Journal of Convex Analysis
Volume 14 (2007), No. 3, 455–463

Direction of Movement of the Element of Minimal Norm in a Moving Convex Set

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Received: October 27, 2005
Revised manuscript received: February 1, 2006

We show that if K is a nonempty closed convex subset of a real Hilbert space H, e is a non-zero arbitrary vector in H and for each $t \in \mathbb{R}$, $z(t)$ is the closest point in $K + te$ to the origin, then the angle $z(t)$ makes with e is a decreasing function of t while $z(t) \neq 0$, and the inner product of $z(t)$ with e is increasing.

Keywords: Moving convex set, nearest point projection

1991 Mathematics Subject Classification: Primary 46C05; Secondary 47H99, 41A65

1. Introduction

Given a nonempty closed convex subset K of a real Hilbert space H, we consider, for t real, the translate

$$K(t) = K + te$$

where $e \neq 0$ is an element of H. Let $z(t) = P_{K(t)}(0)$, the projection of 0 on $K(t)$, i.e. the nearest point in $K(t)$ to 0. We ask the questions: in which direction is $z(t)$ moving, and how is the inner product $\langle z(t), e\rangle$ changing? This problem came up in writing the paper [2], when we wished to study the inner product $\langle (A + tB)^\circ x, Bx\rangle$. Here A is a maximal monotone operator on a real Hilbert space H, B is a single valued monotone operator defined everywhere on H, and for each x in domain of A, $(A+tB)^\circ x$ denotes the element of minimal norm in $(A+tB)x$. The questions are simple and so are the answers, namely, the angle $z(t)$ makes with e is decreasing, and the inner product of $z(t)$ and e is increasing. We also pose the same questions for the related set, $K'(t) = (1-t)K + te$ for $t \in [0, 1]$. If $z'(t) = P_{K'(t)}(0)$, then how does the angle and the inner product with e change with time? Now the angle is decreasing but the inner product need not be increasing.

Other properties of the projection P_K onto K have been studied: for instance, non-expansiveness (see [12]) and differentiability (see [7] and [9]). Zarantonello [14] gave many useful properties of P_K. The book by Dontchev and Zolezzi [6] is a useful reference for the approximation to a given point, using more general sets in more general spaces.

Our translation $K + te$ of K gives a particular case of moving convex sets. By a moving convex set Moreau [10] means a set-valued mapping C from a real interval I to a real Hilbert space H such that $C(t)$ are nonempty closed convex subsets of H. The evolution problem $-\dot{u}(t) \in N_{C(t)}(u(t))$, (the normal cone to $C(t)$ at $u(t)$), is studied in [10] and [11]. The case of nonconvex sets $C(t)$ is studied in [1], [3] and [13].

ISSN 0944-6532 / \$ 2.50

In [4, 5] the authors address a time dependent variational inequality,

$$z(t) \in C(t), \langle \gamma(z(t)), y(t) - z(t) \rangle \geq 0 \quad \forall\, y(t) \in C(t), \text{ i.e. } \gamma(z(t)) + N_{C(t)}z(t) \ni 0,$$

where γ is a mapping. A nice survey of time dependent variational inequalities is given in [8].

2. Main Theorems

Let H be a real Hilbert space and K be a nonempty closed convex subset of H. Define $P_K : H \longrightarrow K$ as, for each $x \in H$, $P_K(x)$ is the nearest point in K to x.

Theorem 2.1. *Let K be a nonempty closed convex subset of a real Hilbert space H and let e be an arbitrary non-zero but fixed vector in H. For each $t \in \mathbb{R}$, let $K(t) = K + te$ and let $z(t)$ be the element of minimal norm in $K(t)$, $z(t) = P_{K(t)}(0)$. Then the angle $z(t)$ makes with e is a decreasing function of t on the set $\{t : z(t) \neq 0\}$. That means for each $t \geq s$; $t, s \in \mathbb{R}$,*

$$\frac{\langle z(t), e \rangle}{\|z(t)\|} \geq \frac{\langle z(s), e \rangle}{\|z(s)\|}, \tag{1}$$

if $z(t)$ and $z(s)$ are not equal to 0.

Lemma 2.2. *The following are equivalent:*

(a) *Theorem 2.1 holds.*

(b) *For all K, e as in Theorem 2.1, and $z(t)$ defined as in Theorem 2.1, if $t > 0$ and $z(t)$ and $z(0)$ are non-zero vectors then*

$$\frac{\langle z(t), e \rangle}{\|z(t)\|} \geq \frac{\langle z(0), e \rangle}{\|z(0)\|}. \tag{2}$$

(c) *For all K, e as in Theorem 2.1, for $t > 0$, if $-te$ and 0 are not in K then the angle $P_K(-te) + te$ makes with e is less than or equal to the angle $P_K(0)$ makes with e.*

Proof. We first show that (a) is equivalent to (b). Obviously, (a) $\Longrightarrow$ (b). To see (b) $\Longrightarrow$ (a), given K, e, s and t, we note that $K(t) = K(s) + (t - s)e$ for $t \geq s$. Applying (2) using the set $K(s)$ for K, and $t - s > 0$ for $t > 0$, we obtain (1).
Now we show that (b) and (c) are equivalent. Let K, e and $t > 0$ be given. There is one to one correspondence between K and $K(t)$ given by $x \mapsto x + te$. Since $z(t) \in K(t)$ there exists a unique $x(t) \in K$ such that $z(t) = x(t) + te$ and thus

$$\frac{\langle z(t), e \rangle}{\|z(t)\|} = \frac{\langle x(t) + te, e \rangle}{\|x(t) + te\|} \quad \text{if } z(t) \neq 0. \tag{3}$$

Note that $z(t)$ is nonzero iff $-te$ is not in K. Also, note that for $t = 0$, $z(0) = x(0) = P_K(0)$ and

$$\frac{\langle z(0), e \rangle}{\|z(0)\|} = \frac{\langle x(0), e \rangle}{\|x(0)\|} \quad \text{if } z(0) \neq 0. \tag{4}$$

From (3) and (4), (b) is equivalent to

$$\frac{\langle x(t) + te, e \rangle}{\|x(t) + te\|} \geq \frac{\langle x(0), e \rangle}{\|x(0)\|} \quad \text{if } z(t) \neq 0, z(0) \neq 0. \tag{5}$$

We note that , $x(0) = P_K(0)$ and $x(t) = P_K(-te)$ so that (5) is equivalent to:

$$\frac{\langle P_K(-te) + te, e\rangle}{\|P_K(-te) + te\|} \geq \frac{\langle P_K(0), e\rangle}{\|P_K(0)\|} \quad \text{if } -te \text{ and } 0 \text{ are not in } K,$$

i.e. (b) is equivalent to (c). □

Proof of Theorem 2.1. In view of Lemma 2.2, we assume $t > 0, z(t) \neq 0, z(0) \neq 0$, and need to show (5).

Let $x(t) := z(t) - te$ (as in the proof of Lemma 2.2) and K_3 be the intersection of K with the vector subspace spanned by $\{e, x(0), x(t)\}$, span$\{e, x(0), x(t)\}$. Then K_3 is a closed convex subset of K. We note that $x(0)$ and $x(t)$ are respectively the projections of the origin and $-te$ on K_3. Depending upon the dimension of span$\{e, x(0), x(t)\}$ three cases arise:

Case 1. Span$\{e, x(0), x(t)\}$ is 3-dimensional.
Let $T(t)$ and $T(0)$ respectively be the supporting planes in span$\{e, x(0), x(t)\}$ to K_3 at $x(t)$ and $x(0)$ such that $T(t)$ is orthogonal to $x(t)+te$ and $T(0)$ is orthogonal to $x(0)$. Let $X(t)$ and $X(0)$ be the closed half spaces of span$\{e, x(0), x(t)\}$ with boundaries $T(t)$ and $T(0)$ respectively, which do not contain $-te$ and 0 respectively. We define $K' = X(t) \cap X(0)$. Then K' is a closed convex subset of span$\{e, x(0), x(t)\}$ and $K' \supseteq K_3$. We note that the planes $T(t)$ and $T(0)$ are neither parallel nor equal to each other, otherwise, the vectors $x(t)+te$ and $x(0)$, their respective normal vectors, would be parallel to each other contradicting the fact that $\{e, x(0), x(t)\}$ are linearly independent. Then the following two cases arise:

Case 1.1. The line L through the origin and the vector e does not pass through K'.
For each $t' \in [0, t]$, let $y(t')$ be the closest point to $-t'e$ in K'. We note that $y(0) = x(0)$ and $y(t) = x(t)$. Then we have for each $t' \in [0, t]$, either

(1.1.1) $y(t') \in T(0)$ and $y(t') + t'e$ orthogonal to $T(0)$, or
(1.1.2) $y(t') \in T(t)$ and $y(t') + t'e$ orthogonal to $T(t)$, or
(1.1.3) $y(t') \in T(0) \cap T(t)$.

Let t_1 and t_2 be in $[0, t]$ such that for $t' \in [0, t_1]$, (1.1.1) holds, for $t' \in [t_1, t_2]$, (1.1.3) holds and for $t' \in [t_2, t]$, (1.1.2) holds.

Then there exists $\lambda_1 > 0$ such that $y(t_1) + t_1 e = \lambda_1 x(0)$ and therefore

$$\frac{\langle y(t_1) + t_1 e, e\rangle}{\|y(t_1) + t_1 e\|} = \frac{\langle \lambda_1 x(0), e\rangle}{\|\lambda_1 x(0)\|} = \frac{\langle x(0), e\rangle}{\|x(0)\|}. \tag{6}$$

Also there exists $\lambda_2 > 0$ such that $y(t_2) + t_2 e = \lambda_2 (x(t) + te)$ and therefore

$$\frac{\langle y(t_2) + t_2 e, e\rangle}{\|y(t_2) + t_2 e\|} = \frac{\langle \lambda_2 (x(t) + te), e\rangle}{\|\lambda_2 (x(t) + te)\|} = \frac{\langle x(t) + te, e\rangle}{\|x(t) + te\|}. \tag{7}$$

So by (6) and (7), to prove (5) it suffices to show that

$$\frac{\langle y(t_2) + t_2 e, e\rangle}{\|y(t_2) + t_2 e\|} \geq \frac{\langle y(t_1) + t_1 e, e\rangle}{\|y(t_1) + t_1 e\|}. \tag{8}$$

To prove (8), we first note down a few useful properties of $y(t_1)$ and $y(t_2)$. Note that $y(t_1)+t_1e$ and $y(t_2)+t_2e$ are both orthogonal to the vector $y(t_1)-y(t_2)$. Therefore,

$$\langle y(t_1)+t_1e, y(t_1)-y(t_2)\rangle = 0, \tag{9}$$

and

$$\langle y(t_2)+t_2e, y(t_1)-y(t_2)\rangle = 0. \tag{10}$$

Note that $t_1 \neq t_2$. Also, we can write e as

$$e = \frac{(y(t_2)+t_2e)-(y(t_1)+t_1e)}{t_2-t_1} + \frac{y(t_1)-y(t_2)}{t_2-t_1}. \tag{11}$$

Then using (10) and (11) we obtain

$$\begin{aligned}
\frac{\langle y(t_2)+t_2e, e\rangle}{\|y(t_2)+t_2e\|} &= \frac{\left\langle y(t_2)+t_2e, \frac{(y(t_2)+t_2e)-(y(t_1)+t_1e)}{t_2-t_1} + \frac{y(t_1)-y(t_2)}{t_2-t_1}\right\rangle}{\|y(t_2)+t_2e\|} \\
&= \frac{\left\langle y(t_2)+t_2e, \frac{(y(t_2)+t_2e)-(y(t_1)+t_1e)}{t_2-t_1}\right\rangle}{\|y(t_2)+t_2e\|} \\
&= \frac{1}{t_2-t_1}\left(\|y(t_2)+t_2e\| - \frac{\langle y(t_2)+t_2e, y(t_1)+t_1e\rangle}{\|y(t_2)+t_2e\|}\right) \\
&\geq \frac{1}{t_2-t_1}\left(\|y(t_2)+t_2e\| - \|y(t_1)+t_1e\|\right).
\end{aligned} \tag{12}$$

Similarly, using (9) and (11) we obtain

$$\begin{aligned}
\frac{\langle y(t_1)+t_1e, e\rangle}{\|y(t_1)+t_1e\|} &= \frac{\left\langle y(t_1)+t_1e, \frac{(y(t_2)+t_2e)-(y(t_1)+t_1e)}{t_2-t_1} + \frac{y(t_1)-y(t_2)}{t_2-t_1}\right\rangle}{\|y(t_1)+t_1e\|} \\
&= \frac{\left\langle y(t_1)+t_1e, \frac{(y(t_2)+t_2e)-(y(t_1)+t_1e)}{t_2-t_1}\right\rangle}{\|y(t_1)+t_1e\|} \\
&= \frac{1}{t_2-t_1}\left(\frac{\langle y(t_1)+t_1e, y(t_2)+t_2e\rangle}{\|y(t_1)+t_1e\|} - \|y(t_1)+t_1e\|\right) \\
&\leq \frac{1}{t_2-t_1}\left(\|y(t_2)+t_2e\| - \|y(t_1)+t_1e\|\right).
\end{aligned} \tag{13}$$

Combining (12) and (13) we get (8).

Case 1.2. The line L through the origin and the vector e passes through K'.
Let $-t_1e$ and $-t_2e$ respectively be the points of intersection of the line L with the planes $T(0)$ and $T(t)$. We note that $0 < t_1 \leq t_2 < t$. The vectors $x(0)$ and $-t_1e$ are in the plane T(0) and thus $x(0)+t_1e$ is orthogonal to $x(0)$, and vectors $x(t)$ and $-t_2e$ are in the plane T(t) and thus $x(t)+t_2e$ is orthogonal to $x(t)+te$. That means

$$\langle x(0), x(0)+t_1e\rangle = 0, \tag{14}$$

and

$$\langle x(t)+te, x(t)+t_2e\rangle = 0. \tag{15}$$

Then using (14) we obtain

$$\frac{\langle x(0),e\rangle}{\|x(0)\|}=\frac{1}{t_1}\frac{\langle x(0),t_1e+x(0)-x(0)\rangle}{\|x(0)\|}=\frac{-1}{t_1}\|x(0)\|<0, \tag{16}$$

and using (15) we obtain

$$\frac{\langle x(t)+te,e\rangle}{\|x(t)+te\|}=\frac{1}{t-t_2}\frac{\langle x(t)+te,te-t_2e+x(t)-x(t)\rangle}{\|x(t)+te\|}=\frac{1}{t-t_2}\|x(t)+te\|>0. \tag{17}$$

Combining (16) and (17) proves (5).

Case 2. Span$\{e,x(0),x(t)\}$ is 2-dimensional.
Let $T(t)$and $T(0)$ respectively be the supporting lines in span$\{e,x(0),x(t)\}$ to K_3 at $x(t)$ and $x(0)$ which are orthogonal to $x(t)+te$ and $x(0)$ respectively. Let $X(t)$ and $X(0)$ be the closed half spaces of span$\{e,x(0),x(t)\}$ with boundaries $T(t)$ and $T(0)$ respectively, which do not contain $-te$ and 0 respectively. We define $K'=X(t)\cap X(0)$. Then K' is a closed convex subset of H and $K'\supseteq K_3$. Then three cases arise.

Case 2.1. The supporting lines $T(t)$ and $T(0)$ are distinct and meet each other in a point and the line L through the origin and the vector e does not pass through K'.
Then inequality (5) follows from the same argument as in Case 1.1. Note that here, $y(t_1)=y(t_2)$.

Case 2.2. The supporting lines $T(t)$ and $T(0)$ are distinct and meet each other in a point, and the line L through the origin and the vector e passes through K'.
Once again, (5) follows from the same argument as in Case 1.2.

Case 2.3. $T(t)$ and $T(0)$ are parallel or equal to each other.
Since $x(0)$ and $x(t)+te$ are respectively orthogonal to the lines $T(0)$ and $T(t)$, $x(t)+te$ and $x(0)$ are parallel i.e., there exists $\lambda\neq 0$ such that $x(t)+te=\lambda x(0)$. Therefore,

$$\frac{\langle x(t)+te,e\rangle}{\|x(t)+te\|}=\frac{\langle \lambda x(0),e\rangle}{\|\lambda x(0)\|}=\frac{\lambda}{|\lambda|}\frac{\langle x(0),e\rangle}{\|x(0)\|}.$$

Clearly for $\lambda>0$,

$$\frac{\langle x(t)+te,e\rangle}{\|x(t)+te\|}=\frac{\langle x(0),e\rangle}{\|x(0)\|},$$

which proves (5). For $\lambda<0$,

$$\frac{\langle x(t)+te,e\rangle}{\|x(t)+te\|}=-\frac{\langle x(0),e\rangle}{\|x(0)\|}. \tag{18}$$

Since $x(t)+te$ and $x(0)$ are in opposite directions there exists $\mu\in(0,1)$ such that

$$\mu\left(x(t)+te\right)+(1-\mu)x(0)=0,$$

implying

$$-\mu te=\mu x(t)+(1-\mu)x(0)=k \quad \text{(say)}.$$

Since $x(t)$ and $x(0)$ are in the closed convex set K_3, $k\in K_3$. As $x(0)$ is the element of minimal norm in K_3, we have

$$0<\|x(0)\|^2\leq\langle x(0),k\rangle=\langle x(0),-\mu te\rangle,$$

implying

$$\langle x(0), e\rangle < 0. \tag{19}$$

Therefore, combining (18) and (19) we get (5).

Case 3. Span$\{e, x(0), x(t)\}$ is a one dimensional space.
Then $x(t) + te$ and $x(0)$ are parallel. Then (5) follows as in Case 2.3. □

Corollary 2.3. *Let K be a nonempty closed convex subset of a real Hilbert space H and let e be an arbitrary non-zero but fixed vector in H. For each $t \in [0, 1]$, let $K'(t) = (1-t)K+te$ and $z'(t) = P_{K'(t)}(0)$. Then the angle $z'(t)$ makes with e is a decreasing function of t on the set $\{t : z'(t) \neq 0\}$. (In fact this corollary readily gives Theorem 2.1, as well.)*

Proof. Let $0 \le s \le t < 1$ and $z'(t), z'(s) \neq 0$. Since $z'(t)$ is the element of minimal norm in $K'(t)$, $\frac{z'(t)}{1-t}$ will be the element of minimal norm in $K + \frac{t}{1-t}e$. Similarly, $\frac{z'(s)}{1-s}$ will be the element of minimal norm in $K + \frac{s}{1-s}e$. Therefore, using (1) we obtain

$$\frac{\left\langle \frac{z'(t)}{1-t}, e\right\rangle}{\left\|\frac{z'(t)}{1-t}\right\|} \ge \frac{\left\langle \frac{z'(s)}{1-s}, e\right\rangle}{\left\|\frac{z'(s)}{1-s}\right\|},$$

implying

$$\frac{\langle z'(t), e\rangle}{\|z'(t)\|} \ge \frac{\langle z'(s), e\rangle}{\|z'(s)\|}.$$

That means the angle $z'(t)$ makes with e is a decreasing function of t on the set $\{t : z'(t) \neq 0\}$. □

Now we study the inner products $\langle z(t), e\rangle$ and $\langle z'(t), e\rangle$, to see how they vary with t.

Theorem 2.4. *Let K be a nonempty closed convex subset of a real Hilbert space H and let e be an arbitrary non-zero but fixed vector in H. For each $t \in \mathbb{R}$, let $K(t) = K + te$ and let $z(t)$ be the element of minimal norm in $K(t)$.*

Then for each $t \ge s$; $t, s \in \mathbb{R}$,

$$\langle z(t), e\rangle \ge \langle z(s), e\rangle, \tag{20}$$

and

$$\langle z(t) - te, e\rangle \le \langle z(s) - se, e\rangle. \tag{21}$$

Proof. Let $t \ge s$; $t, s \in \mathbb{R}$ be given. Since $z(t)$ and $z(s)$ are the elements of minimal norm in $K(t)$ and $K(s)$ respectively we have for all $y \in K$

$$\langle y + te - z(t), z(t)\rangle \ge 0, \tag{22}$$

and

$$\langle y + se - z(s), z(s)\rangle \ge 0. \tag{23}$$

In particular, (22) and (23) hold for $y = z(s) - se$ and $y = z(t) - te$ respectively, giving

$$\langle z(s) - se + te - z(t), z(t)\rangle \ge 0, \tag{24}$$

and

$$\langle z(t) - te + se - z(s), z(s)\rangle \geq 0. \tag{25}$$

Adding (24) and (25) gives

$$-\|z(t) - z(s)\|^2 + (t - s)\langle e, z(t) - z(s)\rangle \geq 0. \tag{26}$$

Hence (20) follows from (26).
To show (21) assume $t > s$ and $\langle z(t) - te, e\rangle > \langle z(s) - se, e\rangle$. Then

$$\langle z(t) - te + se - z(s), (t - s)e\rangle > 0. \tag{27}$$

Adding (25) and (27) gives

$$\langle z(t) - te + se - z(s), z(s) - se + te\rangle > 0,$$

implying $\|z(t)\| > \|z(s) - se + te\|$ which contradicts the fact that $z(t)$ is the element of minimal norm in $K(t)$. Hence (21) holds. □

Remark 2.5. From (26) we get

$$\|z(t) - z(s)\| \leq (t - s)\|e\|,$$

that implies $z(t)$ is Lipschitz continuous and almost everywhere differentiable. The inequality (20) gives $\langle \dot{z}(s), e\rangle \geq 0$ for a.e. s, but we can strengthen this to give (28). We note that if z is differentiable at s, then dividing (26) by $(t - s)^2$ and taking the limit as $t \searrow s$, one obtains

$$\|\dot{z}(s)\|^2 \leq \langle e, \dot{z}(s)\rangle,$$

which can be rewritten as

$$\|\dot{z}(s) - \frac{e}{2}\|^2 = \|\dot{z}(s)\|^2 - \langle e, \dot{z}(s)\rangle + \frac{\|e\|^2}{4} \leq \left(\frac{\|e\|}{2}\right)^2. \tag{28}$$

When we studied the angle, we found the angle $z'(t)$ made with e and the angle $z(t)$ made with e were both decreasing. However, for the inner product, the result is surprisingly different. Although we have Theorem 2.4 showing the inner product $\langle z(t), e\rangle$ increasing, the next corollary gives $\langle z'(t), e\rangle$ strictly decreasing if each $x \in K$ satisfies $\langle x, e\rangle > \|e\|^2$.

Corollary 2.6. *Let K be a nonempty closed convex subset of a real Hilbert space H and let e be an arbitrary non-zero but fixed vector in H. For each $t \in [0, 1]$, let $K'(t) = (1-t)K+te$ and $z'(t)$ be the element of minimal norm in $K'(t)$. Then for each $t, s \in [0, 1)$,*

$$\langle \frac{z'(t) - te}{1 - t}, e\rangle \leq \langle \frac{z'(s) - se}{1 - s}, e\rangle \quad \text{for } t \geq s, \tag{29}$$

and if $\langle x, e\rangle > \|e\|^2$ for all x in K,

$$\langle z'(t), e\rangle < \langle z'(s), e\rangle \quad \text{for } t > s.$$

Proof. Let $t \geq s$; $t, s \in [0, 1)$ be given. Since $z'(t)$ is the element of minimal norm in $K'(t)$, $\frac{z'(t)}{1-t}$ will be the element of minimal norm in $K + \frac{t}{1-t}e$. Similarly, $\frac{z'(s)}{1-s}$ will be the element of minimal norm in $K + \frac{s}{1-s}e$. Therefore, using (21) gives (29).

We note that for each $t \in [0, 1)$, $\frac{z'(t)-te}{1-t} \in K$. Therefore $\langle \frac{z'(t)-te}{1-t}, e\rangle > \|e\|^2$, implying

$$\langle z'(t), e\rangle > \|e\|^2. \tag{30}$$

Also the inequality (29) gives

$$\langle z'(s), e\rangle \geq (1-s)\langle \frac{z'(t)-te}{1-t}, e\rangle + s\|e\|^2.$$

Therefore for $t > s$

$$\begin{aligned} \langle z'(s), e\rangle - \langle z'(t), e\rangle &\geq (1-s)\langle \frac{z'(t)-te}{1-t}, e\rangle + s\|e\|^2 - \langle z'(t), e\rangle \\ &= \left(\frac{1-s}{1-t} - 1\right)\langle z'(t), e\rangle + \left(s - \frac{t(1-s)}{1-t}\right)\|e\|^2 \\ &= \frac{t-s}{1-t}\left(\langle z'(t), e\rangle - \|e\|^2\right) \\ &> 0 \quad \text{(using (30))}. \end{aligned}$$

□

Acknowledgements. I express my thanks to B. Calvert for his help and support.

References

[1] H. Benabdellah: Existence of solutions to the nonconvex sweeping process, J. Differ. Equations 164 (2000) 286–295.

[2] R. Choudhary, Generic convergence of a convex Lyapounov function along trajectories of nonexpansive semigroups in Hilbert space, J. Nonlinear Convex Anal. 7(2) (2006) 245–268.

[3] G. Colombo, V. V. Goncharov: The sweeping process without convexity, Set-Valued Anal. 7 (1999) 357–374.

[4] P. Daniele, A. Maugeri: On dynamical equilibrium problems and variational inequalities, in: Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Methods, F. Giannessi et al. (ed.), Nonconvex Optim. Appl. 58, Kluwer Academic Publishers, Dordrecht (2001) 59–69.

[5] P. Daniele, A. Maugeri, W. Oettli: Time-dependent traffic equilibria, J. Optimization Theory Appl. 103(3) (1999) 543–555.

[6] A. L. Dontchev, T. Zolezzi: Well-Posed Optimization Problems, Springer, Berlin (1993).

[7] S. Fitzpatrick, R. R. Phelps: Differentiability of the metric projection in Hilbert space, Trans. Amer. Math. Soc. 270(2) (1982) 483–501.

[8] J. Gwinner: Time dependent variational inequalities - some recent trends, in: Equilibrium Problems and Variational Models (Erice, 2000), P. Daniele et al. (ed.), Nonconvex Optim. Appl. 68, Kluwer Academic Publishers, Boston (2003) 225–264.

[9] F. Mignot: Contrôle dans les inéquations variationelles elliptiques, J. Funct. Anal. 22(2) (1976) 130–185.

[10] J. J. Moreau: Evolution problem associated with a moving convex set in a Hilbert space, J. Differ. Equations 26 (1977) 347–374.

[11] J. J. Moreau: Numerical aspects of the sweeping process, Comput. Methods Appl. Mech. Eng. 177(3-4) (1999) 329–349.

[12] R. R. Phelps: Convex sets and nearest points I, Proc. Amer. Math. Soc. 9 (1957) 790–797.

[13] L. Thibault: Sweeping process with regular and nonregular sets, J. Differ. Equations 193 (2003) 1–26.

[14] E. H. Zarantonello: Projections on convex sets in Hilbert space and spectral theory, in: Contribution to Nonlinear Functional Analysis, E. H. Zarantonello (ed.), Academic Press, New York (1971) 237–424.