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Weak and Proper Efficiency in Set-Valued Optimization on Real Linear Spaces*

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In this paper, we extend the notion of cone-subconvexlikeness of set-valued maps on topological linear spaces to set-valued maps on linear spaces, (that is, general linear spaces without any particular topology), and we provide several characterizations. An alternative theorem is also established for this kind of maps. Using the notion of vector closure introduced recently by Adán and Novo, we also provide, in this framework, an adaptation of the proper efficiency in the sense of Benson for set-valued maps. The previous notion and results are then applied to obtain optimality conditions of weak efficiency and a characterization of Benson proper efficiency by means of scalarization and multipliers rules.

Keywords: Set-valued maps, set-valued optimization, proper efficiency, generalized convexity

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1. Introduction

In the research area of vector optimization one investigates optimization problems involving more than one objective. This kind of optimization problems appear in mathematical modeling of processes ocurring in management science, operations research, networks, industrial systems and control theory. Since the set of (weak) efficient solutions is often too large and since the condition of convexity is too strong, research about proper efficient solutions (see, for example, Borwein [7], Benson [6]) and weaker convexity assumptions (see, for example, Jeyakumar [17], Jahn [15], Khanh [18], Yang [26, 27]) are two basic topics in vector optimization. The usual framework of the above papers is that of partially ordered

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topological linear spaces. In [2, 3, 4, 5], Adán and Novo have extended weak and proper efficiency and cone-convexity to vector optimization problems on partially ordered linear spaces, that is, general linear spaces without any particular topology, using only algebraic concepts. In most cases, the obtained optimality conditions require some generalization of a separation or alternative theorem.

On the other hand, recently, for its extensive application in many fields such as optimal control, fuzzy programming or stochastic programming, several authors have turned their research work to vector optimization problems with set-valued maps, obtaining optimality conditions in terms of set-valued maps. For instance, Corley [9, 10] established several existence results and a Lagrangian duality theory for vector optimization of set-valued maps, Song [24] proved a cone separation theorem and gave optimality conditions for weakened convex valued-maps in locally convex spaces, Lin [23] generalized a Moreau-Rockafellar type theorem and a Farkas-Minkowski type theorem to set-valued maps, and obtained some necessary and sufficient conditions for vector optimization problems with set-valued maps, Li and Chen [22] gave a multipliers rule for the existence of a weak minimizer under convexlikeness, Li [20, 21] extended the concept of cone subconvexlikeness of single-valued map to set-valued maps in topological linear spaces, obtained an alternative theorem, scalarization theorems and Lagrange multipliers theorems to study the weak efficiency and characterized Benson proper efficiency in vector optimization problems with set-valued maps. The same author in [19], working in ordered topological linear spaces and under cone-subconvexlikeness of set-valued maps, established optimality conditions for weak efficiency by using an alternative theorem. More recently, Huang [13] gave a Farkas-Minkowski type alternative theorem under nearly semiconvexlike for set-valued maps and has obtained optimality conditions for set-valued optimization problems.

In this work, we introduce a new concept of proper efficiency in the sense of Benson for an optimization problem with set-valued maps in real linear spaces, and we characterize this concept by scalarization under suitable conditions. Since, in this work our notion of cone-subconvexlikeness for set-valued maps is weaker than Li's definition [20], who works in topological linear spaces, our results can be considered extensions of Li's results [20, 21, 19].

In order to achieve this goal, in Section 2, we introduce the vectorial Benson efficiency by using concepts and results given by Adán and Novo [1, 2, 3, 4, 5]. In Section 3, we extend the notion of cone-subconvexlikeness of set-valued maps on linear spaces and give several characterizations. We establish separation theorems and an alternative theorem for solid convex cones. We also analyze the behaviour of a cone-subconvexlike set-valued map via a positive linear operator. In Section 4, two necessary optimality conditions for weak efficiency are presented under cone-subconvexlikeness for set-valued maps. We prove scalarization theorems and characterize the vectorial Benson efficiency for cone-subconvexlikeness of set-valued maps. In addition, in Section 5, we obtain optimality conditions for some other vectorial efficiency concepts. In Section 6, using a new generalized Slater constraint qualification, we obtain a Lagrange multiplier rule of algebraic type for vector optimization problems with set-valued maps.

2. Notations and preliminaries

Throughout this paper we consider a real linear space Y partially ordered by a convex cone K in Y. Let $A \subset Y$, we denote by $\text{cone}(A)$, $\text{conv}(A)$, $\text{aff}(A)$, $\text{span}(A)$ and $L(A) = \text{span}(A - A)$ the generated cone, convex hull, affine hull, linear hull and associated linear subspace of A, respectively.

In order to avoid topological concepts we use analogous algebraic concepts. The core (algebraic interior) and the intrinsic core (relative algebraic interior) are defined as

$$\text{cor}(A) = \{x \in A : \ \forall v \in Y, \ \exists t_0 > 0, \ \forall t \in [0, t_0], \ x + tv \in A\},$$
$$\text{icr}(A) = \{x \in A : \ \forall v \in L(A), \ \exists t_0 > 0, \ \forall t \in [0, t_0], \ x + tv \in A\}.$$

In the case when $\text{cor}(A) \neq \emptyset$ we say that A is solid. Analogously, we say that A is relatively solid if $\text{icr}(A) \neq \emptyset$. It is clear that if $\text{cor}(A) \neq \emptyset$ then $\text{cor}(A) = \text{icr}(A)$ because $L(A) = Y$.

For finite dimensional spaces, obviously, there exist sets which are not solid but they are relatively solid, for example any segment, ray or line in $\mathbb{R}^2$. For an example in infinite dimension see Example 2.2.

The so called algebraic closure of a set A is defined by

$$\text{lin}(A) = A \cup \{x \in Y : \ \exists a \in A, \ [a, x) \subset A\}.$$

Except for solid convex sets, this concept is not satisfactory as a substitute for topological closure. In order to solve this problem, in [4] Adán and Novo have introduced a weaker closure of algebraic type, which was called vector closure. This vector closure coincides with the algebraic closure for convex sets, and coincides with the topological closure for solid convex sets whenever Y is a real topological linear space.

Definition 2.1. Let A be a nonempty subset of Y. The vector closure of A is the set

$$\text{vcl}(A) = \{b \in Y : \ \exists v \in Y, \ \forall \alpha > 0, \ \exists t \in (0, \alpha], \ b + tv \in A\}.$$

It is clear that $b \in \text{vcl}(A)$ if and only if there exist $v \in Y$ and a sequence $t_n \to 0^+$ such that $b + t_n v \in A$ for all $n \in \mathbb{N}$.

The set A is called vectorially closed if $A = \text{vcl}(A)$.

We say that a cone K is pointed if $K \cap (-K) = \{0\}$. It is well-known that for a convex cone K, whose relative algebraic interior is non-empty, $\text{icr}(K) \cup \{0\}$ is a convex cone, $\text{icr}(K) + K = \text{icr}(K)$ and $\text{icr}(\text{icr}(K)) = \text{icr}(\text{icr}(K) \cup \{0\}) = \text{icr}(K)$.

The algebraic dual of Y is denoted by Y', the positive dual of a subset A of Y is

$$A^+ = \{\varphi \in Y' \colon \varphi(a) \geq 0, \ \forall a \in A\},$$

and $A^{+s} = \{\varphi \in Y' \colon \varphi(a) > 0, \ \forall a \in A \setminus \{0\}\}$ is the strict positive dual of A. It is known that A^+ is a vectorially closed convex cone and

$$[\text{cone}(A)]^+ = \text{cone}(A^+) = [\text{conv}(A)]^+ = \text{conv}(A^+) = A^+.$$

It is clear that $A \subset \text{lin}(A) \subset \text{vcl}(A)$. So, if A is vectorially closed, it will be algebraically closed too.

Other properties that will be used and appear in Adán and Novo [4, 5] are the following: Let $A, B \subset Y$

$$\text{if } A \subset B \Rightarrow \text{vcl}(A) \subset \text{vcl}(B),$$
$$[\text{vcl}(\text{cone}(\text{conv}(A)))]^+ = A^+.$$

If Y is a topological linear space, the interior and the closure of a set A are denoted by $\text{int}(A)$ and $\text{cl}(A)$, respectively. It is clear that $\text{vcl}(A) \subset \text{cl}(A)$.

To illustrate the notions above we give an example in nonfinite dimension.

Example 2.2. Let Y be the vector space of all sequences of real numbers, let S be the subspace of Y of all bounded sequences:

$$S = \{a = (a_n) \in Y : \ \exists \beta > 0 \quad \text{such that } |a_n| \leq \beta \ \forall n\},$$

and let K be the subset of S of all nonnegative sequences:

$$K = \{a = (a_n) \in S : \ a_n \geq 0 \ \forall n\}.$$

It is clear that K is a pointed convex cone. Furthermore, the vector space generated by K is S, i.e., $L(K) = K - K = S$.

For the cone K we have the following facts:

$$\text{(i) } \ \text{icr}(K) = \{a \in K : \ \inf\{a_n\} > 0\},$$
$$\text{(ii) } K \text{ is vectorially closed, i.e., } \ \text{vcl}(K) = K.$$

(i) Let $a \in K$ such that $\delta = \inf\{a_n\} > 0$, and let us see that $a \in \text{icr}(K)$, i.e., that $\forall v \in S = L(K)$, $\exists t_0 > 0$ such that $a + tv \in K$, $\forall t \in (0, t_0]$.

As the sequence $v = (v_n) \in S$, it is bounded, and then there exists $\beta > 0$ such that $|v_n| \leq \beta \ \forall n$. In consequence,

$$-\delta \leq \frac{\delta}{\beta} v_n \leq \delta \quad \forall n.$$

As δ is the infimum of the numbers a_n, we have $\delta \leq a_n \ \forall n$, and therefore

$$0 \leq a_n - \delta \leq a_n - \delta t \leq a_n + \frac{\delta}{\beta} t v_n \leq a_n + \delta t \quad \forall t \in (0, 1].$$

Defining $t_0 = \delta/\beta$ one has that $a + tv \in K$, $\forall t \in (0, t_0]$.

Now choose $a \in \text{icr}(K)$. As $\text{icr}(K) \subset K$ we have $\inf\{a_n\} \geq 0$. Suppose that $\inf\{a_n\} = 0$. Since $a \in \text{icr}(K)$, given $v \in S$ defined by $v_n = -1$ for all $n \in \mathbb{N}$, there exists $t_0 > 0$ such that $a + tv \in K$, $\forall t \in (0, t_0]$, i.e., $a_n - t \geq 0$ for $0 < t \leq t_0$ and for all n. But since we are supposing $\inf\{a_n\} = 0$, for this $t_0 > 0$ there exists n_0 such that $a_{n_0} < t_0$, and this is a contradiction.

(ii) We only have to prove the inclusion $\text{vcl}(K) \subset K$. Let us choose $a \in \text{vcl}(K)$. Then there exist $v \in S$ and a sequence $t_j \to 0^+$ such that $a + t_j v \in K \ \forall j$. Suppose that for some $n \in \mathbb{N}$, $a_n < 0$. Then $\lim_{j\to\infty}(a_n + t_j v_n) = a_n < 0$. Hence, there exists $j_0 \in \mathbb{N}$ such that $a_n + t_j v_n < 0$ $\forall j \geq j_0$, but this is in contradiction to the fact that $a + t_j v \in K$. Thus, $a \in K$.

The following cone separation theorem is due to Adán and Novo [5, Theorem 2.2].

Theorem 2.3 (Separation Theorem). *Let M, K be two vectorially closed and relatively solid convex cones in Y. If K^+ is solid and $M \cap K = \{0\}$, then there exists a linear functional $\varphi \in Y' \backslash \{0\}$ such that $\forall k \in K$, $m \in M$, $\varphi(k) \geq 0 \geq \varphi(m)$ and furthermore $\forall k \in K\backslash\{0\}$, $\varphi(k) > 0$, i.e., $\varphi \in K^{+s}$.*

Remark 2.4. Notice that if $\text{cor}(K^+) \neq \emptyset$ and K is a convex cone then K is pointed [16, Lemmas 1.25 and 1.27] whenever Y' separates points in Y.

Throughout this paper, we assume that, unless indicated otherwise, X is a set, Y and Z are real linear spaces, $K \subset Y$ and $D \subset Z$ are pointed relatively solid convex cones, and $F\colon X \longrightarrow 2^Y$ and $G\colon X \longrightarrow 2^Z$ are set-valued maps with domain X.

The following unconstrained (P) and constrained (CP) vector optimization problems with set-valued maps will be considered:

$$(P) \quad \begin{cases} K - \text{Min}\{F(x)\} \\ \text{subject to } x \in X, \end{cases}$$

$$(CP) \quad \begin{cases} K - \text{Min}\{F(x)\} \\ \text{subject to } x \in X;\ G(x) \cap (-D) \neq \emptyset. \end{cases}$$

The feasible set of (CP) is

$$\Omega = \{x \in X\colon G(x) \cap (-D) \neq \emptyset\}. \tag{1}$$

The image of a subset $A \subset X$ under F is denoted by $F(A) = \bigcup_{x\in A} F(x)$.

In [3], [4] and [5] Adán and Novo have introduced several concepts of efficient points of a set $S \subset Y$ in the framework of vector optimization problems in partially ordered real linear spaces as follows.

Definition 2.5. (a) The set of Hurwicz-vectorial (HuV) proper efficient points of $S \subset Y$ is defined by

$$\text{HuV}(S) = \{y \in S\colon\ \text{vcl}[\text{conv}(\text{cone}((S - y) \cup K))] \cap (-K) = \{0\}\}.$$

(b) The set of Benson-vectorial (BeV) proper efficient points of $S \subset Y$ is defined by

$$\text{BeV}(S) = \{y \in S\colon\ \text{vcl}(\text{cone}(S - y + K)) \cap (-K) = \{0\}\}.$$

(c) The set of weakly-vectorial (WeV) efficient points of $S \subset Y$ is defined by

$$\text{WeV}(S) = \{y \in S\colon (S - y) \cap (-\,\text{icr}(K)) = \emptyset\}.$$

It is clear that $\mathrm{HuV}(S) \subset \mathrm{BeV}(S) \subset \mathrm{WeV}(S)$. If we assume that Y is a topological linear space, and in this definition we replace the vector closure by the topological closure, and "icr" by "int" we obtain the usual Hurwicz (Hu) (see [14]), Benson (Be) proper efficiency defined in [6] and weak efficiency (We). Because of $\mathrm{vcl}(S) \subset \mathrm{cl}(S)$, it follows that $\mathrm{Hu}(S) \subset \mathrm{HuV}(S)$, $\mathrm{Be}(S) \subset \mathrm{BeV}(S)$ and $\mathrm{We}(S) = \mathrm{WeV}(S)$ (if $\mathrm{int}(K) \neq \emptyset$).

The following diagram illustrates these relationships:

$$\begin{array}{ccccc} \mathrm{Hu} & \Rightarrow & \mathrm{Be} & \Rightarrow & \mathrm{We} \\ \Downarrow & & \Downarrow & & \Updownarrow \\ \mathrm{HuV} & \Rightarrow & \mathrm{BeV} & \Rightarrow & \mathrm{WeV} \end{array}$$

For a vector optimization problem with set-valued maps, we introduce the following concepts of efficient solution.

Definition 2.6. (a) A point $x \in X$ is called a Hurwicz-vectorial (HuV) proper efficient solution of problem (P) if there exists

$$y \in F(x) \cap \mathrm{HuV}(F(X)).$$

The pair (x, y) is called a Hurwicz-vectorial proper minimizer of (P).

(b) A point $x \in X$ is called a Benson-vectorial (BeV) proper efficient solution of problem (P) if there exists

$$y \in F(x) \cap \mathrm{BeV}(F(X)).$$

The pair (x, y) is called a Benson-vectorial proper minimizer of (P).

(c) A point $x \in X$ is called a weak-vectorial (WeV) efficient solution of problem (P) if there exists

$$y \in F(x) \cap \mathrm{WeV}(F(X)).$$

The pair (x, y) is called a weak-vectorial proper minimizer of (P).

3. Cone-subconvexlike set-valued maps

Fan [11] introduced the concept of convexlikeness for a real-valued function and Yu [28] called K-convex to a set A such that $A + K$ is convex. Later Jeyakumar [17] extended this concept in real topological linear spaces, establishing sub-convexlikeness and Yang [26] proposed new concepts of generalized convexity. Adán and Novo [1] extended these concepts to the framework of real linear spaces.

Let X be a set, let $F\colon X \longrightarrow 2^Y$ be a set-valued map whose domain is X and let $K \subset Y$ be a pointed relatively solid convex cone.

Definition 3.1. (a) F is said to be K-convexlike on X if $\forall x, x' \in X$, $\forall \alpha \in (0, 1)$

$$\alpha F(x) + (1 - \alpha) F(x') \subset F(X) + K.$$

(b) F is said to be K-subconvexlike on X if $\exists k_0 \in \mathrm{icr}(K)$ such that $\forall x, x' \in X$, $\forall \alpha \in (0, 1)$, $\forall \varepsilon > 0$,

$$\varepsilon k_0 + \alpha F(x) + (1 - \alpha) F(x') \subset F(X) + K.$$

Notice that, from the definition, if F is K-convexlike on X then F is K-subconvexlike on X (choosing as k_0 any element of $\text{icr}(K)$).

It is well-known that F is K-convexlike on X if and only if $F(X)+K$ is a convex set [22, Proposition 1.(iii)]. Let us see a similar result for K-subconvexlikeness and some other characterizations.

Proposition 3.2. *The following statements are equivalent:*

(a) *F is K-subconvexlike on X.*

(b) $\forall k' \in \text{icr}(K)$, $\forall x, x' \in X$, $\forall \alpha \in (0,1)$,

$$k' + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + \text{icr}(K).$$

(c) $\forall x, x' \in X$, $\forall \alpha \in (0,1)$, $\exists k \in K$ *such that* $\forall \varepsilon > 0$

$$\varepsilon k + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + K. \tag{2}$$

(d) *$F(X) + \text{icr}(K)$ is a convex set.*

Proof. The implications $(b) \Rightarrow (a) \Rightarrow (c)$ are clear. Let us see $(c) \Rightarrow (b)$. Let $k' \in \text{icr}(K)$, $x, x' \in X$, $\alpha \in (0,1)$. Then, by assumption, $\exists k \in K$ such that $\forall \varepsilon > 0$ condition (2) holds. As $k' \in \text{icr}(K) = \text{icr}(\text{icr}(K))$, for $-k \in L(K) = L(\text{icr}(K)) = K - K$ there exists $\varepsilon_0 > 0$ such that $k_0 = k' + \varepsilon_0(-k) \in \text{icr}(K)$. So,

$$\begin{aligned} k' + \alpha F(x) + (1-\alpha)F(x') &= [\varepsilon_0 k + \alpha F(x) + (1-\alpha)F(x')] + k_0 \\ &\subset F(X) + K + k_0 \subset F(X) + \text{icr}(K) \end{aligned}$$

(the last inclusion is true because $K + \text{icr}(K) \subset \text{icr}(K)$).
$(b) \Rightarrow (d)$. Let $u, u' \in F(X) + \text{icr}(K)$, $\alpha \in (0,1)$. Then, $u = y + k$, $u' = y' + k'$ with $y \in F(x)$, $y' \in F(x')$, $k, k' \in \text{icr}(K)$, $x, x' \in X$. Therefore

$$\alpha u + (1-\alpha)u' = \alpha(y+k) + (1-\alpha)(y'+k') = \alpha k + (1-\alpha)k' + \alpha y + (1-\alpha)y'.$$

As $\text{icr}(K)$ is a convex set, $k_0 = \alpha k + (1-\alpha)k' \in \text{icr}(K)$. So,

$$\alpha u + (1-\alpha)u' \in k_0 + \alpha F(x) + (1-\alpha)F(x') \subset F(X) + \text{icr}(K).$$

$(d) \Rightarrow (b)$. Let $k' \in \text{icr}(K)$, $x, x' \in X$, $\alpha \in (0,1)$, $y \in F(x)$, $y' \in F(x')$, then

$$k' + \alpha y + (1-\alpha)y' = \alpha(y+k') + (1-\alpha)(y'+k') \in F(X) + \text{icr}(K)$$

because $F(X) + \text{icr}(K)$ is a convex set by assumption, and $y + k', y' + k' \in F(X) + \text{icr}(K)$. □

In view of proposition (d), a remarkable property of cone-subconvexlikeness is that it is kept by translation: if F is K-subconvexlike on X, then $F - y$ is K-subconvexlike on X for all $y \in Y$.

Remark 3.3. Of course, we may define that F is K-subconvexlike on X in the sense of Li [19] if $\exists k_0 \in \text{icr}(K)$, $\forall x, x' \in X$, $\forall \alpha \in (0,1)$, $\forall \varepsilon > 0$, $\exists x'' \in X$ such that

$$\varepsilon k_0 + \alpha F(x) + (1-\alpha)F(x') \subset F(x'') + K.$$

However, this notion is more restrictive than Definition 3.1(b) (see Example 3.4 below). When $\text{cor}(K) \neq \emptyset$, this notion becomes Definition 2.1 in Li [19] (see also [21, Definition 1.2]).

Example 3.4. Consider $X_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 : \; x_1 < 0, x_2 = -x_1\}$, $a = (1, 0)$, $X = X_0 \cup \{a\}$, $F : X \longrightarrow 2^{\mathbb{R}^2}$ given by $F(x) = \{x, a\}$ and $K = \mathbb{R}^2_+$. We have that $\mathrm{cor}(\mathbb{R}^2_+) = \mathrm{icr}(\mathbb{R}^2_+) = \mathrm{int}(\mathbb{R}^2_+) \neq \emptyset$. It is clear that F is K-subconvexlike on X because

$$F(X) + \mathrm{icr}(K) = \{x \in \mathbb{R}^2 : \; x_2 > 0, \, x_2 > -x_1\}$$

is a convex set (see Proposition 3.2). However, F is not K-subconvexlike on X in the sense of Li [19]. Indeed, given $k_0 = (r, s) \in \mathrm{cor}(K)$, we take $x = (-1, 1) \in X$, $x' = (-3, 3) \in X$, $\alpha = 1/2$, and $\varepsilon = \dfrac{1}{4} \mathrm{Min} \left\{ \dfrac{1}{r}, \dfrac{1}{s} \right\}$, then $\forall x'' \in X$

$$\varepsilon k_0 + \alpha F(x) + (1 - \alpha) F(x') \not\subset F(x'') + K$$

as can be checked.

We obtain another counterexample with the same data changing K for $K = \mathbb{R}_+ \times \{0\}$. Now, $\mathrm{cor}(K) = \emptyset$ and $\mathrm{icr}(K) \neq \emptyset$.

Proposition 3.2 generalizes Lemmas 3.1 and 3.2 in Li [20], which are valid in a topological linear space Y provided with a convex cone K with $\mathrm{int}(K) \neq \emptyset$.

In this paper, the relative solidity of the convex set $F(X) + \mathrm{icr}(K)$ which appears in Proposition 3.2(d) is very important, so we give a new definition.

Definition 3.5. A set-valued map $F \colon X \longrightarrow 2^Y$ is relatively solid K-subconvexlike on X if the following conditions hold:

(i) F is K-subconvexlike on X,
(ii) $\mathrm{icr}(F(X) + \mathrm{icr}(K)) \neq \emptyset$.

Remark 3.6. (a) If Y is finite dimensional, condition (ii) is always true whenever F is K-subconvexlike because $F(X) + \mathrm{icr}(K)$ is a convex set.
(b) If $\mathrm{cor}(K) \neq \emptyset$ then condition (ii) is satisfied because $\mathrm{cor}(F(X) + \mathrm{cor}(K)) = F(X) + \mathrm{cor}(K)$ by [4, Proposition 6(iii)].

In Theorem 3.10 we establish an alternative theorem for K-subconvexlike set-valued maps and K solid. Previously, in Theorem 3.9 we establish a partial result of alternative type when K is only a relatively solid cone, and two lemmas.

Lemma 3.7. *Let A, B be two subsets of Y. If B is a cone, then $(A + B)^+ = A^+ \cap B^+$.*

Proof. The inclusion $A^+ \cap B^+ \subset (A + B)^+$ is clear. Let us see the converse. Let $\varphi \in (A+B)^+$, i.e., $\varphi(a+b) \geq 0 \; \forall a \in A, b \in B$. Then the function $t \mapsto \varphi(a+tb) = \varphi(a)+t\varphi(b)$ is affine and satisfies that $\varphi(a)+t\varphi(b) \geq 0 \; \forall t > 0$ and $\forall a \in A, b \in B$ since $a+tb \in A+B$. So, $\varphi(a) \geq 0$ and $\varphi(b) \geq 0$. □

Note that in the proof of Lemma 3.7 is not necessary to consider $0 \in B$.

Lemma 3.8. *If A is a relatively solid convex subset of Y, then $A^+ = [\mathrm{icr}(A)]^+$.*

Proof. We only have to prove that $[\mathrm{icr}(A)]^+ \subset A^+$. Choose $\varphi \in [\mathrm{icr}(A)]^+$, then $\varphi(a') \geq 0$ $\forall a' \in \mathrm{icr}(A)$. Pick $a_0 \in \mathrm{icr}(A)$, then $[a_0, a) \subset \mathrm{icr}(A) \; \forall a \in A$ [4, Proposition 3(ii)]. Hence, $\varphi(a + t(a_0 - a)) \geq 0 \; \forall t \in (0, 1]$, and therefore $\lim_{t \to 0^+} \varphi(a + t(a_0 - a)) = \varphi(a) \geq 0$. □

Theorem 3.9. *Let $K \subset Y$ be a pointed relatively solid convex cone. Assume that $F : X \to 2^Y$ is relatively solid K-subconvexlike on X. If there is no $x \in X$ such that*

$$F(x) \cap (-\operatorname{icr}(K)) \neq \emptyset, \tag{3}$$

then there exists $\varphi \in K^+ \setminus \{0\}$ such that

$$\varphi(y) \geq 0 \quad \forall y \in F(X). \tag{4}$$

Proof. The set $F(X) + \operatorname{icr}(K)$ is convex by Proposition 3.2. From (3) it follows that $0 \notin F(X) + \operatorname{icr}(K)$. So, $0 \notin \operatorname{icr}(F(X) + \operatorname{icr}(K))$. Using the support theorem [12, Theorem 6.C], there exists $\varphi \in Y' \setminus \{0\}$ such that

$$\varphi(y + k) \geq 0 \quad \forall y \in F(X),\ \forall k \in \operatorname{icr}(K)$$

(and φ is strictly positive on $\operatorname{icr}(F(X) + \operatorname{icr}(K))$), this is $\varphi \in (F(X) + \operatorname{icr}(K))^+$. Applying Lemmas 3.7 and 3.8 we conclude that $\varphi \in K^+$ and $\varphi \in F(X)^+$, i.e., equation (4) holds. □

Theorem 3.10. *Let K be a pointed solid convex cone. If F is K-subconvexlike on X, then exactly one of the following systems is consistent:*

(*i*) $\exists x \in X$ *such that* $F(x) \cap (-\operatorname{cor}(K)) \neq \emptyset$.

(*ii*) $\exists \varphi \in K^+ \setminus \{0\}$ *such that* $\forall y \in F(X)$, $\varphi(y) \geq 0$.

Proof. By Remark 3.6(b), condition (ii) in Definition 3.5 is satisfied. Therefore, by Theorem 3.9, not (*i*) $\Rightarrow$ (*ii*).

If we assume that both (*i*) and (*ii*) are satisfied, then there exist $x \in X$, $y \in F(x) \cap (-\operatorname{cor}(K))$ and $\varphi \in K^+ \setminus \{0\}$ such that $\varphi(y) \geq 0$. But, since $y \in -\operatorname{cor}(K)$ and $\varphi \in K^+ \setminus \{0\}$, we deduce that $\varphi(y) < 0$ (see Corollary 3.15 below), and this is a contradiction. □

Remark 3.11. This theorem is slightly more general than Theorem 2.1 of Li [21] because the notion of K-subconvexlikeness of this author is more restrictive than the our one, even when $\operatorname{cor}(K) \neq \emptyset$ (see Remark 3.3). If we consider that Y is a topological linear space then Theorem 3.10 collapses into Lemma 3.3 in [20]. Indeed, when Y is a topological linear space and $\operatorname{int}(K) \neq \emptyset$, then $\operatorname{int}(K) = \operatorname{cor}(K)$ and the linear functional φ satisfying condition (*ii*) is continuous by Theorem 3.7 in [25]. We can apply this theorem because the open set $\operatorname{int}(K)$ is contained in the set $\{y \in Y : \ \varphi(y) > 0\}$ by Corollary 3.15 since $\varphi \in K^+ \setminus \{0\}$.

Let us note that if $\operatorname{cor}(K) = \emptyset$ and $\operatorname{icr}(K) \neq \emptyset$, then both (*i*) (with $\operatorname{icr}(K)$ instead of $\operatorname{cor}(K)$) and (*ii*) can be true. For instance, in $\mathbb{R}^2$, $K = \mathbb{R}_+ \times \{0\}$, $X = \{(x, 0) \colon x \in (0, 1]\}$, $F(x, 0) = (x, 0) - K$ and $\varphi(x, y) = y$.

Next we analyze the postcomposition of a K-subconvexlike set-valued map with a positive linear map.

Let $\mathcal{L}(Y, Z)$ be the set of all linear maps φ from Y to Z, let $\mathcal{L}_+(Y, Z)$ be the subset of positive linear maps, i.e., $\varphi(K) \subset D$, and

$$\mathcal{L}_{++}(Y, Z) = \{\varphi \in \mathcal{L}_+(Y, Z) : \ \varphi(\operatorname{icr}(K)) \subset \operatorname{icr}(D)\}.$$

Lemma 3.12. *Let $\varphi \in \mathcal{L}_+(Y, Z)$ and let $k_0 \in \operatorname{icr}(K)$ fixed.*

(*i*) *If* $\varphi(k_0) \in \operatorname{icr}(D)$*, then* $\varphi(\operatorname{icr}(K)) \subset \operatorname{icr}(D)$*, i.e.,* $\varphi \in \mathcal{L}_{++}(Y, Z)$*.*

(*ii*) *If* $\varphi(k_0) = 0$*, then* $\varphi(k) = 0\ \forall k \in K$*.*

(*iii*) *If* $\varphi(k_0) \in D \setminus [\operatorname{icr}(D) \cup \{0\}]$*, then*

$$\varphi(\operatorname{icr}(K)) \subset D \setminus [\operatorname{icr}(D) \cup \{0\}] \quad \textit{and} \quad \varphi(K) \subset D \setminus \operatorname{icr}(D).$$

Proof. (*i*) Choose $k_1 \in \operatorname{icr}(K)$. Then $\varphi(k_1) \in D$ since $\varphi(K) \subset D$. As $\varphi(k_0) \in \operatorname{icr}(D)$ it follows that $(\varphi(k_1), \varphi(k_0)] \subset \operatorname{icr}(D)$ [4, Proposition 3(ii)], this is,

$$\varphi(k_1 + t(k_0 - k_1)) = \varphi(k_1) + t(\varphi(k_0) - \varphi(k_1)) \in \operatorname{icr}(D) \quad \forall t \in (0, 1].$$

As $k_1 - k_0 \in L(K)$ and $k_1 \in \operatorname{icr}(K)$, by the definition of $\operatorname{icr}(K)$ and using that K is convex we have $k_1 + t(k_0 - k_1) \in K\ \forall t \in [-\delta, 1]$ for some $\delta > 0$. Hence $\varphi(k_1 - \delta(k_0 - k_1)) \in D$ since $\varphi(K) \subset D$. But $(\varphi(k_1 - \delta(k_0 - k_1)), \varphi(k_0)] \subset \operatorname{icr}(D)$, and consequently, $\varphi(k_1) \in \operatorname{icr}(D)$ because $k_1 \in (k_1 - \delta(k_0 - k_1), k_0]$.

(*ii*) Assume that $\varphi(k) \neq 0$ for some $k \in K$. As $k_0 \in \operatorname{icr}(K)$, for $k_0 - k \in L(K)$ there exists $t_0 > 0$ such that $k_0 + t_0(k_0 - k) \in K$. So, $\varphi(k_0 + t_0(k_0 - k)) \in D$ because $\varphi(K) \subset D$. But

$$\varphi(k_0 + t_0(k_0 - k)) = -t_0\varphi(k) \in -D \setminus \{0\},$$

and we have a contradiction because D is pointed.

(*iii*) The first assertion follows from parts (*i*) and (*ii*). Let us show the second one. If for some $k \in K$, $\varphi(k) \in \operatorname{icr}(D)$, then $(k_0, k) \subset \operatorname{icr}(K)$ and $\varphi[(k_0, k)] = (\varphi(k_0), \varphi(k)) \subset \operatorname{icr}(D)$. As $(k_0, k) \neq \emptyset$, from (*i*) it follows that $\varphi(\operatorname{icr}(K)) \subset \operatorname{icr}(D)$ and this is a contradiction with assumption because $k_0 \in \operatorname{icr}(K)$. □

Example 3.13. To illustrate the above lemma, consider the following data:

(a) $\varphi_1 : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\varphi_1(x, y, z) = (x, y)$, $K = \mathbb{R}^3_+$ and $D = \mathbb{R}^2_+$.

(b) $\varphi_2 : \mathbb{R}^2 \to \mathbb{R}$ given by $\varphi_2(x, y) = y$, $K = \mathbb{R}_+ \times \{0\}$ and $D = \mathbb{R}_+$.

(c) $\varphi_3 : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\varphi_3(x, y) = (x, y, 0)$, $K = \mathbb{R}^2_+$ and $D = \mathbb{R}^3_+$.

These data provide instances of types (*i*), (*ii*) and (*iii*), respectively, in Lemma 3.12.

Lemma 3.12 allows us to split the set $\mathcal{L}_+(Y, Z)$ into three classes:

1) the class $\mathcal{L}_{++}(Y, Z)$,
2) the class $\mathcal{L}_{+0}(Y, Z) = \{\varphi \in \mathcal{L}_+(Y, Z) :\ \varphi(k) = 0\ \forall k \in K\}$, and
3) the class $\mathcal{L}_{+b}(Y, Z) = \{\varphi \in \mathcal{L}_+(Y, Z) :\ \varphi(\operatorname{icr}(K)) \subset D \setminus [\operatorname{icr}(D) \cup \{0\}]\}$.

We have

$$\mathcal{L}_+(Y, Z) = \mathcal{L}_{++}(Y, Z) \sqcup \mathcal{L}_{+0}(Y, Z) \sqcup \mathcal{L}_{+b}(Y, Z), \tag{5}$$

where $\sqcup$ denotes disjoint cup.

Remark 3.14. When $Z = \mathbb{R}$ and $D = \mathbb{R}_+$, the last set in (5) is empty, therefore if we denote

$$K^{+i} := \mathcal{L}_{++}(Y, \mathbb{R}) = \{\varphi \in K^+ :\ \varphi(y) > 0\ \forall y \in \operatorname{icr}(K)\} \quad \text{and} \quad K^{+0} := \mathcal{L}_{+0}(Y, \mathbb{R}),$$

we have

$$K^+ = K^{+i} \sqcup K^{+0}.$$

If, in addition, $\operatorname{cor}(K) \neq \emptyset$, then $K^{+0} = \{0\}$. Indeed, let $\varphi \in K^{+0}$ and let us choose $k \in \operatorname{cor}(K)$. Then for all $v \in Y$ there exists $t_0 > 0$ such that $k' = k + t_0 v \in K$. Hence, $v = t_0^{-1}(k' - k)$ and $\varphi(v) = t_0^{-1}(\varphi(k') - \varphi(k)) = 0$. Therefore, $\varphi = 0$. Thus we have proved the following result, whose part *(ii)* is Lemma 2.2 in [19].

Corollary 3.15. *Assume that K is a relatively solid convex cone.*

(i) *If $\varphi \in K^+ \setminus K^{+0}$ then $\varphi(k) > 0$ $\forall k \in \operatorname{icr}(K)$.*
(ii) *If K is solid, $\varphi \in K^+$ and $\varphi \neq 0$, then $\varphi(k) > 0$ $\forall k \in \operatorname{cor}(K)$.*

It is clear that in general the inclusion $\{0\} \subset K^{+0}$ is strict if K is only relatively solid. In order to see this fact, we consider $Y = \mathbb{R}^3$, $K = \{(x, y, 0) : x, y \geq 0\}$ and $\varphi(x, y, z) = z$. Then $\varphi \in K^{+0}$ and $\varphi \neq 0$.

The relationship between K^{+i} and K^{+s} is immediate from the definitions:

$$K^{+s} \subset K^{+i}.$$

Notice that $K^{+s} = K^{+i} = \emptyset$ whenever $0 \in \operatorname{icr}(K)$.

To study the composition of a set-valued map with a linear map, we need Proposition 3.17, whose proof will be easier with a previous result.

Lemma 3.16. *Let S_1 be a relatively solid convex set of Y and $S_2 \subset Y$. If $S_1 \subset S_2$ and $\operatorname{vcl}(S_1) = \operatorname{vcl}(S_2)$, then $\operatorname{icr}(S_1) = \operatorname{icr}(S_2)$.*

Proof. One has $\operatorname{aff}(S_1) = \operatorname{aff}(S_2)$ because by assumption $\operatorname{vcl}(S_1) = \operatorname{vcl}(S_2)$ and for any set $S \subset Y$, $\operatorname{aff}(S) = \operatorname{aff}(\operatorname{vcl}(S))$. Hence, as $S_1 \subset S_2$ we deduce that $\operatorname{icr}(S_1) \subset \operatorname{icr}(S_2)$. On the other hand, $S_2 \subset \operatorname{vcl}(S_2) = \operatorname{vcl}(S_1)$ and as S_2 and $\operatorname{vcl}(S_1)$ have the same affine hull, we get that $\operatorname{icr}(S_2) \subset \operatorname{icr}(\operatorname{vcl}(S_1)) = \operatorname{icr}(S_1)$. The last equality is true by Proposition 4(i) in [4]. Consequently, the conclusion follows. □

Proposition 3.17. *Let S be a relatively solid convex subset of Y and $\varphi : Y \to Z$ a linear map. Then*

$$\varphi(\operatorname{icr}(S)) = \operatorname{icr}(\varphi(S)).$$

Proof. Firstly let us see that

$$\varphi(\operatorname{icr}(S)) \subset \operatorname{icr}(\varphi(S)). \tag{6}$$

(as a consequence, $\varphi(S)$ is relatively solid).

It is obvious that $\varphi(L(S)) = L(\varphi(S))$. Take $a \in \operatorname{icr}(S)$ and let us prove that $\varphi(a) \in \operatorname{icr}(\varphi(S))$. Given $w \in L(\varphi(S))$, there exists $v \in L(S)$ satisfying $\varphi(v) = w$. As $a \in \operatorname{icr}(S)$, for $v \in L(S)$ there exists $t_0 > 0$ such that $a + tv \in S$ $\forall t \in (0, t_0]$. From here,

$$\varphi(a) + tw = \varphi(a + tv) \in \varphi(S) \quad \forall t \in (0, t_0],$$

and therefore, $\varphi(a) \in \operatorname{icr}(\varphi(S))$.

Now, the reverse inclusion: $\operatorname{icr}(\varphi(S)) \subset \varphi(\operatorname{icr}(S))$. For this aim, let us see that $\varphi(S)$ and $\varphi(\operatorname{icr}(S))$ have the same vector closure. We have that

$$\varphi(\operatorname{vcl}(S)) \subset \operatorname{vcl}(\varphi(S)). \tag{7}$$

Indeed, choose $b \in \text{vcl}(S)$, then there exists $v \in Y$ such that $\forall \alpha' > 0\ \exists \alpha \in (0, \alpha']$ such that $b + \alpha v \in S$. Hence, $\varphi(b) + \alpha\varphi(v) \in \varphi(S)$. This means that $\varphi(b) \in \text{vcl}(\varphi(S))$.

The following inclusions are clear in view of (7):

$$\varphi(S) \subset \varphi(\text{vcl}(S)) = \varphi(\text{vcl}(\text{icr}(S))) \subset \text{vcl}[\varphi(\text{icr}(S))] \subset \text{vcl}(\varphi(S)).$$

From this chain, we select the following:

$$\varphi(S) \subset \text{vcl}[\varphi(\text{icr}(S))] \subset \text{vcl}(\varphi(S)).$$

Taking vector closure and using that $\text{vcl}(\text{vcl}(B)) = \text{vcl}(B)$, if B is a relatively solid convex set, by [4, Proposition 3(iii)] (as $\varphi(\text{icr}(S)) = \varphi(\text{icr}(\text{icr}(S))) \subset \text{icr}[\varphi(\text{icr}(S))]$, by condition (6) and as S is relatively solid, $\varphi(\text{icr}(S))$ is relatively solid too) we have that:

$$\text{vcl}(\varphi(S)) \subset \text{vcl}(\varphi(\text{icr}(S))) \subset \text{vcl}(\varphi(S)).$$

Therefore, $\text{vcl}(\varphi(S)) = \text{vcl}[\varphi(\text{icr}(S))]$, and by Lemma 3.16, $\text{icr}(\varphi(S)) = \text{icr}[\varphi(\text{icr}(S))] \subset \varphi(\text{icr}(S))$. Using (6), the conclusion follows. □

Proposition 3.18. *Let $F : X \longrightarrow 2^Y$ be K-subconvexlike on X and let $\varphi \in \mathcal{L}_+(Y, Z)$.*

(*i*) *If $\varphi \in \mathcal{L}_{++}(Y, Z)$, then $\varphi \circ F$ is D-subconvexlike on X.*
(*ii*) *If $\varphi \in \mathcal{L}_{+0}(Y, Z)$, then $\varphi \circ F$ is $\{0\}$-convexlike on X.*
(*iii*) *If $\varphi \in \mathcal{L}_{+b}(Y, Z)$, then $\varphi \circ F$ is $\varphi(K)$-subconvexlike on X.*

Furthermore, if F is relatively solid K-subconvexlike on X, then $\varphi \circ F$ is relatively solid $\varphi(K)$-subconvexlike on X in part (iii) and $\text{icr}((\varphi \circ F)(X)) \neq \emptyset$ *in part (ii). If, in addition, $\varphi(\text{icr}(K)) = \text{icr}(D)$, then $\varphi \circ F$ is relatively solid D-subconvexlike on X in part (i).*

Proof. (*i*) By assumption, $\exists k_0 \in \text{icr}(K)$, $\forall x, x' \in X$, $\forall y \in F(x)$, $\forall y' \in F(x')$, $\forall \alpha \in (0, 1)$, $\forall \varepsilon > 0$ we have

$$\varepsilon k_0 + \alpha y + (1 - \alpha) y' \in F(X) + K,$$

and therefore,

$$\varepsilon\varphi(k_0) + \alpha\varphi(y) + (1 - \alpha)\varphi(y') \in (\varphi \circ F)(X) + \varphi(K) \subset (\varphi \circ F)(X) + D. \tag{8}$$

As $\varphi \in \mathcal{L}_{++}(Y, Z)$, $\varphi(k_0) \in \text{icr}(D)$, and the conclusion follows.

(*ii*) It is similar to (*i*).

(*iii*) It is obvious that $\varphi(K)$ is a convex cone. It is pointed because $\varphi(K) \cap -\varphi(K) \subset D \cap -D = \{0\}$. Now, the conclusion follows from (8) and Proposition 3.17 because of $\varphi(k_0) \in \varphi(\text{icr}(K)) = \text{icr}(\varphi(K))$.

The last claims follow from Definition 3.5 since

$$\begin{aligned} \emptyset \neq \varphi[\text{icr}(F(X) + \text{icr}(K))] &= \text{icr}[(\varphi \circ F)(X) + \varphi(\text{icr}(K))] \\ &= \text{icr}[(\varphi \circ F)(X) + \text{icr}(\varphi(K))] \end{aligned}$$

by Proposition 3.17. □

Proposition 3.19. *Let $(F, G) : X \longrightarrow 2^{Y \times Z}$ be $K \times D$-subconvexlike on X.*

(*i*) *If* $\varphi \in K^{+i}$ *then* $(\varphi \circ F, G)$ *is* $\mathbb{R}_+ \times D$*-subconvexlike on* X.
(*ii*) *If* $\varphi \in K^+ \setminus K^{+i} = K^{+0}$ *then* $(\varphi \circ F, G)$ *is* $\{0\} \times D$*-subconvexlike on* X.
(*iii*) *If* $\psi \in \mathcal{L}_+(Z, Y)$ *then* $F + \psi \circ G$ *is* K*-subconvexlike on* X.

Furthermore, if (F, G) *is relatively solid, then* $(\varphi \circ F, G)$ *is relatively solid in parts* (*i*) *and* (*ii*)*, and* $F + \psi \circ G$ *is relatively solid in part* (*iii*).

Proof. (*i*) Define $(\varphi \times i) : Y \times Z \to \mathbb{R} \times Z$ given by $(\varphi \times i)(y, z) = (\varphi(y), z)$. It is easy to verify that $(\varphi \times i)$ is a positive linear map with respect to the cones $K \times D$ and $\mathbb{R}_+ \times D$. As $\varphi \in K^{+i}$, by definition of K^{+i}, we have that $\varphi(\mathrm{icr}(K)) = \mathrm{icr}(\mathbb{R}_+)$, and therefore, since $\mathrm{icr}(K \times D) = \mathrm{icr}(K) \times \mathrm{icr}(D)$, we obtain $(\varphi \times i)(\mathrm{icr}(K) \times \mathrm{icr}(D)) = \mathrm{icr}(\mathbb{R}_+) \times \mathrm{icr}(D)$. Consequently, $(\varphi \times i) \in \mathcal{L}_{++}(Y \times Z, \mathbb{R} \times Z)$, and Proposition 3.18(*i*) is applicable to $(\varphi \times i) \circ (F, G) = (\varphi \circ F, G)$.

The proof of part (*ii*) is similar to the one above (we only have to take into account the definition of K^{+0}) and apply Proposition 3.18(*iii*) because $(\varphi \times i)(K \times D) = \{0\} \times D$ and $(\varphi \times i) \in \mathcal{L}_{+b}(Y \times Z, \mathbb{R} \times Z)$.

(*iii*) Let $\varphi : Y \times Z \to Y$ be defined by $\varphi(y, z) = y + \psi(z)$. It is easy to verify that φ is a positive linear map (in $Y \times Z$ we consider the cone $K \times D$). Let us see that $\varphi \in \mathcal{L}_{++}(Y \times Z, Y)$. Let $(y, z) \in \mathrm{icr}(K \times D) = \mathrm{icr}(K) \times \mathrm{icr}(D)$, then $\varphi(y, z) = y + \psi(z) \in \mathrm{icr}(K)$ because $\psi(z) \in K$ (since $\psi \in \mathcal{L}_+(Z, Y)$) and $\mathrm{icr}(K) + K \subset \mathrm{icr}(K)$.

Finally, $F + \psi \circ G = \varphi \circ (F, G)$ and we can apply Proposition 3.18(*i*).

The last part of this proposition follows from the last part of Proposition 3.18. For the set-valued map $F + \psi \circ G$ we have to take into account that $\varphi(\mathrm{icr}(K \times D)) = \mathrm{icr}(K)$ since $\varphi(K \times D) = K$ and $\varphi(\mathrm{icr}(K \times D)) = \mathrm{icr}(\varphi(K \times D)) = \mathrm{icr}(K)$, where φ is the linear map used in the proof of part (*iii*). □

4. Necessary optimality conditions

In the following results we establish necessary conditions of weak vectorial efficiency for the unconstrained (P) and constrained (CP) optimization problems.

Theorem 4.1. *Let* K *be a pointed relatively solid convex cone. Assume that* $F : X \to 2^Y$ *is relatively solid* K*-subconvexlike on* X*. If* $x_0 \in X$ *is a weak-vectorial efficient solution for problem* (P)*, then there exist* $\varphi \in K^+ \setminus \{0\}$*,* $y_0 \in F(x_0)$ *such that*

$$\mathrm{Min}_{y \in F(X)}\, \varphi(y) = \varphi(y_0).$$

Proof. By hypothesis there exists $y_0 \in F(x_0)$ such that $y_0 \in \mathrm{WeV}(F(X))$, i.e.

$$(F(X) - y_0) \cap -\,\mathrm{icr}(K) = \emptyset. \tag{9}$$

Let $H\colon X \longrightarrow 2^Y$ be defined by $H(x) = F(x) - y_0$. Then it is clear that H is relatively solid K-subconvexlike on X because $H(X) + \mathrm{icr}(K) = F(X) + \mathrm{icr}(K) - y_0$ is a relatively solid convex set. So applying Theorem 3.9, taking into account condition (9), there exists $\varphi \in K^+ \setminus \{0\}$ such that $\varphi(z) \geq 0$ for all $z \in H(X) = F(X) - y_0$, and therefore

$$\varphi(y) \geq \varphi(y_0) \quad \text{for all } y \in F(X).$$

Hence, $\mathrm{Min}_{y \in F(X)}\, \varphi(y) = \varphi(y_0)$. □

Notice that $\varphi \circ F$ is relatively solid $\mathbb{R}_+$-subconvexlike or $\{0\}$-convexlike by Proposition 3.18 taking into account Remark 3.14.

Theorem 4.2. *Assume the following:*

(a) (x_0, y_0) *is a weak-vectorial minimizer for problem* (CP).
(b) $(F, G) : X \longrightarrow 2^{Y\times Z}$ *is relatively solid* $K \times D$*-subconvexlike on* X.

Then there exists $(\varphi, \psi) \in K^+ \times D^+$, $(\varphi, \psi) \neq (0, 0)$, *such that*

$$\mathrm{Min}_{x\in X}\{(\varphi \circ F)(x) + (\psi \circ G)(x)\} = \varphi(y_0),$$
$$\mathrm{Min}\{(\psi \circ G)(x_0)\} = 0.$$

Proof. By hypothesis

$$-(F(\Omega) - y_0) \cap \mathrm{icr}(K) = \emptyset. \tag{10}$$

We consider the following set-valued map

$$H(x) = (F(x) - y_0) \times G(x) = F(x) \times G(x) - (y_0, 0)$$

defined from X to $2^{Y\times Z}$. Since $(F, G) : X \longrightarrow 2^{Y\times Z}$ is a $K \times D$-subconvexlike set-valued map on X and $\mathrm{icr}((F, G)(X) + \mathrm{icr}(K \times D)) \neq \emptyset$, we have that H is $K \times D$-subconvexlike on X and $\mathrm{icr}(H(X) + \mathrm{icr}(K \times D)) \neq \emptyset$, i.e. H is relatively solid $K \times D$-subconvexlike on X. Furthermore

$$H(x) \cap (-\,\mathrm{icr}(K \times D)) \neq \emptyset \quad \text{for all } x \in X. \tag{11}$$

Suppose that this is false. Then there exists $(y - y_0, z) \in H(x)$ such that $(y - y_0, z) \in -\,\mathrm{icr}(K \times D)$ with $y \in F(x)$ and $z \in G(x)$ for some $x \in X$. By Lemma 2.2 in [1], $\mathrm{icr}(K \times D) = \mathrm{icr}(K) \times \mathrm{icr}(D)$. Hence $z \in -\,\mathrm{icr}(D) \subset -D$ and $(F(x) - y_0) \cap -\,\mathrm{icr}(K) \neq \emptyset$, which contradicts (10) since $x \in \Omega$.

Taking into account condition (11), if we apply Theorem 3.9 there exists $(\varphi, \psi) \in (K \times D)^+ \setminus \{(0, 0)\} = K^+ \times D^+ \setminus \{(0, 0)\}$ such that $\varphi(F(x) - y_0) + \psi(G(x)) \geq 0$ for all $x \in X$. Therefore

$$(\varphi \circ F)(x) + (\psi \circ G)(x) \geq \varphi(y_0) \quad \text{for all } x \in X. \tag{12}$$

Since $x_0 \in \Omega$ then there exists $z_0 \in G(x_0)$ such that $z_0 \in -D$. Choosing $x = x_0$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0)$ in (12) we derive

$$\varphi(y_0) + \psi(z_0) \geq \varphi(y_0),$$

and therefore $\psi(z_0) \geq 0$. But $\psi \in D^+$, so $\psi(z_0) = 0$. It follows that

$$\mathrm{Min}_{x\in X}\{(\varphi \circ F)(x) + (\psi \circ G)(x)\} = \varphi(y_0).$$

If we choose again $x = x_0$ in (12), we obtain

$$(\varphi \circ F)(x_0) + (\psi \circ G)(x_0) \geq \varphi(y_0);$$

so $(\psi \circ G)(x_0) \geq 0$, and then we deduce that

$$\mathrm{Min}\{(\psi \circ G)(x_0)\} = 0.$$

□

If in the theorems above we consider K solid, taking into account Theorem 3.10, we obtain Theorems 3.1 and 3.2 in Li [19].

5. Scalarization

In this section we characterize the Benson-vectorial efficiency, for a pointed relatively solid convex cone and cone-subconvexlike set-valued maps, through scalarization. We also show necessary conditions for a Hurwicz-vectorial proper minimizer. As in the previous section, X is a set, Y is a linear space and $K \subset Y$ is a pointed relatively solid convex cone.

Let $\varphi \in \mathcal{L}(Y, \mathbb{R})$. We can associate to problem (P) the following scalar optimization problem with a set-valued map:

$$(SP_\varphi) \quad \begin{cases} \text{Min } \{(\varphi \circ F)(x)\} \\ \text{subject to } x \in X. \end{cases}$$

Definition 5.1. If $x_0 \in X$, $y_0 \in F(x_0)$ and

$$\varphi(y_0) \leq \varphi(y) \quad \forall y \in F(X),$$

then x_0 is called a minimal solution of problem (SP_φ), and (x_0, y_0) is called a minimizer of problem (SP_φ).

Theorem 5.2. *Let $\varphi \in K^{+s}$. If (x_0, y_0) is a minimizer of (SP_φ) then (x_0, y_0) is a Hurwicz-vectorial proper minimizer of (P) and therefore (x_0, y_0) is a Benson-vectorial proper minimizer of (P).*

Proof. As (x_0, y_0) is a minimizer of (SP_φ) we have $\varphi(y) \geq 0$ for all $y \in F(X) - y_0$ and the same is true for all $y \in K$ because $\varphi \in K^{+s}$. Since φ is linear, we deduce that

$$\varphi \in [\text{cone}(\text{conv}((F(X) - y_0) \cup K))]^+.$$

As $[\text{vcl}(S)]^+ = S^+$ and $\text{cone}(\text{conv}(S)) = \text{conv}(\text{cone}(S))$ for any $S \subset Y$ (see [5] and [1]) it follows that

$$\varphi \in \{\text{vcl}[\text{conv}(\text{cone}((F(X) - y_0) \cup K))]\}^+. \tag{13}$$

On the other hand, $\varphi(y) > 0$ for all $y \in K \backslash \{0\}$, then we obtain

$$\text{vcl}[\text{conv}(\text{cone}((F(X) - y_0) \cup K))] \cap (-K) = \{0\}.$$

In fact, if $z \in \text{vcl}[\text{conv}(\text{cone}((F(X) - y_0) \cup K))] \cap (-K)$ and $z \neq 0$ then $\varphi(z) \geq 0$ by (13) which is a contradiction because $\varphi \in K^{+s}$ and $z \in -K \setminus \{0\}$ imply $\varphi(z) < 0$.
Therefore (x_0, y_0) is a Hurwicz-vectorial proper minimizer of (P). □

As a consequence of the previous result, if we consider a topological linear space Y and we replace the vector closure by the topological closure, the previous proof is valid too whenever φ is continuous. Therefore, the result above includes as a particular case Theorem 4.1 in Li [20].

To establish sufficient conditions we need some convexity properties and the following lemma.

Lemma 5.3. *Let S be a relatively solid convex set of Y. Then*

$$\text{icr}(S) \subset \text{icr}(\text{cone}(S)). \tag{14}$$

Proof. Firstly, let us prove that

$$L(\mathrm{cone}(S)) = \mathrm{aff}(S \cup \{0\}) = \begin{cases} L(S) & \text{if } 0 \in \mathrm{aff}(S) \\ L(S) + \mathrm{span}(s_0) & \text{if } 0 \notin \mathrm{aff}(S), \end{cases} \tag{15}$$

where s_0 is an arbitrary element of S and $\mathrm{span}(s_0)$ is the linear subspace generated by s_0.

Indeed, the statement is obvious when $0 \in \mathrm{aff}(S)$. Thus, assume that $0 \notin \mathrm{aff}(S)$. The vector subspace $L(S) + \mathrm{span}(s_0)$ is the smallest affine variety containing $S \cup \{0\}$ because:

1) $S \subset L(S) + s_0 \subset L(S) + \mathrm{span}(s_0)$ and $\{0\} \subset L(S) + \mathrm{span}(s_0)$.

2) If V is an affine variety containing $S \cup \{0\}$, then $\mathrm{aff}(S) = L(S) + s_0 \subset V$ and V is a linear subspace of Y. So, $L(S) \subset V - s_0 = V$ and $\mathrm{span}(s_0) \subset V$ since $s_0 \in S \subset V$. Therefore, $L(S) + \mathrm{span}(s_0) \subset V$.

Secondly, let us see that equation (14) holds. Let $a \in \mathrm{icr}(S)$, we have to prove that $\forall u \in L(\mathrm{cone}(S))$

$$\exists t_0 > 0 \text{ such that } a + tu \in \mathrm{cone}(S) \quad \forall t \in (0, t_0]. \tag{16}$$

Taking into account equation (15), it is enough to prove (16) in the following cases: (i) $u \in L(S)$, (ii) $u = s_0$ and (iii) $u = -s_0$.

(i) Let $u \in L(S)$. As $a \in \mathrm{icr}(S)$, then there is $t_0 > 0$ such that $a + tu \in S \subset \mathrm{cone}(S)$ $\forall t \in (0, t_0]$, i.e., (16) is satisfied.

(ii) Now, $u = s_0$. Then, as $a, s_0 \in \mathrm{cone}(S)$ we have $a + ts_0 \in \mathrm{cone}(S)$ $\forall t \geq 0$ since $\mathrm{cone}(S)$ is a convex cone.

(iii) Finally, $u = -s_0$. As $a \in \mathrm{icr}(S) \subset S$ and $s_0 \in S$ (so $a - s_0 \in L(S)$), there exists $\gamma > 0$ such that

$$s_1 := s_0 + (1 + \gamma)(a - s_0) = a + \gamma(a - s_0) \in S.$$

The equation $a + t(-s_0) = \rho s_1$ in the unknown (t, ρ) has solution (t_0, ρ_0) where $t_0 = \gamma/(1+\gamma) > 0$ and $\rho_0 = 1/(1+\gamma) > 0$. Hence $a + t_0(-s_0) = \rho_0 s_1 \in \mathrm{cone}(S)$, and therefore $[a, a + t_0(-s_0)] \subset \mathrm{cone}(S)$ (i.e., (16) is true). □

Theorem 5.4. *Assume that K is vectorially closed and $\mathrm{cor}(K^+) \neq \emptyset$. Let F be relatively solid K-subconvexlike on X. If (x_0, y_0) is a Benson-vectorial proper minimizer of (P) then there exists $\varphi \in K^{+s}$ such that (x_0, y_0) is a minimizer of (SP_φ).*

Proof. Since (x_0, y_0) is a Benson-vectorial proper minimizer then

$$-\mathrm{vcl}[\mathrm{cone}(F(X) - y_0 + K)] \cap K = \{0\}. \tag{17}$$

As $\mathrm{vcl}[\mathrm{cone}(F(X) - y_0 + \mathrm{icr}(K))] \subset \mathrm{vcl}[\mathrm{cone}(F(X) - y_0 + K)]$ then

$$-\mathrm{vcl}[\mathrm{cone}(F(X) - y_0 + \mathrm{icr}(K))] \cap K = \{0\}. \tag{18}$$

Let us see that $\mathrm{vcl}[\mathrm{cone}(F(X) - y_0 + \mathrm{icr}(K))]$ is vectorially closed and relatively solid convex cone. It is clear that $\mathrm{vcl}[\mathrm{cone}(F(X) - y_0 + \mathrm{icr}(K))]$ is a cone.

Because of F is relatively solid K-subconvexlike on X, $\operatorname{icr}[F(X) + \operatorname{icr}(K)] \neq \emptyset$ and $F(X) + \operatorname{icr}(K)$ is a convex set, then $\operatorname{icr}[F(X) - y_0 + \operatorname{icr}(K)] \neq \emptyset$ and $F(X) - y_0 + \operatorname{icr}(K)$ is convex too. Therefore, $\operatorname{cone}(F(X) - y_0 + \operatorname{icr}(K))$ is convex and applying Lemma 5.3 we obtain that

$$\operatorname{icr}[\operatorname{cone}(F(X) - y_0 + \operatorname{icr}(K))] \neq \emptyset.$$

From here, applying Proposition 3(iii)-(iv) in [4], we obtain that $\operatorname{vcl}[\operatorname{cone}(F(X) - y_0 + \operatorname{icr}(K))]$ is vectorially closed and convex. On the other hand, by Proposition 4(i) in [4], $\operatorname{vcl}[\operatorname{cone}(F(X) - y_0 + \operatorname{icr}(K))]$ is a relatively solid set.

In these conditions we can apply the separation Theorem 2.3, so taking into account condition (18), there exists $\varphi \in K^{+s} \setminus \{0\}$ such that

$$\varphi(v) \geq 0 \quad \text{for all } v \in \operatorname{vcl}[\operatorname{cone}(F(X) - y_0 + K)].$$

Since $F(X) - y_0 \subset \operatorname{cone}(F(X) - y_0 + K) \subset \operatorname{vcl}[\operatorname{cone}(F(X) - y_0 + K)]$ we have

$$\varphi(y - y_0) \geq 0 \quad \text{for all } y \in F(X).$$

Therefore (x_0, y_0) is a minimizer of (SP_φ). □

From the theorems above we obtain the following corollary, which gives us a characterization of Benson-vectorial minimality under cone-subconvexlikeness.

Corollary 5.5. *Let K^+ be solid and K be vectorially closed. Let F be relatively solid K-subconvexlike on X. Then (x_0, y_0) is a Benson-vectorial proper minimizer of (P) if and only if (x_0, y_0) is a minimizer of (SP_φ) for some $\varphi \in K^{+s}$.*

Therefore if we consider a topological linear space Y and $\operatorname{int}(K) \neq \emptyset$ then Theorem 5.4 and Corollary 5.5 can be considered extensions of Theorem 4.2 and Corollary 4.1, respectively, in Li [20].

6. Lagrange multiplier rules

We shall consider the same notations as in the previous section. X is a set, Y is a linear space and $K \subset Y$ is a vectorially closed and pointed relatively solid convex cone.

Definition 6.1. We say that the optimization problem (CP) satisfies the generalized Slater constraint qualification if there exists $x \in X$ such that $G(x) \cap -\operatorname{icr}(D) \neq \emptyset$.

Theorem 6.2. *Consider problem (CP). Assume that $\operatorname{cor}(K^+) \neq \emptyset$, (F, G) is relatively solid $K \times D$-subconvexlike on X, F is relatively solid K-subconvexlike on Ω and*

$$\operatorname{aff}(D) = \operatorname{aff}(G(X) + \operatorname{icr}(D)). \tag{19}$$

If (CP) satisfies the generalized Slater constraint qualification and (x_0, y_0) is a Benson-vectorial proper minimizer of (CP) then there exists $T \in \mathcal{L}_+(Z, Y)$ such that $0 \in T(G(x_0))$ and (x_0, y_0) is a Hurwicz-vectorial proper minimizer of the following unconstrained problem

$$\begin{cases} K - \operatorname{Min}\{(F + T \circ G)(x)\} \\ \textit{subject to } x \in X. \end{cases} \tag{20}$$

Proof. Since F is relatively solid K-subconvexlike on Ω, we can apply Theorem 5.4, then there exists a linear functional $\varphi \in K^{+s}$ such that (x_0, y_0) is a minimizer of the scalar problem

$$\mathrm{Min}\{\varphi[F(x)] \colon x \in \Omega\},$$

i.e.

$$\varphi(y) \geq \varphi(y_0) \quad \text{for all } y \in F(\Omega). \tag{21}$$

Let $H \colon X \longrightarrow 2^{\mathbb{R}\times Z}$ be the set-valued map defined by

$$H(x) = [\varphi(F(x)) - \varphi(y_0)] \times G(x) = \varphi(F(x)) \times G(x) - (\varphi(y_0), 0).$$

As a consequence of (21) we have

$$H(X) \cap -\mathrm{icr}(\mathbb{R}_+ \times D) = \emptyset. \tag{22}$$

Since $\varphi \in K^{+s} \subset K^{+i}$ and $(F - y_0, G)$ is relatively solid $K \times D$-subconvexlike on X then, by Proposition 3.19, we have that $H = (\varphi \circ (F - y_0), G) = (\varphi \circ F - \varphi(y_0), G)$ is relatively solid $\mathbb{R}_+ \times D$-subconvexlike on X. Together with (22), by Theorem 3.9 applied to H, we obtain that there exists $(r, \psi) \in \mathbb{R}_+ \times D^+ \backslash \{(0, 0)\}$ such that

$$r[\varphi(F(x) - y_0)] + \psi[G(x)] \geq 0 \quad \text{for all } x \in X \tag{23}$$

and (see the proof of Theorem 3.9)

$$(r, \psi)(y', z') > 0 \quad \text{for all } (y', z') \in \mathrm{icr}(M), \tag{24}$$

where $M = (\varphi \circ (F - y_0), G)(X) + \mathrm{icr}(\mathbb{R}_+ \times D)$.

We note that $r > 0$. Otherwise, if $r = 0$ then from condition (24) it results

$$\psi(z') > 0 \quad \text{for all } (y', z') \in \mathrm{icr}(M). \tag{25}$$

Let $\pi : \mathbb{R} \times Z \to Z$ be the projection on second space. By Proposition 3.17,

$$\pi(\mathrm{icr}(M)) = \mathrm{icr}[\pi(M)] = \mathrm{icr}[G(X) + \mathrm{icr}(D)] \tag{26}$$

as $\pi(M) = G(X) + \mathrm{icr}(D)$. In view of (25) and (26) we derive that

$$\psi(z') > 0 \quad \text{for all } z' \in \mathrm{icr}[G(X) + \mathrm{icr}(D)]. \tag{27}$$

As a consequence of the generalized Slater constraint qualification, $0 \in G(X) + \mathrm{icr}(D)$, so

$$D = 0 + D \subset G(X) + \mathrm{icr}(D) + D = G(X) + \mathrm{icr}(D).$$

Using assumption (19) it follows

$$\mathrm{icr}(D) \subset \mathrm{icr}[G(X) + \mathrm{icr}(D)]$$

and by (27) we obtain that

$$\psi(\mathrm{icr}(D)) > 0. \tag{28}$$

Again, because of the generalized Slater constraint qualification, there exist $x' \in X$ and $z' \in G(x') \cap -\operatorname{icr}(D)$, and consequently, by (28), $\psi(z') < 0$ and by (23), $\psi(z') \geq 0$, which is a contradiction. Thus, $r > 0$.

Since $x_0 \in \Omega$ there exists $z_0 \in G(x_0) \cap -D$, and as $\psi \in D^+$ then $\psi(z_0) \leq 0$. Taking $x = x_0$, $y_0 \in F(x_0)$ and $z_0 \in G(x_0)$ in (23) we have that $\psi(z_0) \geq 0$, so $\psi(z_0) = 0$. Hence,

$$0 \in \psi[G(x_0)]. \tag{29}$$

As $r \neq 0$ and $\varphi \in K^{+s}$, we can choose $k \in K$ such that

$$r\varphi(k) = 1. \tag{30}$$

We define the linear operator $T\colon Z \longrightarrow Y$ by

$$T(z) = \psi(z)k. \tag{31}$$

It is clear that $T(D) \subset K$, i.e., $T \in \mathcal{L}_+(Z, Y)$. By (29), $0 \in T(G(x_0))$ and consequently

$$y_0 \in F(x_0) \subset F(x_0) + T(G(x_0)). \tag{32}$$

Now, from (23), (30) and (31) we have that for all $x \in X$

$$r\varphi[F(x) + T(G(x))] = r\varphi[F(x)] + \psi[G(x)]r\varphi(k) = r\varphi[F(x)] + \psi[G(x)] \geq r\varphi(y_0).$$

So if we divide this inequality by $r > 0$ we obtain that (x_0, y_0) is a minimizer of the scalar problem

$$K - \operatorname{Min}\{(\varphi \circ (F + T \circ G))(x) :\ x \in X\}.$$

According to Theorem 5.2, (x_0, y_0) is a Hurwicz-vectorial proper minimizer of the unconstrained problem

$$K - \operatorname{Min}\{F(x) + T[G(x)] :\ x \in X\}.$$

□

Remark 6.3. (a) Since (F, G) is relatively solid $K \times D$-subconvexlike on X, it follows from Proposition 3.18 (considering that $\varphi : Y \times Z \to Z$ is the projection) that G is relatively solid D-subconvexlike on X. So, $G(X) + \operatorname{icr}(D)$ is a relatively solid convex set.
(b) Condition (19) is equivalent to

$$\operatorname{aff}(\operatorname{icr}(D)) = \operatorname{aff}(\operatorname{icr}[G(X) + \operatorname{icr}(D)]).$$

because:

If $S \subset Y$ is a relatively solid convex set, then $\operatorname{aff}(S) = \operatorname{aff}(\operatorname{icr}(S))$.

Indeed, the inclusion "$\subset$" is clear as $\operatorname{icr}(S) \subset S$. Let us see the converse. Let $v \in L(S)$ and pick $a \in \operatorname{icr}(S)$. Then there is $t_0 > 0$ such that $a + t_0 v \in S$, and by Proposition 3(ii) in [4], $[a, a + t_0 v) \subset \operatorname{icr}(S)$. So if we choose $t \in (0, t_0)$, we have that $b = a + tv \in \operatorname{icr}(S)$, and therefore $v = t^{-1}(b - a) \in L(\operatorname{icr}(S))$. Thus, $L(S) = L(\operatorname{icr}(S))$, and consequently, $\operatorname{aff}(S) = \operatorname{aff}(\operatorname{icr}(S))$.
(c) Condition (19) is weaker than $\operatorname{cor}(D) \neq \emptyset$ since in this case $\operatorname{aff}(D) = \operatorname{aff}(G(X) + \operatorname{cor}(D)) = Z$.

Theorem 6.4. *Assume that there exists a pair* (x_0, y_0) *and a positive linear operator* $T \in \mathcal{L}_+(Z, Y)$ *such that*

(*i*) $x_0 \in \Omega$ *and* $y_0 \in F(x_0)$,

(*ii*) $0 \in T(G(x_0))$,

(*iii*) (x_0, y_0) *is a Benson-vectorial proper minimizer of the unconstrained problem*

$$K - \text{Min}\{(F + T \circ G)(x)\} \text{ subject to } x \in X.$$

Then (x_0, y_0) *is a Benson-vectorial proper minimizer of the problem* (CP).

If, in addition, one of the following conditions holds:

(*a*) $F + T \circ G$ *is* K*-convexlike on* X,

(*b*) $\text{cor}(K^+) \neq \emptyset$ *and* $F + T \circ G$ *is relatively solid* K*-subconvexlike on* X,

then (x_0, y_0) *is a Hurwicz-vectorial proper minimizer of the problem* (CP).

Proof. The proof of the first part ((x_0, y_0) is BeV of (CP)) is similar to case (a) and we omit it. Let us see that (x_0, y_0) is HuV of (CP) when (i)-(iii) and (a) are satisfied.

Firstly, let us observe that $y_0 \in (F+T\circ G)(x_0)$ by assumptions (i) and (ii), so assumption (iii) makes sense. Condition (iii) is equivalent to

$$\text{vcl}\left\{ \text{cone}\left[\left(\bigcup_{x\in X}((F + T \circ G)(x))\right) + K - y_0\right]\right\} \cap (-K) = \{0\}. \tag{33}$$

Now, if $x \in \Omega$, then there exists $z \in G(x) \cap (-D)$. Since $T(D) \subset K$ we have $0 \in T(z) + K \subset T(G(x)) + K$, and consequently

$$K \subset T(G(x)) + K + K = T(G(x)) + K \quad \forall x \in \Omega.$$

Using this and assumption (i), it follows that

$$K \subset F(x_0) - y_0 + T(G(x_0)) + K \subset \bigcup_{x\in\Omega}(F(x) - y_0 + (T \circ G)(x) + K). \tag{34}$$

On the other hand,

$$\left(\bigcup_{x\in\Omega} F(x)\right) - y_0 = \bigcup_{x\in\Omega}(F(x) - y_0) \subset \bigcup_{x\in\Omega}(F(x) - y_0 + (T \circ G)(x) + K). \tag{35}$$

But for the last set in (34) and (35) we have

$$\begin{aligned}\bigcup_{x\in\Omega}(F(x) - y_0 + (T \circ G)(x) + K) &= (F + T \circ G)(\Omega) + K - y_0 \\ &\subset (F + T \circ G)(X) + K - y_0.\end{aligned}$$

Therefore,

$$\begin{aligned}\text{conv}((F(\Omega) - y_0) \cup K) &\subset \text{conv}((F + T \circ G)(\Omega) + K - y_0) \\ &\subset \text{conv}((F + T \circ G)(X) + K - y_0).\end{aligned}$$

But, $(F + T \circ G)(X) + K$ is a convex set because $F + T \circ G$ is K-convexlike on X, so $(F + T \circ G)(X) + K - y_0$ is also a convex set, and then

$$\mathrm{conv}((F(\Omega) - y_0) \cup K) \subset (F + T \circ G)(X) + K - y_0.$$

Hence

$$\mathrm{vcl}\{\mathrm{cone}[\mathrm{conv}((F(\Omega) - y_0) \cup K)]\} \subset \mathrm{vcl}\{\mathrm{cone}[(F + T \circ G)(X) + K - y_0]\}.$$

From (33) it follows

$$\mathrm{vcl}\{\mathrm{cone}[\mathrm{conv}((F(\Omega) - y_0) \cup K)]\} \cap (-K) = \{0\},$$

with this we have concluded.

Now, let us see that (x_0, y_0) is HuV of (CP) when (i)-(iii) and (b) are satisfied.

Since $F + T \circ G$ is relatively solid K-subconvexlike on X, applying Theorem 5.4 there exists $\varphi \in K^{+s}$ such that

$$\varphi(F(x)) + \varphi(T(G(x))) \geq \varphi(y_0) \quad \text{for all } x \in X. \tag{36}$$

If $x \in \Omega$, there exists $z \in G(x)$ such that $z \in -D$. On the other hand, as $T \in \mathcal{L}_+(Z, Y)$, $T(z) \in -K$, and since $\varphi \in K^{+s}$ we obtain that

$$\varphi(T(z)) \leq 0.$$

From this, according to (36) and taking $z \in G(x)$, for each $y \in F(x)$ we obtain that

$$\varphi(y) \geq \varphi(y) + \varphi(T(z)) \geq \varphi(y_0).$$

Hence, for all $y \in F(\Omega)$

$$\varphi(y) \geq \varphi(y_0).$$

As $y_0 \in F(x_0) \subset F(\Omega)$ by assumption (i), applying Theorem 5.2, (x_0, y_0) is a Hurwicz-vectorial proper minimizer of problem (CP). □

Once again, our results extend Theorems 5.1 and 5.2 in Li [20] which are done in the framework of topological linear spaces with solid cones.

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