1) is globally calm at 0. More precisely, $\operatorname{Calm} b^a(0) \geq -C$, where $C = (1/\delta_0)(f(0)-a)$ with $\delta_0 > 0$ defined so that $f(x) > a$ for $\|x\| \leq \delta_0$ (the existence of δ_0 follows from $a < f(0)$ and from f lsc at 0).*

Proof. Fix $x \in \mathbb{R}^n \setminus \{0\}$. We show that

$$\frac{b^a(x) - b^a(0)}{\|x\|} = \frac{b^a(x)}{\|x\|} \geq -C. \tag{8}$$

For this purpose we prove first that

$$f(tx) \geq -\frac{f(0) - a}{\delta_0}\, t\|x\| + f(0) \quad \text{for all } t > \frac{\delta_0}{\|x\|}. \tag{9}$$

Suppose on the contrary, that for some $t > \delta_0/\|x\|$ the opposite inequality is true. From the convexity of f on the ray $\{sx \mid s \geq 0\}$ we get

$$\begin{aligned} f\left(\frac{\delta_0}{\|x\|}\, x\right) &= f\left(\left(1 - \frac{\delta_0}{t\|x\|}\right) 0 + \frac{\delta_0}{t\|x\|}\, tx\right) \leq \left(1 - \frac{\delta_0}{t\|x\|}\right) f(0) + \frac{\delta_0}{t\|x\|}\, f(tx) \\ &< \left(1 - \frac{\delta_0}{t\|x\|}\right) f(0) + \frac{\delta_0}{t\|x\|} \left(-\frac{f(0) - a}{\delta_0}\, t\|x\| + f(0)\right) = a. \end{aligned}$$

The obtained inequality contradicts however the inequality $f((\delta_0/\|x\|)\, x) > a$ which is a consequence of $\|\, (\delta_0/\|x\|)\, x\, \| = \delta_0$.

Now we show that

$$\frac{1}{t}\, (f(tx) - a) \geq -\frac{f(0) - a}{\delta_0}\, \|x\| \quad \text{for all } t > 0. \tag{10}$$

For $t > \delta_0/\|x\|$ this follows straightforward from (9). For $0 < t \leq \delta_0/\|x\|$ this follows from $f(tx) - a > 0$. Inequality (10) gives $b^a(x) \geq -((f(0) - a)/\delta_0)\, \|x\|$ and straightforward (8). In consequence $\operatorname{Calm} b^a(0) \geq -C$. □

Corollary 4.8. *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be PH and let f be lsc at 0. Then*

$$\operatorname{Calm} f(0) := \inf_{x \neq 0} \frac{f(x)}{\|x\|} > -\infty. \tag{11}$$

Proof. Since f is PH, it follows that $b^a = f$ for all $a < 0 = f(0)$. Now our assertion follows from Proposition 4.6. This assertion can be proved easily also in a direct way. Looking again at the proof of Proposition 4.2, we see that in fact we have demonstrated there not only the local calmness, but also the global calmness of f at zero. □

The lower semicontinuity of a PH function which satisfies (11) holds as a direct consequence of Propositions 4.4 and 4.1.

Proposition 4.9. *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be PH, $0 \in \operatorname{dom} f$ and let (11) hold. Then f is lsc at zero.*

The property of a PH function to be lsc does not imply the calmness at the points $x \neq 0$. We give now a simple example that confirms this assertion.

Example 4.10. The function

$$f : \mathbb{R}^2 \to \mathbb{R}, \quad f(x_1, x_2) = \begin{cases} -\sqrt{|x_1 x_2|}\,, & x_1 \geq 0, \\ \sqrt{|x_1 x_2|}\,, & x_1 < 0, \end{cases}$$

is continuous and CAL. Moreover, f is linear on each 1-dimensional subspace of $\mathbb{R}^2$ (in other words f is linear-along-lines), but f is not locally calm at the nonzero points of the positive x_1-semiaxis and the x_2-axis (and it is locally calm at all the remaining points of $\mathbb{R}^2$).

It is obvious that f is continuous and linear-along-lines. From the following estimation it is clear that f is not locally calm at the points $(x_1^0, 0)$, $x_1^0 > 0$.

$$\operatorname{calm} f(x_1^0, 0) \leq \liminf_{x_2 \to 0^+} \frac{f(x_1^0, x_2) - f(x_1^0, 0)}{x_2} = \liminf_{x_2 \to 0^+} -\left(\frac{x_1^0}{x_2}\right)^{1/2} = -\infty.$$

Similar estimations show that f is not locally calm at the nonzero points of the x_2-axis. For $x^0 = (x_1^0, 0)$, $x_1^0 < 0$, we have $\operatorname{calm} f(x^0) = 0$. This equality follows from $f(x) \geq 0$ for $x = (x_1, x_2)$ satisfying $\|x - x^0\| \leq -x_1^0$, while $f(x) = 0$ for x on the x_1-axis. At the origin $x^0 = (0, 0)$ we get straightforward from the definition that $\operatorname{calm} f(0, 0) = -1/\sqrt{2}$. At all the remaining points of $\mathbb{R}^2$ the function f is differentiable, hence locally calm.

5. $\mathcal{L}_n$-subdifferentiability: necessary conditions

First we prove a statement showing that the global calmness is a necessary condition for $\mathcal{L}_k$-subdifferentiability.

Proposition 5.1. *Let function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be $\mathcal{L}_k$-subdifferentiable at $x^0 \in \operatorname{dom} f$ with $k \geq 1$. Then f is globally calm (hence locally calm) at x^0.*

Proof. Let $l \in \partial_{\mathcal{L}_k} f(x^0)$. Then there exist $l_1, \dots l_k \in \mathbb{R}^n$ such that $l(x) = \min_{i=1,\dots,k} \langle l_i, x\rangle$. We have also $f(x) - f(x^0) \geq l(x) - l(x^0)$. The function $l(x) = \min_{i=1,\dots,k} \langle l_i, x\rangle$ is superlinear. Let $\|l\| = \max_{\|x\| \leq 1} |l(x)|$. It is well-known that $|l(x) - l(y)| \leq \|l\| \|x - y\|$ for all $x, y \in \mathbb{R}^n$. Since $f(x) - f(x^0) \geq l(x) - l(x^0)$ for all $x \in \mathbb{R}^n$, we have:

$$\begin{aligned} \operatorname{Calm} f(x^0) &= \inf_{x \neq x^0} \frac{f(x) - f(x^0)}{\|x - x^0\|} \geq \inf_{x \neq x^0} \frac{l(x) - l(x^0)}{\|x - x^0\|} \\ &\geq -\sup_{x \neq x^0} \frac{|l(x) - l(x^0)|}{\|x - x^0\|} \geq -\|l\| > -\infty. \end{aligned}$$

□

The following statement gives a necessary condition for a function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ with $0 \in \operatorname{dom} f$ to be $\mathcal{H}_n$-convex with nonempty $\mathcal{L}_n$-subdifferential.

Theorem 5.2. *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be such that $f(0) < +\infty$ and f is $\mathcal{H}_n$-convex at the points at which it is infinite. Then if f is $\mathcal{L}_n$-subdifferentiable on $\mathbb{R}^n$, it is CAR and globally calm (hence locally calm) at each $x^0 \in \operatorname{dom} f$ and there exists a 1-dimensional subspace $L \subset \mathbb{R}^n$, such that the restriction $f|_L$ of f on L is a convex function.*

Proof. 1^0. Global calmness follows from Proposition 5.1.

2^0. The function f is CAR according to Proposition 2.4.

3^0. Since f is $\mathcal{L}_n$-subdifferentiable at 0, it holds $f(x) - f(0) \geq \ell(x) - \ell(0) = \ell(x)$ for all $x \in \mathbb{R}^n$, where $\ell(x) = \min_{1 \leq i \leq n} \langle l_i, x \rangle$. The homogeneous system of $(n-1)$ linear equations of n variables

$$\langle l_1, x \rangle = \langle l_2, x \rangle = \cdots = \langle l_n, x \rangle. \tag{12}$$

possesses a solution $u^0 \neq 0$. Put $L = \{tu^0 \mid t \in \mathbb{R}\}$. Then L is 1-dimensional subspace, for which $f(x) \geq f(0) + \langle l, x \rangle$ for $x \in L$, where l stands for any of the vectors l_i, $i = 1, \ldots, n$. We get from here for $x \in L$ the inequality $f(x) + f(-x) \geq 2f(0)$. According to Proposition 3.2 we get that the restriction of f to L is convex. □

The next example shows that the necessary conditions for $\mathcal{L}_n$-subdifferentiability obtained in Theorem 5.2 are not sufficient, that is Theorem 5.2 does not admit a reversal. We use there the following proposition.

Proposition 5.3 ([9], Proposition 7.15, p. 292). *Let f be a PH function and L be a set of PH functions. Then f is H_L-convex if and only if f is L-convex.*

Example 5.4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1, x_2) = \begin{cases} x_1^2 / \sqrt{x_1^2 + x_2^2}, & x_1 > 0, \\ 0, & x_1 = 0, \\ -2x_1^2 / \sqrt{x_1^2 + x_2^2}, & x_1 < 0. \end{cases}$$

The function f is finite and PH. It is CAR, globally calm at each $x^0 \in \mathbb{R}^2$, and its restriction to the 1-dimensional subspace $L = \{(x_1, x_2) \mid x_1 = 0\}$ is identically zero, hence convex. However, f is not $\mathcal{L}_2$-subdifferentiable at the points of the set $L_+ = \{(x_1, x_2) \mid x_1 > 0\}$. Moreover, f is not $\mathcal{H}_2$-convex at these points.

It is obvious that f is finite, PH and with a restriction to L being identically zero. The global calmness of f can be proved by means of Proposition 4.4. Indeed, at $x = 0$, one has

$$\frac{f(x) - f(0)}{\|x - 0\|} = \begin{cases} \dfrac{x_1^2}{x_1^2 + x_2^2}, & x_1 > 0, \\ 0, & x_1 = 0, \\ \dfrac{-2x_1^2}{x_1^2 + x_2^2}, & x_1 < 0. \end{cases}$$

Hence

$$\frac{f(x) - f(0)}{\|x - 0\|} \geq \frac{-2x_1^2}{x_1^2 + x_2^2} \quad \forall x \in \mathbb{R}^2,$$

which implies $\liminf_{x \to 0} \dfrac{f(x) - f(0)}{\|x - 0\|} \geq -2 > -\infty$, hence calm $f(0) > -\infty$. By Proposition 4.4, the latter means also Clam $f(0) > -\infty$. Further f is locally calm at every $x \neq 0$, since it is Fréchet differentiable. Again, by Proposition 4.4, we conclude global camlmness at every $x \in \mathbb{R}^2$.

Now we prove that f is not $\mathcal{H}_2$-convex at the points $x \in L_+$. Since any $\mathcal{H}_2$-convex PH function is $\mathcal{L}_2$-convex (see Proposition 5.3), it is enough to show that f is not $\mathcal{L}_2$-convex. This follows from the following property: *If $\ell(x) = \min_{1\le i\le 2} \langle l_i, x\rangle$ satisfies the inequality $\ell(x) \le f(x)$ for all $x \in \mathbb{R}^2$, then $\ell(x) \le 0$ at the points $x \in L_+$.* Indeed, for $x \in L_+$ obviously $f(x) > 0$, therefore $f(x)$ cannot be a supremum of the set $\{\ell(x) \mid \ell \in \mathcal{L}_2\}$. To prove the property, let

$$\langle l_1, \bar{x}\rangle = \langle l_2, \bar{x}\rangle \quad \text{for some } \bar{x} = (\bar{x}_1, \bar{x}_2). \tag{13}$$

Then also $\langle l_1, t\bar{x}\rangle = \langle l_2, t\bar{x}\rangle$ for all $t \in \mathbb{R}$. Therefore we may assume without loss of generality that $\bar{x}_1 \ge 0$. Now we have

$$-f(-\bar{x}) \le \ell(\bar{x}) \le f(\bar{x}), \tag{14}$$

whence $2\bar{x}_1^2/\sqrt{\bar{x}_1^2 + \bar{x}_2^2} \le \bar{x}_1^2/\sqrt{\bar{x}_1^2 + \bar{x}_2^2}$. This inequality can hold only if $\bar{x}_1 = 0$. Let $l_i = (\alpha_i, \beta_i)$, $i = 1, 2$. From (13) we get $\beta_1 = \beta_2 := \beta$ and (14) gives $\beta = 0$. Therefore $\ell(x) = \min\{\alpha_1 x_1, \alpha_2 x_1\}$. For $x \in L_+$ applying (14) we obtain

$$\ell(x) = \min\{\alpha_1 x_1, \alpha_2 x_1\} \le \frac{x_1^2}{\sqrt{x_1^2 + x_2^2}}.$$

Since $\ell(x)$ does not depend on x_2, we may consider x_1 on the left-hand side of the above inequality a fixed point and vary with x_2 on the right-hand side. Passing to a limit with $x_2 \to \infty$ we get $\ell(x) \le 0$.

A natural generalization of the appearing in Theorem 5.2 conclusion that *the restriction $f|_L$ of f to some 1-dimensional subspace is convex*, or in other words that the defined in Section 3 *function f^x is convex for some $x \ne 0$*, is the condition that *f^x is convex for all x*, which means that f is CAL. This observation leads to the question, whether the assumptions of Theorem 5.2 imply that f is CAL. The next example gives a negative answer to this question.

Example 5.5. Let $l_1, \dots, l_n$ with $n > 1$ be linearly independent vectors in $\mathbb{R}^n$. Then the function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = \min_{1\le i\le n}\langle l_i, x\rangle$ is $\mathcal{L}_n$-subdifferentiable, but not CAL.

Since $f \in \mathcal{L}_n$, f is $\mathcal{L}_n$-subdifferentiable. Let $\bar{x} \ne 0$ be a solution of system (12) and let L be the 1-dimensional linear space spanned on $\bar{x}$. Then there exist indexes i_0, i_1 and $x^0 \ne \bar{x}$ such that $f(x^0) = \min_{1\le i\le n}\langle l_i, x^0\rangle = \langle l_{i_0}, x^0\rangle$ and $\langle l_{i_1}, x^0\rangle > \langle l_{i_0}, x^0\rangle$. (Indeed, the system of linear equations $\langle l_i - l_n, x\rangle = 0$, $i = 1, \dots, n-1$, is of rank $n-1$, whence its solutions form 1-dimensional subspace. The rank is $n-1$, since the vectors $l_i - l_n$, $i = 1, \dots, n-1$, are linearly independent as a consequence of the linear independence of the vectors l_i, $i = 1, \dots, n$.) Now $f(-x^0) = \min_{1\le i\le n}\langle l_i, -x^0\rangle \le \langle l_{i_1}, -x^0\rangle$ and

$$f(-x^0) + f(x^0) \le \langle l_{i_1}, -x^0\rangle + \langle l_{i_0}, x^0\rangle < 0.$$

The function f is however PH and in virtue of Corollary 3.3 the above inequality shows that f is not CAL.

Thus, an $\mathcal{L}_n$-subdifferentiable function need not be CAL. The things change however, if instead we confine to $\mathcal{L}_n^0$-subdifferentiability.

Theorem 5.6. *Let the function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be such that $f(0) < +\infty$ and f is $\mathcal{H}_n$-convex at the points at which it is infinite. Suppose that f is $\mathcal{L}_n^0$-subdifferentiable on $\mathbb{R}^n$. Then f is CAL and globally calm (hence locally calm) at each $x^0 \in \operatorname{dom} f$.*

Proof. Since f is $\mathcal{L}_n^0$-subdifferentiable function, it is also $\mathcal{L}_n$-subdifferentiable and according to Theorem 5.2 also CAR and globally calm at each $x^0 \in \operatorname{dom} f$. In particular f is globally calm and lsc at $0 \in \operatorname{dom} f$. Take arbitrary $u \in \mathbb{R}^n \setminus \{0\}$. In order to prove that f is CAL according to Proposition 3.2 it is enough to show that $f'(0,u) + f'(0,-u) \geq 0$. If on at least one of the sets $\{tu \mid t > 0\}$ or $\{tu \mid t < 0\}$ the function f is identically $+\infty$, then from f CAR we get that f restricted to the set $\{tu \mid t \in \mathbb{R}\}$ is convex and the above inequality follows from Proposition 3.2 (the inequality is a necessary condition for the restricted function to be convex). Assume now that f is not identically $+\infty$ on any of the two sets. Then there exists $t_0 > 0$, such that $[-t_0 u, t_0 u] \subset \operatorname{int} \operatorname{dom} f$. Let $0 < t < t_0$. Since f is $\mathcal{L}_n^0$-subdifferentiable at $tu \in \operatorname{dom} f$, there exists $\ell^t \in \mathcal{L}_n$ with $\ell^t(x) = \min_{1 \leq i \leq n} \langle l_i^t, x\rangle$ such that $\ell^t(x) - \ell^t(tu) \leq f(x) - f(tu)$ and $\langle l_1^t, tu\rangle = \cdots = \langle l_n^t, tu\rangle$. These equalities show that ℓ^t restricted to L is linear, where L is the linear space spanned by u. Adding the inequalities

$$\ell^t(t_0 u - tu) \leq f(t_0 u) - f(tu),$$
$$\ell^t(-t_0 u - tu) \leq f(-t_0 u) - f(tu).$$

we get

$$2f(tu) \leq f(t_0 u) + f(-t_0 u) + 2t\, \ell^t(u),$$

From

$$(t_0 - t)\ell^t(u) = \ell^t(t_0 u) - \ell^t(tu) \leq f(t_0 u) - f(tu)$$

we obtain

$$\ell^t(u) \leq -\frac{f(t_0 u - (t_0 - t)u) - f(t_0 u)}{t_0 - t} \leq -f'(t_0 u, -u) < +\infty.$$

The estimation with the directional derivative follows from standard properties of the convex function $f(\lambda u)$, $0 \leq \lambda \leq 1$. The finiteness of the directional derivative comes from $t_0 u \in \operatorname{int} \operatorname{dom} f$. Therefore

$$2f(tu) \leq f(t_0 u) + f(-t_0 u) - 2t\, f'(t_0 u, -u).$$

Passing to a limit with $t \to 0^+$ and using that f is lsc at 0 we get

$$2f(0) \leq 2 \liminf_{t \to 0^+} f(tu) \leq f(t_0 u) + f(-t_0 u).$$

The obtained inequality can be written as

$$\frac{1}{t_0}\,(f(t_0 u) - f(0)) + \frac{1}{t_0}\,(f(-t_0 u) - f(0)) \geq 0.$$

Passing to a limit with $t_0 \to 0^+$ we get $f'(0,u) + f'(0,-u) \geq 0$. We have shown that property 2^0 in Proposition 3.2 is satisfied, whence f is CAL. □

Thus, Theorem 5.6 shows that the $\mathcal{L}_n^0$-subdifferentiable functions with $0 \in \operatorname{dom} f$ which are $\mathcal{H}_n$-convex at the points where they are infinite are necessary CAL functions. A natural question arises whether Theorem 5.6 can be reverted. This problem for CAL PH functions is investigated in Section 6.

6. $\mathcal{L}_n^0$-subgradients and $\mathcal{L}_n^0$-convexity for CAL PH functions

In this section we prove existence of $\mathcal{L}_n^0$-subgradients for CAL PH functions and investigate $\mathcal{H}_n^0$-convexity for such functions, or more precisely, their $\mathcal{L}_n^0$-convexity. Let us however underline, that as in Proposition 5.3 it holds: A PH function is $\mathcal{H}_n^0$-convex at x^0 if and only if it is $\mathcal{L}_n^0$-convex at x^0.

We need the following construction. Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a PH function and $x^0 \in \operatorname{dom} f \setminus \{0\}$. Consider the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ defined by

$$\tilde{f}(x) = \begin{cases} -f(-x), & x = \lambda x^0,\ \lambda < 0, \\ f(x), & \text{otherwise.} \end{cases} \tag{15}$$

If f is CAL then due to Corollary 3.3 we have $f(x) \geq -f(-x)$ for all x, hence $\tilde{f} \leq f$. Clearly $\tilde{f}$ is also a CAL and PH function. Note that $-x^0 \in \operatorname{dom} \tilde{f}$. Assume that the function $\tilde{f}$ has a $\mathcal{L}_n^0$-subgradient ℓ at the point x_0. Since $\tilde{f} \leq f$ and $\tilde{f}(x^0) = f(x^0)$ it follows that ℓ is also a $\mathcal{L}_n^0$-subgradient of f at x^0. At the same time the equalities $\ell(-x^0) = -\ell(x^0) = -\tilde{f}(x^0) = \tilde{f}(-x^0)$ demonstrate that ℓ is $\mathcal{L}_n^0$-subgradient of $\tilde{f}$ at $-x^0$. Assume that f is $\mathcal{L}_n^0$-subdifferentiable at $x^0 \in \operatorname{dom} f \setminus \{0\}$. Then $\tilde{f}$ is $\mathcal{L}_n^0$-subdifferentiable at both x^0 and $-x^0$, hence (see Proposition 5.1) $\tilde{f}$ is locally calm at x^0 and $-x^0$. The local calmness at $-x^0$ means that

$$\liminf_{x \to -x^0} \frac{f(x) + f(x^0)}{\|x + x^0\|} > -\infty.$$

In Theorem 6.2 below we prove that the local calmness of $\tilde{f}$ at x^0 and $-x^0$ implies $\mathcal{L}_n^0$-subdifferentiability of f at x^0. In the next theorem we establish first a special case of this result.

Theorem 6.1. *Let f be a lsc at zero CAL PH function. Let $x^0 \neq 0$ be such that $x^0 \in \operatorname{dom} f$, $-x^0 \in \operatorname{dom} f$ and f is locally calm at both x^0 and $-x^0$. Then f is $\mathcal{L}_n^0$-subdifferentiable at x^0.*

Proof. Since the PH function f is lsc at zero it follows from Corollary 4.8 that f is globally calm at zero. From Proposition 4.4 and the made assumptions it follows that f is globally calm at x^0 and $-x^0$. It follows from (5) that a $\mathcal{L}_n^0$-subgradient of f at a point x is a $\mathcal{L}_n^0$-subgradient of f at a point μx with $\mu > 0$. Thus we can assume without loss of generality that $\|x^0\| = 1$. Let $L = \{\lambda x^0 : \lambda \in \mathbb{R}\}$ be the straight line going through zero and x^0. Due to Corollary 3.3 we have $-f(-x^0) \leq f(x^0)$. It follows from this inequality that the linear function p defined on $\mathbb{R}$ by $p(\lambda) = f(x^0)\lambda$ enjoys the following properties:

$$p(\lambda) \leq f(\lambda x^0), \quad \lambda \in \mathbb{R} \quad \text{and} \quad p(1) = f(x^0).$$

Consider now function (15) and set along the proof for notational simplicity $g = \tilde{f}$. The function g is PH and CAL with $g(x) \leq f(x)$ for all x and $g(x) = f(x)$ for $x \notin L$. We have

$$\operatorname{Calm} g(0) = \inf_{x \neq 0} \frac{g(x)}{\|x\|} = \min\left(\inf_{x \notin L} \frac{f(x)}{\|x\|},\ f(x^0), -f(x^0) \right) > -\infty.$$

In view of Proposition 4.9 we can assert that g is lsc at zero. Having in mind that $\|x^0\| = 1$ we get:

$$\operatorname{Calm} g(x^0) = \min\left(\inf_{x\in L,\, x\neq x^0} \frac{g(x)-g(x^0)}{\|x-x^0\|},\ \inf_{x\notin L} \frac{g(x)-g(x^0)}{\|x-x^0\|} \right)$$
$$= \min\left(\inf_{\lambda\neq 1} \frac{(\lambda-1)f(x^0)}{|\lambda-1|},\ \inf_{x\notin L} \frac{f(x)-f(x^0)}{\|x-x^0\|} \right) \geq \min\left(-|f(x^0)|, \operatorname{Calm} f(x^0)\right) > -\infty.$$

Now we estimate $\operatorname{Calm} g(-x^0)$. Since f is CAL and PH, we have $-g(-x^0) = f(x^0) \geq -f(-x^0)$, so

$$\operatorname{Calm} g(-x^0) = \min\left(\inf_{x\in L,\, x\neq x^0} \frac{g(x)-g(-x^0)}{\|x+x^0\|},\ \inf_{x\notin L} \frac{g(x)-g(-x^0)}{\|x+x^0\|} \right)$$
$$\geq \min\left(\inf_{\lambda\neq -1} \frac{(\lambda+1)f(x^0)}{|\lambda+1|},\ \inf_{x\notin L} \frac{f(x)-f(-x^0)}{\|x-(-x^0)\|} \right) \geq \min\left(-|f(x^0)|, \operatorname{Calm} f(-x^0)\right) > -\infty.$$

Due to Proposition 4.6 we have

$$\inf_{x\in L} \operatorname{Calm} g(x) \geq -C > -\infty. \tag{16}$$

Consider the subspace $M = \{x \in \mathbb{R}^n \mid \langle x^0, x\rangle = 0\}$ of $\mathbb{R}^n$ orthogonal to the vector x^0. Since M is a $n-1$ dimensional space we can find n vectors $m_1, \dots, m_n$ in M such that their convex hull S, which is a simplex, contains the ball $B = \{x \in M : \|x\| \leq \varepsilon\}$. Let $q(x) = \max_{i=1,\dots,n}\langle m_i, x\rangle$ be the support function of S. Since $S \supset B$ and the support function of B is equal to $\varepsilon\|x\|$ for $x \in M$, it follows that

$$q(x) := \max_{1\leq i\leq n} \langle m_i, x\rangle \geq \varepsilon\|x\|, \quad x \in M. \tag{17}$$

Fix $x \in \mathbb{R}^n$ and let $\bar{x} = \langle x^0, x\rangle\, x^0$ be the orthogonal projection of x on L. We have

$$g(\bar{x}) = f(x^0)\,\langle x^0, x\rangle. \tag{18}$$

Since $\bar{x} \in L$ we can apply (16). Then

$$g(x) - g(\bar{x}) \geq -C\,\|x-\bar{x}\|.$$

Due to (17) and $x - \bar{x} \in M$ we get

$$\|x-\bar{x}\| \leq \frac{1}{\varepsilon} \max_{1\leq i\leq n} \langle m_i, x-\bar{x}\rangle,$$

so

$$g(x) - g(\bar{x}) \geq -C\,\|x-\bar{x}\| \geq -\frac{C}{\varepsilon} \max_{1\leq i\leq n} \langle m_i, x-\bar{x}\rangle.$$

Since $m_i \in M$, $i = 1, \dots, n$, and $\bar{x}$ belongs to the subspace L being orthogonal to M, it follows that $\langle m_i, \bar{x}\rangle = 0$ or $i = 1, \dots, n$. Using these equalities and (18) we get

$$f(x) \geq g(x) = (g(x) - g(\bar{x})) + g(\bar{x}) \geq -\frac{C}{\varepsilon} \max_{1\leq i\leq n} \langle m_i, x\rangle + f(x^0)\langle x^0, x\rangle.$$

Let $l_i = -(C/\varepsilon)\, m_i + f(x^0)x^0$, $i = 1, \dots, n$. The above inequalities can be written as

$$f(x) \geq \min_{1\leq i\leq n} \langle l_i, x\rangle, \quad x \in \mathbb{R}^n.$$

and from the definition of l_i we have

$$\langle l_1,\, x^0\rangle = \cdots = \langle l_n,\, x^0\rangle = f(x^0)\langle x^0,\, x^0\rangle = f(x^0).$$

Therefore, we have proved that the function $\ell(x) = \min_{1\leq i\leq n}\langle l_i, x\rangle$ is a $\mathcal{L}_n^0$-subgradient of f at the point x^0. □

The following theorem rejects some of the limitations of Theorem 6.1.

Theorem 6.2. *Let $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a lsc at zero CAL PH function. Let $x^0 \in \operatorname{dom} f\backslash\{0\}$ be a point such that both* $\operatorname{calm} f(x^0) = \operatorname{calm} \tilde{f}(x^0) > -\infty$ *and* $\operatorname{calm} \tilde{f}(-x^0) > -\infty$*, where $\tilde{f} : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is defined by* (15) *(turn attention that if $x^0 \in \operatorname{dom} f$, then both x^0 and $-x^0$ belong to $\tilde{f}$). Then there exists a $\mathcal{L}_n^0$-subgradient of f at x^0. Consequently, if f is locally calm at each point $x \in \operatorname{dom} f$, then f is $\mathcal{L}_n^0$-subdifferentiable on $\mathbb{R}^n$.*

Proof. Consider the function $\tilde{f}$ defined by (15). Since f is lsc at zero, it easily follows that $\tilde{f}$ is also lsc at zero. Hence, with regard to the assumptions on f, we can apply Theorem 6.1 to the function $\tilde{f}$ obtaining that $\tilde{f}$ has a $\mathcal{L}_n^0$-subgradient ℓ at x^0. It follows from the discussion in the beginning of this section that ℓ is a $\mathcal{L}_n^0$-subgradient of f at x^0.

Let f be locally calm at each point in $\operatorname{dom} f$. To prove its $\mathcal{L}_n^0$-subdifferentiability it remains to show that f is $\mathcal{L}_n^0$-subdifferentiable at zero. If $\operatorname{dom} f = \{0\}$, $\ell \equiv 0$ is a $\mathcal{L}_n^0$-subgradient at 0. Otherwise, if there exists at least one $x^0 \neq 0$ in $\operatorname{dom} f$, then f possesses a $\mathcal{L}_n^0$-subgradients at x^0, and from the PH property it follows that each such $\mathcal{L}_n^0$-subgradient is also a $\mathcal{L}_n^0$-subgradient at zero. We get again that f is $\mathcal{L}_n^0$-subdifferentiable at zero. □

The following property of a PH function f has been used in the proof of Theorem 6.2 and needs specially to be underlined. If ℓ is $\mathcal{L}_n^0$-subgradient of f at x^0 then ℓ is also a $\mathcal{L}_n^0$-subgradient of f at 0. Therefore the $\mathcal{L}_n^0$-subdifferentiability of f at any point $x^0 \neq 0$ implies $\mathcal{L}_n^0$-subdifferentiability of f at 0. Obviously, this property remains true also for other type of abstract subdifferentials as far as the set $\mathcal{L}$ of abstract linear functions consists of PH functions.

The next result involves $\mathcal{L}_n^0$-convexity of PH functions and illustrates an application of Theorem 6.2.

Theorem 6.3. *Let f be a lsc at zero CAL PH function. Let $x^0 \neq 0$ be a point such that f is lsc at x^0 and $-f(x^0) < \liminf_{x\to -x^0} f(x)$. Assume also $f(x^0) + f(-x^0) > 0$. Then f is $\mathcal{L}_n^0$-convex at x^0. If in addition $x^0 \in \operatorname{dom} f$ and f is locally calm at x^0, then f is $\mathcal{L}_n^0$-subdifferentiable at x^0.*

Proof. Choose $y^0 \in \mathbb{R}$ such that

$$-f(x^0) < -y^0 < \liminf_{x\to -x^0} f(x). \tag{19}$$

Consider the function g defined on $\mathbb{R}^n$ by

$$g(x) = \begin{cases} \lambda y^0, & x = \lambda x^0,\ \lambda \in \mathbb{R}, \\ f(x), & \text{otherwise.} \end{cases}$$

Then g is a PH function. We have $g(x)+g(-x)\geq 0$ for all $x\in\mathbb{R}^n$. Hence by Corollary (3.3) g is a CAL function. We indicate some properties of g.

1) $g(x)\leq f(x)$ *for all* $x\in\mathbb{R}^n$.

Indeed, we need only to check that $g(\lambda x^0)\leq f(\lambda x^0)$ for $\lambda\in\mathbb{R}$. Using (19) we derive the following: if $\lambda\geq 0$ then $g(\lambda x^0)=\lambda y^0\leq\lambda f(x^0)=f(\lambda x^0)$; if $\lambda\leq 0$ then $g(\lambda x^0)=-|\lambda|y^0\leq|\lambda|f(-x^0)=f(\lambda x^0)$.

2) g *is lsc at zero.*

This property follows easily from f lsc at zero, the definition of g, Corollary 4.8 and Proposition 4.9.

3) $\operatorname{calm} g(-x^0)>-\infty$.

Choose $\delta>0$ such that $\|x+x^0\|<\delta$ implies $-y^0<f(x)$. If $0<\|x-x^0\|<\delta$ we have the estimations

$$\frac{g(x)-g(-x^0)}{\|x+x^0\|}\geq\begin{cases}\dfrac{(\lambda+1)y^0}{|\lambda+1|\,\|x^0\|}\geq-\dfrac{|y^0|}{\|x^0\|}, & x=\lambda x^0,\\ \dfrac{f(x)+y^0}{\|x+x^0\|}>0, & \textit{otherwise.}\end{cases}$$

Therefore $\operatorname{calm} g(-x^0)\geq-(|y^0|/\|x^0\|)>-\infty$.

4) $\operatorname{calm} g(x^0)>-\infty$.

We have $y^0<f(x^0)$. Since f is lsc at x^0, there exists $\delta>0$, such that $\|x-x^0\|<\delta$ implies $y^0<f(x)$. If $0<\|x-x^0\|<\delta$ as above we have the estimations

$$\frac{g(x)-g(x^0)}{\|x-x^0\|}\geq\begin{cases}\dfrac{(\lambda-1)y^0}{|\lambda-1|\,\|x^0\|}\geq-\dfrac{|y^0|}{\|x^0\|}, & x=\lambda y^0,\\ \dfrac{f(x)-y^0}{\|x-x^0\|}>0, & \textit{otherwise.}\end{cases}$$

Therefore $\operatorname{calm} g(x^0)\geq-(|y^0|/\|x^0\|)>-\infty$.

Due to Theorem 6.2 applied to the function g, we see that there exist vectors $l_1,\dots,l_n$ such that $\langle l_i,x^0\rangle=y^0$ for all $i=1,\dots,n$ and $\min_{1\leq i\leq n}\langle l_i,x\rangle\leq g(x)\leq f(x)$ for all x. Since in the inequality $-f(x^0)<-y^0$, or equivalently $y^0<f(x^0)$, we can choose y^0 arbitrary close to $f(x^0)$, we see that f is $\mathcal{L}_n^0$-convex at x^0.

Let f be $\mathcal{L}_n$-convex at x^0. We apply the same construction putting

$$-f(x^0)=-y^0<\liminf_{x\to-x^0}f(x).$$

The latter implies that $g(x)=f(x)$, sinse f is PH. Hence we can follow the same steps which led to prove g is $\mathcal{L}_n^0$-subdifferentiable at x^0. The only thing that changes in the above reasonings is the demonstration of 4). Choosing $\varepsilon > 0$ we will have for sufficiently small $\delta > 0$ the inequality

$$\frac{g(x) - g(x^0)}{\|x - x^0\|} \le \operatorname{calm} f(x^0) - \varepsilon,$$

whence we will get $\operatorname{calm} g(x^0) \ge \min\left(\operatorname{calm} f(x^0), -(|y^0|/\|x^0\|)\right) > -\infty$. □

The next theorem is in fact a direct corollary of Theorem 6.3.

Theorem 6.4. *Let a PH function $f : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be lsc on $\mathbb{R}^n$, $0 \in \operatorname{dom} f$ and let $f(x) + f(-x) > 0$ for all $x \neq 0$ (this assumption implies that f is CAL). Then f is $\mathcal{L}_n^0$-convex. If in addition f is locally calm at the points $x \in \operatorname{dom} f$, then f is $\mathcal{L}_n^0$-subdifferentiable on $\mathbb{R}^n$.*

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References

[1] J. M. Borwein, A. S. Lewis: Convex Analysis and Nonlinear Optimization. Theory and Examples, Springer, Berlin (2000).

[2] J. Dutta, J.-E. Martinez-Legaz, A. Rubinov: Monotonic analysis over cones, Optimization, to appear.

[3] I. Ginchev, V. I. Ivanov: Second-order characterizations of convex and pseudoconvex functions, J. Appl. Anal. 9(2) (2003) 261–273.

[4] J.-J. Moreau: Inf-convolution, sous-additivité, convexité des fonctions numériques, J. Math. Pures Appl. 49 (1970) 109–154.

[5] D. Pallaschke, S. Rolewicz: Foundation of Mathematical Optimization (Convex Analysis without Linearity), Kluwer Academic Publishers, Dordrecht (1997).

[6] J.-P. Penot: A duality for starshaped functions, Bull. Pol. Acad. Sci., Math. 50(2) (2002) 127–139.

[7] R. T. Rockafellar: Convex Analysis, Princeton University Press, Princeton (1970).

[8] R. T. Rockafellar, R. J.-B. Wets: Variational Analysis, Springer, Berlin (1998).

[9] A. M. Rubinov: Abstract Convexity and Global Optimization, Kluwer Academic Publishers, Dordrecht (2000).

[10] A. Rubinov, Z. Dzalilov: Abstract convexity of positively homogeneous functions, J. Stat. Manag. Syst. 5(1–3) (2002) 1–20.

[11] A. P. Shveidel: Separability of star-shaped sets and its application to an optimization problem, Optimization 40(3) (1997) 207–227.

[12] A. P. Shveidel: Abstract convex sets with respect to the class of general min-type functions, Optimization 52 (2003) 571–580.

[13] I. Singer: Abstract Convex Analysis, Wiley-Interscience, New York (1997).