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Minimal Area Tetrahedron Circumscribing a Strictly Convex Body in $\mathbb{R}^3$

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We are interested in the geometric characterization of tangency points of a minimal area tetrahedron circumscribed around a strictly convex compact body in $\mathbb{R}^3$.

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1. Introduction

Mark Levi in [5] considered minimal perimeter triangles circumscribing a strictly convex set in the plane with a regular boundary. The author proved that for the minimal perimeter triangle the segments connecting the vertices of the triangle with the opposite tangency points are concurrent, as well as the lines perpendicular to the sides of the triangle at these points. The similar theorem is well-known for the minimal area triangle circumscribing a strictly convex set in the plane ([1], [5]). Note, that in this case the tangency points are the midpoints of the sides of the triangle ([7], [4]). The similar statement holds also for minimal volume tetrahedron circumscribing a strictly convex body in $\mathbb{R}^3$ ([4]). On the other hand the formulas of Levi (see also [7]) show that tangency points of a minimal perimeter triangle are the tangency points of its excircles. The segments connecting the vertices with these points concur at the Nagel point of the triangle.

In this paper we are interested in a similar geometric characterization of tangency points of a minimal area tetrahedron circumscribed about a strictly convex compact body. We prove the following theorem:

Theorem 1.1. *Let $T(w, x, y, z)$ be a minimal area tetrahedron with vertices w, x, y, z circumscribed about a strictly convex compact body $C \subset \mathbb{R}^3$ with nonempty interior. Let*

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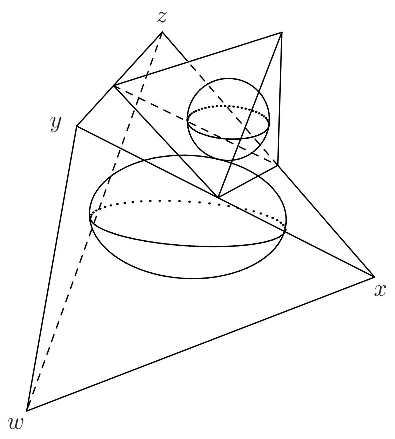

Figure 1.1: Geometric determination of tangency points of strictly convex body with a minimal area tetrahedron

$J_{c_w}^{-\frac{1}{2}}$, $J_{c_x}^{-\frac{1}{2}}$, $J_{c_y}^{-\frac{1}{2}}$, $J_{c_z}^{-\frac{1}{2}}$ be the homotheties with ratio $-\frac{1}{2}$ and centers c_w, c_x, c_y, c_z at centroids of the corresponding faces $\Delta(x,y,z)$, $\Delta(w,y,z)$, $\Delta(w,x,z)$, $\Delta(w,x,y)$. Then the tangency points of C with the tetrahedron $T(w,x,y,z)$ are exactly the tangency points of spheres inscribed in images of the tetrahedron under these homotheties (see Fig. 1.1).

2. Proof of the theorem

2.1. Basic facts

Let us consider a strictly convex compact body $C \subset \mathbb{R}^3$ such that $\operatorname{int}(C) \neq \emptyset$ and let $\sigma(x) = \max\limits_{u \in C} \langle x, u \rangle$, $x \in \mathbb{R}^3$ be the support function of C.

We are looking for minimal area tetrahedron $T(w,x,y,z)$ with vertices w,x,y,z circumscribed about C. Put

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad x, a, b \in \mathbb{R}^3$$

We define a linear mapping $L_a : \mathbb{R}^3 \to \mathbb{R}^3$

$$L_a x = x \times a = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and a product

$$x \wedge a \wedge b = \det \begin{bmatrix} x_1 & a_1 & b_1 \\ x_2 & a_2 & b_2 \\ x_3 & a_3 & b_3 \end{bmatrix}.$$

Then we have

$$\frac{\partial}{\partial x}(x \wedge a \wedge b) = a \times b, \ \frac{\partial}{\partial x}(x \times a) = L_a$$

and for any differentiable function $f : \mathbb{R}^3 \to \mathbb{R}$

$$\frac{\partial}{\partial x} f(x \times a) = \left(\frac{\partial}{\partial x}(x \times a)\right)^T \cdot f'(x \times a) = L_a^T f'(x \times a) = -L_a f'(x \times a) = -f'(x \times a) \times a.$$

The double area $P = P(w, x, y, z)$ of $T(w, x, y, z)$ is equal to

$$P(w, x, y, z) = ||N_w|| + ||N_x|| + ||N_y|| + ||N_z||,$$

where

$$\begin{aligned} N_w &= (z - x) \times (y - x) = -y \times z + y \times x + x \times z, \\ N_x &= (y - w) \times (z - w) = -w \times z + w \times y + y \times z, \\ N_y &= (z - w) \times (x - w) = -x \times z + x \times w + w \times z, \\ N_z &= (x - w) \times (y - w) = -w \times y + w \times x + x \times y \end{aligned}$$

are the normal vectors to the suitable faces pointing outward for right choice of vertices w, x, y, z and $||\ \ ||$ denotes the Euclidean norm in $\mathbb{R}^3$. Using the above definitions we have

$$\begin{array}{llll} \dfrac{\partial N_w}{\partial w} = 0, & \dfrac{\partial N_x}{\partial w} = L_{y-z}, & \dfrac{\partial N_y}{\partial w} = L_{z-x}, & \dfrac{\partial N_z}{\partial w} = L_{x-y}, \\ \dfrac{\partial N_w}{\partial x} = L_{z-y}, & \dfrac{\partial N_x}{\partial x} = 0, & \dfrac{\partial N_y}{\partial x} = L_{w-z}, & \dfrac{\partial N_z}{\partial x} = L_{y-w}, \\ \dfrac{\partial N_w}{\partial y} = L_{x-z}, & \dfrac{\partial N_x}{\partial y} = L_{z-w}, & \dfrac{\partial N_y}{\partial y} = 0, & \dfrac{\partial N_z}{\partial y} = L_{w-x}, \\ \dfrac{\partial N_w}{\partial z} = L_{y-x}, & \dfrac{\partial N_x}{\partial z} = L_{w-y}, & \dfrac{\partial N_y}{\partial z} = L_{x-w}, & \dfrac{\partial N_z}{\partial z} = 0. \end{array}$$

2.2. Lagrange multipliers

Suppose now, that $T(w, x, y, z)$ is a minimal area tetrahedron circumscribing C. In this case we have

$$\begin{aligned} \sigma(N_w) &= \langle x, N_w \rangle = \langle x, -y \times z \rangle = -x \wedge y \wedge z \\ \sigma(N_x) &= \langle y, N_x \rangle = \langle y, -w \times z \rangle = -y \wedge w \wedge z \\ \sigma(N_y) &= \langle x, N_y \rangle = \langle z, x \times w \rangle = -w \wedge x \wedge z \\ \sigma(N_z) &= \langle x, N_z \rangle = \langle w, x \times y \rangle = -x \wedge w \wedge y. \end{aligned}$$

Then from the Lagrange multipliers rule it follows that there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned}
&\frac{\partial P}{\partial w} - \lambda_2 \frac{\partial}{\partial w}\left(\sigma\left(N_x\right) + y \wedge w \wedge z\right) - \lambda_3 \frac{\partial}{\partial w}\left(\sigma\left(N_y\right) + z \wedge w \wedge x\right) - \\
&\qquad - \lambda_4 \frac{\partial}{\partial w}\left(\sigma\left(N_z\right) + w \wedge y \wedge x\right) = 0, \\
&\frac{\partial P}{\partial x} - \lambda_1 \frac{\partial}{\partial x}\left(\sigma\left(N_w\right) + x \wedge y \wedge z\right) - \lambda_3 \frac{\partial}{\partial x}\left(\sigma\left(N_y\right) + z \wedge w \wedge x\right) - \\
&\qquad - \lambda_4 \frac{\partial}{\partial x}\left(\sigma\left(N_z\right) + w \wedge y \wedge x\right) = 0, \\
&\frac{\partial P}{\partial y} - \lambda_1 \frac{\partial}{\partial y}\left(\sigma\left(N_w\right) + x \wedge y \wedge z\right) - \lambda_2 \frac{\partial}{\partial y}\left(\sigma\left(N_x\right) + y \wedge w \wedge z\right) - \\
&\qquad - \lambda_4 \frac{\partial}{\partial y}\left(\sigma\left(N_z\right) + w \wedge y \wedge x\right) = 0, \\
&\frac{\partial P}{\partial z} - \lambda_1 \frac{\partial}{\partial z}\left(\sigma\left(N_w\right) + x \wedge y \wedge z\right) - \lambda_2 \frac{\partial}{\partial z}\left(\sigma\left(N_x\right) + y \wedge w \wedge z\right) - \\
&\qquad - \lambda_3 \frac{\partial}{\partial z}\left(\sigma\left(N_y\right) + z \wedge w \wedge x\right) = 0.
\end{aligned}$$

Let us denote

$$n_w = \frac{N_w}{\|N_w\|},\ n_x = \frac{N_x}{\|N_x\|},\ n_y = \frac{N_y}{\|N_y\|},\ n_z = \frac{N_z}{\|N_z\|}.$$

Using the formula for differentiation from Section 2.1 we get

$$\begin{aligned}
\frac{\partial P}{\partial w} &= L_{y-z}^T n_x + L_{z-x}^T n_y + L_{x-y}^T n_z = n_x \times (z-y) + n_y \times (x-z) + n_z \times (y-x), \\
\frac{\partial P}{\partial x} &= L_{z-y}^T n_w + L_{w-z}^T n_y + L_{y-w}^T n_z = n_w \times (y-z) + n_y \times (z-w) + n_z \times (w-y), \\
\frac{\partial P}{\partial y} &= L_{x-z}^T n_w + L_{z-w}^T n_x + L_{w-x}^T n_z = n_w \times (z-x) + n_x \times (w-z) + n_z \times (x-w), \\
\frac{\partial P}{\partial z} &= L_{y-x}^T n_w + L_{w-y}^T n_x + L_{x-w}^T n_y = n_w \times (x-y) + n_x \times (y-w) + n_y \times (w-x).
\end{aligned}$$

By the similar reasoning we get

$$\begin{aligned}
\frac{\partial}{\partial w}\left(\sigma\left(N_x\right) + y \wedge w \wedge z\right) &= L_{y-z}^T \sigma'\left(N_x\right) - y \times z = \left(\sigma'\left(N_x\right) - y\right) \times (z-y), \\
\frac{\partial}{\partial w}\left(\sigma\left(N_y\right) + z \wedge w \wedge x\right) &= L_{z-x}^T \sigma'\left(N_y\right) - z \times x = \left(\sigma'\left(N_y\right) - z\right) \times (x-z), \\
\frac{\partial}{\partial w}\left(\sigma\left(N_z\right) + w \wedge y \wedge x\right) &= L_{x-y}^T \sigma'\left(N_z\right) - x \times y = \left(\sigma'\left(N_z\right) - x\right) \times (y-x), \\
\frac{\partial}{\partial x}\left(\sigma\left(N_w\right) + x \wedge y \wedge z\right) &= L_{z-y}^T \sigma'\left(N_w\right) - z \times y = \left(\sigma'\left(N_w\right) - z\right) \times (y-z), \\
\frac{\partial}{\partial x}\left(\sigma\left(N_y\right) + z \wedge w \wedge x\right) &= L_{w-z}^T \sigma'\left(N_y\right) - w \times z = \left(\sigma'\left(N_y\right) - w\right) \times (z-w), \\
\frac{\partial}{\partial x}\left(\sigma\left(N_z\right) + w \wedge y \wedge x\right) &= L_{y-w}^T \sigma'\left(N_z\right) - y \times w = \left(\sigma'\left(N_z\right) - y\right) \times (w-y),
\end{aligned}$$

$$\frac{\partial}{\partial y}\left(\sigma\left(N_w\right)+x\wedge y\wedge z\right)=L^T_{x-z}\sigma'\left(N_w\right)-x\times z=\left(\sigma'\left(N_w\right)-x\right)\times(z-x),$$
$$\frac{\partial}{\partial y}\left(\sigma\left(N_x\right)+y\wedge w\wedge z\right)=L^T_{z-w}\sigma'\left(N_x\right)-z\times w=\left(\sigma'\left(N_x\right)-z\right)\times(w-z),$$
$$\frac{\partial}{\partial y}\left(\sigma\left(N_z\right)+w\wedge y\wedge x\right)=L^T_{w-x}\sigma'\left(N_z\right)-w\times x=\left(\sigma'\left(N_z\right)-w\right)\times(x-w),$$
$$\frac{\partial}{\partial z}\left(\sigma\left(N_w\right)+x\wedge y\wedge z\right)=L^T_{y-x}\sigma'\left(N_w\right)-y\times x=\left(\sigma'\left(N_w\right)-y\right)\times(x-y),$$
$$\frac{\partial}{\partial z}\left(\sigma\left(N_x\right)+y\wedge w\wedge z\right)=L^T_{w-y}\sigma'\left(N_x\right)-w\times y=\left(\sigma'\left(N_x\right)-w\right)\times(y-w),$$
$$\frac{\partial}{\partial z}\left(\sigma\left(N_y\right)+z\wedge w\wedge x\right)=L^T_{x-w}\sigma'\left(N_y\right)-x\times w=\left(\sigma'\left(N_y\right)-x\right)\times(w-x).$$

2.3. Some auxiliary considerations

Let $\Delta(a,b,c)$ denote the triangle with vertices $a,b,c\in\mathbb{R}^3$. Since

$$\sigma'\left(N_w\right)\in\operatorname{int}(\Delta(x,y,z)),$$

then there exist $\mu_{w,x},\mu_{w,y},\mu_{w,z}>0$, such that $\mu_{w,x}+\mu_{w,y}+\mu_{w,z}=1$ and

$$\sigma'\left(N_w\right)=\mu_{w,x}\cdot x+\mu_{w,y}\cdot y+\mu_{w,z}\cdot z.$$

Analogously, we have

$$\sigma'\left(N_x\right)=\mu_{x,w}\cdot w+\mu_{x,y}\cdot y+\mu_{x,z}\cdot z,\ \text{ where } \mu_{x,w},\mu_{x,y},\mu_{x,z}>0,\ \mu_{x,w}+\mu_{x,y}+\mu_{x,z}=1,$$
$$\sigma'\left(N_y\right)=\mu_{y,w}\cdot w+\mu_{y,x}\cdot x+\mu_{y,z}\cdot z,\ \text{ where } \mu_{y,w},\mu_{y,x},\mu_{y,z}>0,\ \mu_{y,w}+\mu_{y,x}+\mu_{y,z}=1,$$
$$\sigma'\left(N_z\right)=\mu_{z,w}\cdot w+\mu_{z,x}\cdot x+\mu_{z,y}\cdot y,\ \text{ where } \mu_{z,w},\mu_{z,x},\mu_{z,y}>0,\ \mu_{z,w}+\mu_{z,x}+\mu_{z,y}=1.$$

It is easy to see that

$$\frac{\partial}{\partial w}\left(\sigma\left(N_x\right)+y\wedge w\wedge z\right)=-\mu_{x,w}\cdot N_x,$$
$$\frac{\partial}{\partial w}\left(\sigma\left(N_y\right)+z\wedge w\wedge x\right)=-\mu_{y,w}\cdot N_y,$$
$$\frac{\partial}{\partial w}\left(\sigma\left(N_z\right)+w\wedge y\wedge x\right)=-\mu_{z,w}\cdot N_z,$$
$$\frac{\partial}{\partial x}\left(\sigma\left(N_w\right)+x\wedge y\wedge z\right)=-\mu_{w,x}\cdot N_w,$$
$$\frac{\partial}{\partial x}\left(\sigma\left(N_y\right)+z\wedge w\wedge x\right)=-\mu_{y,x}\cdot N_y,$$
$$\frac{\partial}{\partial x}\left(\sigma\left(N_z\right)+w\wedge y\wedge x\right)=-\mu_{z,x}\cdot N_z,$$
$$\frac{\partial}{\partial y}\left(\sigma\left(N_w\right)+x\wedge y\wedge z\right)=-\mu_{w,y}\cdot N_w,$$
$$\frac{\partial}{\partial y}\left(\sigma\left(N_x\right)+y\wedge w\wedge z\right)=-\mu_{x,y}\cdot N_x,$$
$$\frac{\partial}{\partial y}\left(\sigma\left(N_z\right)+w\wedge y\wedge x\right)=-\mu_{z,y}\cdot N_z,$$

$$\frac{\partial}{\partial z}\left(\sigma\left(N_w\right)+x\wedge y\wedge z\right)=-\mu_{w,z}\cdot N_w,$$
$$\frac{\partial}{\partial z}\left(\sigma\left(N_x\right)+y\wedge w\wedge z\right)=-\mu_{x,z}\cdot N_x,$$
$$\frac{\partial}{\partial z}\left(\sigma\left(N_y\right)+z\wedge w\wedge x\right)=-\mu_{y,z}\cdot N_y.$$

Thus, if $T(w,x,y,z)$ is the minimal area tetrahedron circumscribing C, then

$$\begin{aligned}
-\lambda_2\mu_{x,w}\cdot N_x-\lambda_3\mu_{y,w}\cdot N_y-\lambda_4\mu_{z,w}\cdot N_z&=n_x\times(z-y)+n_y\times(x-z)+n_z\times(y-x)\,,\\
-\lambda_1\mu_{w,x}\cdot N_w-\lambda_3\mu_{y,x}\cdot N_y-\lambda_4\mu_{z,x}\cdot N_z&=n_w\times(y-z)+n_y\times(z-w)+n_z\times(w-y)\,,\\
-\lambda_1\mu_{w,y}\cdot N_w-\lambda_2\mu_{x,y}\cdot N_x-\lambda_4\mu_{z,y}\cdot N_z&=n_w\times(z-x)+n_x\times(w-z)+n_z\times(x-w)\,,\\
-\lambda_1\mu_{w,z}\cdot N_w-\lambda_2\mu_{x,z}\cdot N_x-\lambda_3\mu_{y,z}\cdot N_y&=n_w\times(x-y)+n_x\times(y-w)+n_y\times(w-x)\,.
\end{aligned}$$

From the above equations we get

$$\begin{aligned}
\mu_{x,w}&=\frac{1}{2}\cdot\frac{\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle}{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle},\\
\mu_{x,y}&=\frac{1}{2}\cdot\frac{\langle n_w-n_x,N_w\rangle+\langle n_z-n_x,N_z\rangle}{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle},\\
\mu_{x,z}&=\frac{1}{2}\cdot\frac{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle}{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle}.
\end{aligned}$$

2.4. Final step

We shall denote by ξ the center of the sphere inscribed in $T(w,x,y,z)$. Let ξ_x be the common point of the sphere with the face $\Delta(w,y,z)$. Then

$$\xi_x=\mu_w\cdot w+\mu_y\cdot y+\mu_z\cdot z,\ \text{ where } \mu_w,\mu_y,\mu_z>0 \text{ and } \mu_w+\mu_y+\mu_z=1.$$

and after some calculations we get

$$\begin{aligned}
\mu_w&=\frac{\langle n_w-n_x,N_w\rangle}{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle},\\
\mu_y&=\frac{\langle n_y-n_x,N_y\rangle}{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle},\\
\mu_z&=\frac{\langle n_z-n_x,N_z\rangle}{\langle n_w-n_x,N_w\rangle+\langle n_y-n_x,N_y\rangle+\langle n_z-n_x,N_z\rangle}.
\end{aligned}$$

For fixed $a\in\mathbb{R}^3$ we define a homothety

$$J_a^{-\frac{1}{2}}(x)=\frac{3}{2}\cdot a-\frac{1}{2}\cdot x,\quad x\in\mathbb{R}^3.$$

Let ξ_w, ξ_x, ξ_y, ξ_z be the common points of $T(w,x,y,z)$ with the faces

$$\Delta\left(x,y,z\right),\ \Delta\left(w,y,z\right),\ \Delta\left(w,x,z\right),\ \Delta\left(w,x,y\right),$$

respectively, and let c_w, c_x, c_y, c_z be the barycenters of these faces. If the tetrahedron $T(w,x,y,z)$ is circumscribed about C and has minimal area, then

$$J_{c_w}^{-\frac{1}{2}}(\xi_w) \in C,\ J_{c_x}^{-\frac{1}{2}}(\xi_x) \in C,\ J_{c_y}^{-\frac{1}{2}}(\xi_y) \in C,\ J_{c_z}^{-\frac{1}{2}}(\xi_z) \in C.$$

Indeed, for example, for the face $\Delta(w,y,z)$ we have

$$\begin{aligned} J_{c_x}^{-\frac{1}{2}}(\xi_x) &= \frac{3}{2} \cdot c_x - \frac{1}{2} \cdot \xi_x = \frac{1}{2} \cdot (w+y+z) - \frac{1}{2} \cdot (\mu_w \cdot w + \mu_y \cdot y + \mu_z \cdot z) \\ &= \frac{1}{2}(1-\mu_w) \cdot w + \frac{1}{2}(1-\mu_y) \cdot y + \frac{1}{2}(1-\mu_z) \cdot z \end{aligned}$$

and

$$\begin{aligned} 1-\mu_w &= \frac{\langle n_y - n_x, N_y \rangle + \langle n_z - n_x, N_z \rangle}{\langle n_w - n_x, N_w \rangle + \langle n_y - n_x, N_y \rangle + \langle n_z - n_x, N_z \rangle} = 2\mu_{x,w}, \\ 1-\mu_y &= \frac{\langle n_w - n_x, N_w \rangle + \langle n_z - n_x, N_z \rangle}{\langle n_w - n_x, N_w \rangle + \langle n_y - n_x, N_y \rangle + \langle n_z - n_x, N_z \rangle} = 2\mu_{x,y}, \\ 1-\mu_z &= \frac{\langle n_w - n_x, N_w \rangle + \langle n_y - n_x, N_y \rangle}{\langle n_w - n_x, N_w \rangle + \langle n_y - n_x, N_y \rangle + \langle n_z - n_x, N_z \rangle} = 2\mu_{x,z}. \end{aligned}$$

Thus the theorem is proved. □

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