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# On Semicontinuity of Convex-Valued Multifunctions and Cesari's Property (Q)

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We investigate two types of semicontinuity for set-valued maps, Painlevé–Kuratowski semicontinuity and Cesari's property (Q). It is shown that, in the context of convex-valued maps, the concepts related to Cesari's property (Q) have better properties than the concepts in the sense of Painlevé–Kuratowski. In particular we give a characterization of Cesari's property (Q) in terms of upper semicontinuity of a family of scalar functions $\sigma_{f(\,\cdot\,)}(y^*) : X \to \overline{\mathbb{R}}$, where $\sigma_{f(x)} : Y^* \to \overline{\mathbb{R}}$ is the support function of the set $f(x)$. We compare both types of semicontinuity and show their coincidence in special cases.

*Keywords:* Semi-continuity, set-valued maps, property (Q)

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## 1. Introduction

Working in the framework of convex-valued multifunctions one expects that an appropriate notion of an upper semicontinuous hull generates a convex-valued multifunction being upper semicontinuous. This cannot be ensured by upper semicontinuity in the sense of Painlevé and Kuratowski, in [18] called outer semicontinuity, as the following examples show. We denote by $\operatorname{Lim\,sup}_{x' \to x} f(x')$ the Painlevé–Kuratowski upper limit of a set-valued map $f$ at $x$ and by $(\operatorname{Usc} f)(x) = \operatorname{Lim\,sup}_{x' \to x} f(x')$ the corresponding upper semicontinuous hull, see Section 2 for the precise definitions.

**Example 1.1.** Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$, $f(x) := \{\operatorname{sgn} x\}$, where $\operatorname{sgn} x = x \,/\, |x|$ if $x \neq 0$ and $\operatorname{sgn} 0 = 0$. Then the upper semicontinuous hull of $f$, namely $(\operatorname{Usc} f) : \mathbb{R} \rightrightarrows \mathbb{R}$, $(\operatorname{Usc} f)(x) = f(x)$ if $x \neq 0$ and $(\operatorname{Usc} f)(0) = \{-1, 0, 1\}$, is not convex-valued.

This could inspire us to redefine the Painlevé–Kuratowski upper semicontinuous hull in the framework of convex-valued multifunctions as follows:

$$(\widetilde{\operatorname{Usc}} f)(x) := \operatorname{cl}\operatorname{conv} \operatorname*{Lim\,sup}_{x' \to x} f(x').$$

However, $(\widetilde{\operatorname{Usc}} f)$ is not necessarily upper semicontinuous as the following example shows.

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**Example 1.2.** Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$,

$$f(x) := \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } \exists n \in \mathbb{N} : x \in \left[2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if } \exists n \in \mathbb{N} : x \in \left[2^{-2n+1}, 2^{-2n+2}\right) \\ \emptyset & \text{else.} \end{cases}$$

The modified upper semicontinuous hull $(\widetilde{\mathrm{Usc}}\, f)$ of $f$ is obtained as

$$(\widetilde{\mathrm{Usc}}\, f)(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } \exists n \in \mathbb{N} : x \in \left(2^{-2n}, 2^{-2n+1}\right) \\ \left\{-\frac{1}{x}\right\} & \text{if } \exists n \in \mathbb{N} : x \in \left(2^{-2n+1}, 2^{-2n+2}\right) \\ \left[-\frac{1}{x}, \frac{1}{x}\right] & \text{if } \exists n \in \mathbb{N} : x = 2^{-n} \\ \emptyset & \text{else.} \end{cases}$$

The graph of $(\widetilde{\mathrm{Usc}}\, f)$ is not closed. Indeed, the members of the sequence $((2^{-n}, 0))$ belong to the graph of $(\widetilde{\mathrm{Usc}}\, f)$, but the limit $(0, 0)$ does not. Hence $(\widetilde{\mathrm{Usc}}\, f)$ is not Painlevé–Kuratowski upper semicontinuous.

Let us illuminate another aspect. An important idea of Convex Analysis is the relationship between a convex set $A \subset \mathbb{R}^p$ and its support function $\sigma_A : \mathbb{R}^p \to \overline{\mathbb{R}}$. In particular, for closed convex sets $A, B \subset \mathbb{R}^p$ and $\alpha \in \mathbb{R}_+$ we have the following relationships (in particular, we set $-\infty + \infty = -\infty$, $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$, $0 \cdot \emptyset = \{0\}$):

$$\Big(A \subset B \Leftrightarrow \sigma_A \leq \sigma_B\Big), \qquad \sigma_A + \sigma_B = \sigma_{A+B}, \qquad \alpha\sigma_A = \sigma_{\alpha A}.$$

This implies that a set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is concave (i.e. graph-convex) if and only if the functions $\sigma_{f(\cdot)}(y^*) : \mathbb{R}^n \to \overline{\mathbb{R}}$ have the same property for all $y^* \in \mathbb{R}^p$. But, what can we say about a corresponding relationship for continuity properties? The usual Painlevé–Kuratowski upper and lower semicontinuities don't yield a positive result, as the following example shows.

**Example 1.3.** Let $f : \mathbb{R} \rightrightarrows \mathbb{R}$, $f(x) := \left\{\frac{1}{x}\right\}$ if $x \neq 0$ and $f(0) := \{0\}$. Then $f$ is Painlevé–Kuratowski upper semicontinuous (in particular at $x = 0$), but $\sigma_{f(\cdot)}(y^*)$ is not upper semicontinuous at $x = 0$ for all $y^* \neq 0$.

Motivated by these examples we look for an alternative semicontinuity concept for multifunctions having better properties in this framework. In this article we show that *Cesari's property* (Q), which plays an important role in Optimal Control and is well-known in this field, fits all our requirements. Property (Q) was investigated by Cesari [3, 4], Cesari and Suryanarayana [5], Goodman [9], Denkowski [6], Suryanarayana [20], Papageorgiou [15] and others. As a main result we give here a characterization of Cesari's property (Q) in terms of scalarizations by the support function. Our investigations are based on some results on $\mathcal{C}$-convergence (in connection to property (Q) usually called Q-convergence), which were recently obtained by C. Zălinescu and the author [13, 14][1]. For other notions of convergence of convex sets see e.g. [1, 2, 7, 19] and the references given there.

[1]After a discussion with J.-P. Penot we realized that the convergence introduced in [13, 14] is related to Cesari's property (Q) introduced in [3] and studied in several articles. By Proposition 3.1 below, it is clear that this convergence coincides with the Q-convergence considered in [9, 6]. For the origin of this convergence see the remarks in [9, 6].

This article is organized as follows. In the next section we shortly recall some facts on the two types of semicontinuity, Painlevé–Kuratowski semicontinuity and Cesari's property (Q), and we propose our main tools. In Section 3 we present our main result and draw some conclusions. Section 4 is devoted to a comparison of Cesari's property (Q) and the Painlevé–Kuratowski semicontinuity. We show their coincidence under certain assumptions. Finally, in Section 5 we discuss the special case of concave (i.e. graph-convex) maps.

## 2. Preliminaries

Throughout the paper let $Y$ be a finite dimensional normed vector space with dimension $p \geq 1$. For the standard concepts of Convex Analysis we mainly use the notation of Rockafellar's "Convex Analysis" [17].

We denote by $\mathcal{F} := \mathcal{F}(Y)$ the family of closed subsets of $Y$ and by $\mathcal{C} := \mathcal{C}(Y)$ the family of closed convex subsets of $Y$. It is well-known that $(\mathcal{F}, \subset)$ and $(\mathcal{C}, \subset)$ provide complete lattices, i.e., every nonempty subset of $\mathcal{F}$ (resp. of $\mathcal{C}$) has a supremum and an infimum, denoted by $\operatorname{Sup} \mathcal{A}$ ($\sup \mathcal{A}$) and $\operatorname{Inf} \mathcal{A}$ ($\inf A$). Of course, for nonempty sets $\mathcal{A} \subset \mathcal{F}$ and $\mathcal{B} \subset \mathcal{C}$ we have $\operatorname{Sup} \mathcal{A} = \operatorname{cl} \bigcup \{A|\ A \in \mathcal{A}\}$, $\operatorname{Inf} \mathcal{A} = \bigcap \{A|\ A \in \mathcal{A}\}$, $\sup \mathcal{B} = \operatorname{cl} \operatorname{conv} \bigcup \{B|\ B \in \mathcal{B}\}$ and $\inf \mathcal{B} = \bigcap \{B|\ B \in \mathcal{B}\}$. Further, we set $\operatorname{Inf} \emptyset = \operatorname{Sup} \mathcal{F} = Y$, $\operatorname{Sup} \emptyset = \operatorname{Inf} \mathcal{F} = \emptyset$, $\inf \emptyset = \sup \mathcal{C} = Y$ and $\sup \emptyset = \inf \mathcal{C} = \emptyset$.

We frequently use the following notation of [18] (but omitting the index $\infty$):

$$\mathcal{N} := \{N \subset \mathbb{N}|\ \mathbb{N} \setminus N \text{ finite}\} \quad \text{and} \quad \mathcal{N}^{\#} := \{N \subset \mathbb{N}|\ N \text{ infinite}\}.$$

For a sequence $(A_n)$ in $\mathcal{F}$ the *upper* and *lower $\mathcal{F}$-limits* are defined, respectively, by

$$\operatorname*{Lim\,sup}_{n \to \infty} A_n = \operatorname*{Inf}_{N \in \mathcal{N}} \operatorname*{Sup}_{n \in N} A_n, \qquad \operatorname*{Lim\,inf}_{n \to \infty} A_n = \operatorname*{Inf}_{N \in \mathcal{N}^{\#}} \operatorname*{Sup}_{n \in N} A_n.$$

Of course, the *upper and lower $\mathcal{F}$-limits* coincide with the upper and lower limits in the sense of Painlevé–Kuratowski (in [18] called outer and inner limits). See for instance [18] for alternative definitions.

In the following, all concepts related to upper and lower $\mathcal{F}$-limits are indicated by the prefix $\mathcal{F}$, because $\mathcal{F}$ is the underlying lattice. In formulas we don't use this prefix, instead a term begins with a capital letter.

A sequence $(A_n)$ in $\mathcal{F}$ is *$\mathcal{F}$-convergent* to some $A \in \mathcal{F}$ if $A = \operatorname{Lim\,sup}_{n \to \infty} A_n = \operatorname{Lim\,inf}_{n \to \infty} A_n$. Then we write $A = \operatorname{Lim}_{n \to \infty} A_n$ or $(A_n) \xrightarrow{\mathcal{F}} A$.

We proceed analogously in the complete lattice $\mathcal{C}$. The *upper* and *lower $\mathcal{C}$-limits* of a sequence $(A_n)$ in $\mathcal{C}$ are defined, respectively, by

$$\limsup_{n \to \infty} A_n := \inf_{N \in \mathcal{N}} \sup_{n \in N} A_n \quad \text{and} \quad \liminf_{n \to \infty} A_n := \inf_{N \in \mathcal{N}^{\#}} \sup_{n \in N} A_n.$$

Upper and lower $\mathcal{C}$-limits and related concepts were used in the field of Optimal Control, see e.g. [3, 4, 5, 6, 9, 15, 20]. In this area, one speaks about (upper and lower) $Q$-limits, $Q$-convergence and so on, because these concepts are related to Cesari's property (Q) (the definition is given in the next section). Also, what in [15] is called Cesari's limit of a sequence of subsets $A_n$ is nothing else but the upper $\mathcal{C}$-limit (in our finite dimensional setting), see [15, Proposition 2.1]. In the present article we use the prefix $\mathcal{C}$ instead of $Q$, because $\mathcal{C}$ is the underlying lattice. In formulas we consequently use small letters.

We say a sequence $(A_n)$ in $\mathcal{C}$ is *$\mathcal{C}$-convergent* to some $A \in \mathcal{C}$ if $A = \limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$ and we write $A = \lim_{n\to\infty} A_n$ or $(A_n) \xrightarrow{\mathcal{C}} A$.

We next summarize some results related to $\mathcal{C}$-limits. The following initial result is an immediate consequence of [17, Cor. 16.5.1]. By $\sigma_A : Y \to \overline{\mathbb{R}}$, we denote the support function of a set $A \subset Y$.

**Proposition 2.1.** *Let $\mathcal{A} \subset \mathcal{C}$. Then $\sigma_{\inf \mathcal{A}} \leq \inf_{A\in\mathcal{A}} \sigma_A$ and $\sigma_{\sup \mathcal{A}} = \sup_{A\in\mathcal{A}} \sigma_A$.*

As an easy consequence, for any sequence $(A_n)$ in $\mathcal{C}$ one has

$$\sigma_{\limsup_n A_n} \leq \limsup_n \sigma_{A_n} \qquad \text{and} \qquad \sigma_{\liminf_n A_n} \leq \liminf_n \sigma_{A_n}. \tag{1}$$

The next two theorems give us sufficient conditions for the coincidence of $\mathcal{F}$- and $\mathcal{C}$-convergence. Our condition here is weaker than the one in [15], but the underlying space is a Banach space there. Let $K \subset Y$ be a nonempty closed convex cone. By $\mathcal{C}_K$ we denote the family of all members $A$ of $\mathcal{C}_0 := \mathcal{C} \setminus \{\emptyset\}$ satisfying $0^+A = K$, where $0^+A$ denotes the recession cone of a convex set $A \subset Y$.

**Theorem 2.2 ([14]).** *Let $(A_n)$ be a sequence in $\mathcal{C}_K$ such that $\sup_{n\in\mathbb{N}} A_n \in \mathcal{C}_K$. Then, $\limsup_{n\to\infty} A_n = \operatorname{cl}\operatorname{conv} \operatorname{Lim\,sup}_{n\to\infty} A_n$.*

**Theorem 2.3 ([14]).** *Let $(A_n)$ be a sequence in $\mathcal{C}$ such that for all $\bar{N} \in \mathcal{N}^\#$ there exists some $\widetilde{N} \in \mathcal{N}^\#$ with $\widetilde{N} \subset \bar{N}$ and some nonempty closed convex cone $K \subset Y$ such that $A_n \in \mathcal{C}_K$ for all $n \in \widetilde{N}$ and $\sup_{n\in\widetilde{N}} A_n \in \mathcal{C}_K$. Then it holds $\liminf_{n\to\infty} A_n = \operatorname{Lim\,inf}_{n\to\infty} A_n$.*

The following lemmas provide the main tools in our investigations.

**Lemma 2.4 ([14]).** *Let $A, B \subset Y$ be nonempty closed and convex. Then,*

$$A \subset B \quad \Leftrightarrow \quad \forall y^* \in \operatorname{ri}(0^+B)^\circ,\ \sigma_A(y^*) \leq \sigma_B(y^*).$$

**Lemma 2.5 ([14]).** *For any sequence $(A_n)$ in $\mathcal{C}$ with $A := \limsup_{n\to\infty} A_n \neq \emptyset$ it holds*

$$\forall y^* \in \operatorname{ri}(0^+A)^\circ, \quad \limsup_{n\to\infty} \sigma_{A_n}(y^*) = \sigma_A(y^*).$$

We next turn to upper and lower $\mathcal{F}$-limits for set-valued maps and collect some well-known results. Analogous concepts and results for the lattice $\mathcal{C}$ are discussed in the next section. Let $(X, d)$ be a metric space. The *upper* and *lower $\mathcal{F}$-limits* of $f : X \to \mathcal{F}$ at $\bar{x} \in X$ are defined, respectively, by

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} f(x) := \bigcup_{x_n\to\bar{x}} \operatorname*{Lim\,sup}_{n\to\infty} f(x_n) \qquad \text{and} \qquad \operatorname*{Lim\,inf}_{x\to\bar{x}} f(x) := \bigcap_{x_n\to\bar{x}} \operatorname*{Lim\,inf}_{n\to\infty} f(x_n).$$

where the index "$x_n \to \bar{x}$" stands for the union and intersection over all sequences in $X$ converging to $\bar{x}$, respectively. The limit of $f : X \to \mathcal{F}$ at $\bar{x}$ exists if the upper and lower $\mathcal{F}$-limits coincide. Then we write

$$\operatorname*{Lim}_{x\to\bar{x}} f(x) = \operatorname*{Lim\,sup}_{x\to\bar{x}} f(x) = \operatorname*{Lim\,inf}_{x\to\bar{x}} f(x).$$

The function $f$ is said to be *upper $\mathcal{F}$-semicontinuous* (*$\mathcal{F}$-usc*), *lower $\mathcal{F}$-semicontinuous* (*$\mathcal{F}$-lsc*), *$\mathcal{F}$-continuous* at $\bar{x} \in X$ if $f(\bar{x}) \supset \operatorname{Lim\,sup}_{x\to\bar{x}} f(x)$, $f(\bar{x}) \subset \operatorname{Lim\,inf}_{x\to\bar{x}} f(x)$, $f(\bar{x}) = \operatorname{Lim}_{x\to\bar{x}} f(x)$, respectively. If $f$ is $\mathcal{F}$-usc, $\mathcal{F}$-lsc, $\mathcal{F}$-continuous at every $\bar{x} \in X$ we just say $f$ is $\mathcal{F}$-usc, $\mathcal{F}$-lsc, $\mathcal{F}$-continuous, respectively. The *epigraph* and the *hypograph* of $f : X \to \mathcal{F}$ are defined, respectively, by

$$\operatorname{Epi} f := \{(x, A) \in X \times \mathcal{F} |\ A \supset f(x)\}, \qquad \operatorname{Hyp} f := \{(x, A) \in X \times \mathcal{F} |\ A \subset f(x)\}.$$

Note that for all $x \in X$ we have $(x, \emptyset) \in \operatorname{Hyp} f$ and $(x, Y) \in \operatorname{Epi} f$. For a characterization of $\mathcal{F}$-semicontinuity we need to know what is meant by closedness of the epigraph and hypograph. A subset $\mathcal{A} \subset X \times \mathcal{F}$ is said to be (sequentially) $\mathcal{F}$-closed if for every sequence $((x_n, F_n))$ in $\mathcal{A}$ with $(x_n) \to \bar{x} \in X$ and $(F_n) \xrightarrow{\mathcal{F}} \bar{F} \in \mathcal{F}$ it is true that $(\bar{x}, \bar{F}) \in \mathcal{A}$. The (sequential) *$\mathcal{F}$-closure* of a set $\mathcal{A} \subset X \times \mathcal{F}$, denoted by $\operatorname{Cl} \mathcal{A}$, is the set of all such limits $(\bar{x}, \bar{F}) \in X \times \mathcal{F}$ of sequences $((x_n, F_n))$ in $\mathcal{A}$. In the following we omit the term "sequential", because the whole article is written in this context.

We have the following characterization of $\mathcal{F}$-upper semicontinuity

$$\operatorname{Hyp} f \text{ is } \mathcal{F}\text{-closed} \quad \Leftrightarrow \quad f \text{ is } \mathcal{F}\text{-usc} \quad \Leftrightarrow \quad \operatorname{gr} f \subset X \times Y \text{ is closed.} \tag{2}$$

The first equivalence is a consequence of the cluster point description of upper $\mathcal{F}$-limits, see e.g. [18, Proposition 4.19], the second one is easy to show.

Likewise (compare [18, Exercise 5.6 (d)]), lower $\mathcal{F}$-semicontinuity of $f$ is equivalent to the $\mathcal{F}$-closedness of the epigraph. Note that the description by the graph fails in this case, i.e., a function $f : X \to \mathcal{F}$ that is $\mathcal{F}$-lsc has not necessarily a closed graph, see [18, Fig. 5–3 (b)].

Let us collect some basic properties of the *upper $\mathcal{F}$-semicontinuous hull* of $f$, defined by $(\operatorname{Usc} f) : X \to \mathcal{F}$, $(\operatorname{Usc} f)(x) := \operatorname{Lim\,sup}_{x'\to x} f(x')$.

**Proposition 2.6.** *Let $f : X \to \mathcal{F}$. Then it holds*

(*i*) $\operatorname{gr}(\operatorname{Usc} f) = \operatorname{cl}(\operatorname{gr} f)$,
(*ii*) $\operatorname{Hyp}(\operatorname{Usc} f) \supset \operatorname{Cl}(\operatorname{Hyp} f)$,
(*iii*) $(\operatorname{Usc} f)$ *is $\mathcal{F}$-usc,*
(*iv*) $\forall x \in X :\ (\operatorname{Usc} f)(x) \supset f(x)$,
(*v*) *$f$ is $\mathcal{F}$-usc at $\bar{x} \in X$* $\Leftrightarrow$ $(\operatorname{Usc} f)(\bar{x}) = f(\bar{x})$,
(*vi*) $\operatorname{gr} f$ *convex* $\Rightarrow$ $\operatorname{gr}(\operatorname{Usc} f)$ *convex,*

*where $X$ is assumed to be a normed space in* (*vi*).

**Proof.** (*i*) Let $(\bar{x}, \bar{y}) \in \operatorname{gr}(\operatorname{Usc} f)$, i.e., there exists a sequence $(x_n) \to \bar{x}$ such that $\bar{y} \in \operatorname{Lim\,sup}_{n\to\infty} f(x_n)$. Recall that $y \in \operatorname{Lim\,sup}_{n\to\infty} f(x_n)$ if and only if $y$ is a cluster point of some sequence $(y_n)$ with $y_n \in f(x_n)$ for all $n \in \mathbb{N}$, see e.g. [12, page 243]. Hence, $(\bar{x}, \bar{y}) \in \operatorname{gr}(\operatorname{Usc} f)$ is equivalent to $(\bar{x}, \bar{y}) \in \operatorname{cl}(\operatorname{gr} f)$.

(*ii*) Let $(\bar{x}, \bar{A}) \in \operatorname{Cl}(\operatorname{Hyp} f)$. Then there exist sequences $(x_n) \to \bar{x}$ and $(A_n) \xrightarrow{\mathcal{F}} \bar{A}$ such that $A_n \subset f(x_n)$ for all $n \in \mathbb{N}$. Hence, $(\operatorname{Usc} f)(\bar{x}) = \operatorname{Lim\,sup}_{x\to\bar{x}} f(x) \supset \operatorname{Lim\,sup}_{n\to\infty} f(x_n) \supset \operatorname{Lim\,sup}_{n\to\infty} A_n = \operatorname{Lim}_{n\to\infty} A_n = \bar{A}$, i.e., $(\bar{x}, \bar{A}) \in \operatorname{Hyp}(\operatorname{Usc} f)$.

(*iii*) By (*i*), $\operatorname{gr}(\operatorname{Usc} f)$ is closed. Thus $(\operatorname{Usc} f)$ is $\mathcal{F}$-usc, by (2).

(*iv*) Choosing the special sequence $(x_n)$ with $x_n = x$ for all $n \in \mathbb{N}$, we obtain $(\operatorname{Usc} f)(x) = \operatorname{Lim\,sup}_{x' \to x} f(x') \supset \operatorname{Lim\,sup}_{n\to\infty} f(x_n) = \operatorname{Lim\,sup}_{n\to\infty} f(x) = f(x)$.

(*v*) By definition, $f$ is $\mathcal{F}$-usc at $\bar{x}$ if and only if $f(\bar{x}) \supset (\operatorname{Usc} f)(\bar{x})$. By (*iv*) we obtain equality.

(*vi*) Since $\operatorname{gr} f$ is convex, $\operatorname{cl}(\operatorname{gr} f)$ is convex, too. Hence, the convexity of $\operatorname{gr}(\operatorname{Usc} f)$ follows from (*i*). □

The next example shows that the opposite inclusion in assertion (*ii*) of the previous proposition is not true, in general.

**Example 2.7.** Let $f : \mathbb{R} \to \mathcal{F}(\mathbb{R})$, $f(x) := \{x/|x|\}$ if $x \neq 0$, $f(0) := \emptyset$. Then, $(0, \{-1, 1\})$ belongs to $\operatorname{Hyp}(\operatorname{Usc} f)$ but it does not belong to $\operatorname{Cl}(\operatorname{Hyp} f)$.

**Remark 2.8.** As noticed in [18], an analogous definition of the $\mathcal{F}$-lower semicontinuous hull, namely by $(\operatorname{Lsc} f)(x) := \operatorname{Lim\,inf}_{x'\to x} f(x')$, is not constructive in the sense that $(\operatorname{Lsc} f)$ is not necessarily $\mathcal{F}$-lsc. In the framework of $\mathcal{C}$-valued functions we will have similar problems, see Example 3.9 below.

## 3. Upper and lower $\mathcal{C}$-Semicontinuity

In this section we deal with upper and lower limits for functions with values in $\mathcal{C}$. The *upper* and *lower $\mathcal{C}$-limits* (compare [3, 4, 6, 9, 20]) of a function $f : X \to \mathcal{C}$ at $\bar{x} \in X$ are defined, respectively, by

$$\limsup_{x\to\bar{x}} f(x) := \bigcup_{x_n\to\bar{x}} \limsup_{n\to\infty} f(x_n) \quad \text{and} \quad \liminf_{x\to\bar{x}} f(x) := \bigcap_{x_n\to\bar{x}} \liminf_{n\to\infty} f(x_n).$$

The *$\mathcal{C}$-limit* of $f$ at $\bar{x}$ exists if the upper and lower limits coincide. Then we write

$$\lim_{x\to\bar{x}} f(x) = \limsup_{x\to\bar{x}} f(x) = \liminf_{x\to\bar{x}} f(x).$$

In case of upper $\mathcal{F}$-limits it is well-known that

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} f(x) = \operatorname*{Sup}_{x_n\to\bar{x}} \operatorname*{Lim\,sup}_{n\to\infty} f(x_n) = \bigcap_{\delta>0} \operatorname{cl} \bigcup_{x\in B_\delta(\bar{x})} f(x), \tag{3}$$

where $B_\delta(\bar{x}) := \{x \in X|\, d(x, \bar{x}) < \delta\}$; see e.g. [10, Proposition 2.3.5 (ii)] (with a slightly different definition of Limsup) or modify the proof of Proposition 3.1 below. An analogous result is valid for upper $\mathcal{C}$-limits.

**Proposition 3.1.** *Let $f : X \to \mathcal{C}$ and $\bar{x} \in X$. Then it holds*

$$\limsup_{x\to\bar{x}} f(x) = \sup_{x_n\to\bar{x}} \limsup_{n\to\infty} f(x_n) = \bigcap_{\delta>0} \operatorname{cl\,conv} \bigcup_{x\in B_\delta(\bar{x})} f(x).$$

**Proof.** Since $A := \bigcap_{\delta>0} \operatorname{cl\,conv} \bigcup_{x\in B_\delta(\bar{x})} f(x)$ is closed and convex, it remains to prove $A = \limsup_{x\to\bar{x}} f(x) =: B$.

(i) $A \subset B$. Letting $y \in A$, for all $n \in \mathbb{N}$ we have $y \in \operatorname{conv} \bigcup_{x \in B_{1/n}(\bar{x})} f(x) + \bar{B}_{1/n}$, where $\bar{B}_{1/n} := \left\{y \in Y \mid \|y\| < \frac{1}{n}\right\}$. Hence

$$\forall n \in \mathbb{N}, \exists b_n \in \bar{B}_{1/n}, \exists k_n \in \mathbb{N}, \forall i \in \{1,2,\dots,k_n\}, \exists \lambda_n^i \geq 0, \exists x_n^i \in B_{1/n}(\bar{x}),$$

$$y - b_n \in \sum_{j=1}^{k_n} \lambda_n^j f(x_n^j), \quad \sum_{j=1}^{k_n} \lambda_n^j = 1.$$

Consider the sequence $(x_n) \to \bar{x}$ defined as

$$(x_n)_{n \in \mathbb{N}} := (x_1^1, x_1^2, \dots, x_1^{k_1}, x_2^1, x_2^2, \dots, x_2^{k_2}, \dots, x_m^1, x_m^2 \dots, x_m^{k_m}, \dots).$$

We conclude that

$$\forall n \in \mathbb{N}, \forall m \geq n: \quad y - b_m \in \operatorname{conv} \bigcup_{k \geq n} f(x_k),$$

hence for all $n \in \mathbb{N}$ we have $y \in \operatorname{cl}\operatorname{conv} \bigcup_{k \geq n} f(x_k)$. That means we have found a sequence $(x_n) \to \bar{x}$ such that $y \in \operatorname{cl}\operatorname{conv} \bigcup_{n \in N} f(x_n)$ for all $N \in \mathcal{N}$, i.e., $y \in B$.

(ii) $B \subset A$. For arbitrary $y \in B$ there is a sequence $(x_n) \to \bar{x}$ in $X$ such that $y \in \operatorname{cl}\operatorname{conv} \bigcup_{n \in N} f(x_n)$ for all $N \in \mathcal{N}$. For all $\delta > 0$ there exists $N_\delta \in \mathcal{N}$ such that $\bigcup_{n \in N_\delta} x_n \subset B_\delta(\bar{x})$. Thus for all $\delta > 0$ we have $y \in \operatorname{cl}\operatorname{conv} \bigcup_{n \in N_\delta} f(x_n) \subset \operatorname{cl}\operatorname{conv} \bigcup_{x \in B_\delta(\bar{x})} f(x)$, i.e., $y \in A$. □

As an easy consequence of the definition we have the following relationship between upper and lower $\mathcal{F}$- and $\mathcal{C}$-limits: $\limsup_{x \to \bar{x}} f(x) \supset \operatorname{Lim\,sup}_{x \to \bar{x}} f(x)$, $\liminf_{x \to \bar{x}} f(x) \supset \operatorname{Lim\,inf}_{x \to \bar{x}} f(x)$.

A function $f : X \to \mathcal{C}$ is said to be *upper $\mathcal{C}$-semicontinuous* (*$\mathcal{C}$-usc*), *lower $\mathcal{C}$-semicontinuous* (*$\mathcal{C}$-lsc*), *$\mathcal{C}$-continuous* at $\bar{x} \in X$ if $f(\bar{x}) \supset \limsup_{x \to \bar{x}} f(x)$, $f(\bar{x}) \subset \liminf_{x \to \bar{x}} f(x)$, $f(\bar{x}) = \lim_{x \to \bar{x}} f(x)$, respectively. If $f$ is $\mathcal{C}$-lsc, $\mathcal{C}$-usc, $\mathcal{C}$-continuous at every $\bar{x} \in X$ we just say $f$ is $\mathcal{C}$-lsc, $\mathcal{C}$-usc, $\mathcal{C}$-continuous, respectively. It is easy to see that $\mathcal{C}$-usc implies $\mathcal{F}$-usc and $\mathcal{F}$-lsc implies $\mathcal{C}$-lsc. By Proposition 3.1 it is clear that upper $\mathcal{C}$-semicontinuity is the same as Cesari's property (Q) (A multifunction $f : X \rightrightarrows Y$ satisfies property (Q) at $\bar{x}$ if $\bigcap_{\delta > 0} \operatorname{cl}\operatorname{conv} \bigcup_{x \in B_\delta(\bar{x})} f(x) = f(\bar{x})$).

**Remark 3.2.** Of course, $\mathcal{C}$-semicontinuity can also be defined for arbitrary set-valued maps rather than $\mathcal{C}$-valued functions, and the following results can be easily rewritten in this case. We prefer to suppose $\mathcal{C}$-valued functions, because this makes the notation easier and underlines the role of the complete lattice $\mathcal{C}$.

With the aid of Lemmas 2.4 and 2.5 we obtain our main result, a characterization of upper $\mathcal{C}$-semicontinuity or in other words a characterization of Cesari's property (Q).

**Theorem 3.3.** *Let $f : X \to \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$. Then the following statements are equivalent:*

*(i)* *$f$ is $\mathcal{C}$-usc at $\bar{x}$,*
*(ii)* *For all $y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ$ the function $\sigma_{f(\cdot)}(y^*) : X \to \overline{\mathbb{R}}$ is usc at $\bar{x}$.*

**Proof.** Let be given an arbitrary sequence $(x_n) \to \bar{x}$ in $X$.

$(i) \Rightarrow (ii)$. Let the sequence $(\tilde{x}_n) \to \bar{x}$ be defined by $\tilde{x}_{2n} := x_n$ and $\tilde{x}_{2n+1} := \bar{x}$. From $(i)$ and by the special choice of the sequence, we deduce that $f(\bar{x}) = \limsup_{n\to\infty} f(\tilde{x}_n)$. Lemma 2.5 implies

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ, \;\; \sigma_{f(\bar{x})}(y^*) = \limsup_{n\to\infty} \sigma_{f(\tilde{x}_n)}(y^*) \geq \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*).$$

$(ii) \Rightarrow (i)$. We have

$$\forall y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ, \;\; \sigma_{f(\bar{x})}(y^*) \geq \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*) \overset{(1)}{\geq} \sigma_{\limsup_n f(x_n)}(y^*).$$

Lemma 2.4 yields $f(\bar{x}) \supset \limsup_{n\to\infty} f(x_n)$. □

**Remark 3.4.** By standard arguments one can show: If $\sigma_{f(\cdot)}(y^*) : X \to \overline{\mathbb{R}}$ is upper semicontinuous at $\bar{x}$ for all $y^* \in Y^*$, then $f$ is upper $\mathcal{C}$-semicontinuous at $\bar{x}$. A result of this type, even for $Y$ being an arbitrary Banach space, can be found in [15, Proposition 2.1]. The proof of the above result, however, is based on additional arguments, see the proof of Lemma 2.4 and Lemma 2.5, which can be found in [14]. Therefore we restrict ourselves to a finite dimensional space $Y$.

The following example shows the importance of the the restriction $y^* \in \operatorname{ri}\left(0^+ f(\bar{x})\right)^\circ$ in statement $(ii)$ of the previous theorem.

**Example 3.5.** Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$, $f(x) := \left\{\lambda \cdot (x, 1)^T \mid \lambda \geq 0\right\}$. Then, $f$ is $\mathcal{C}$-usc at 0, but for $\bar{y}^* := (1, 0) \notin \operatorname{ri}\left(0^+ f(0)\right)^\circ$ we have $\sigma_{f(x)}(\bar{y}^*) = \infty$ if $x > 0$ and $\sigma_{f(x)}(\bar{y}^*) = 0$ else, i.e., $x \longmapsto \sigma_{f(x)}(\bar{y}^*)$ is not usc at 0.

The next assertion about nested upper $\mathcal{C}$-limits is essential for an expedient definition of the *upper $\mathcal{C}$-semicontinuous hull*. An analogous assertion for the lower $\mathcal{C}$-limit is not true, see Example 3.9 below.

**Proposition 3.6.** *Let $f : X \to \mathcal{C}$ and $\bar{x} \in X$. Then it holds*

$$\limsup_{x\to\bar{x}} f(x) = \limsup_{x\to\bar{x}} \; \limsup_{w\to x} f(w).$$

**Proof.** Clearly we have $f(x) \subset \limsup_{w\to x} f(w)$ for all $x \in X$. This implies the inclusion "$\subset$". It remains to show that for an arbitrarily given sequence $(x_n) \to \bar{x}$, one has $A := \limsup_{n\to\infty} \limsup_{w\to x_n} f(w) \subset \limsup_{n\to\infty} f(x_n) =: B$. For all $y^* \in \operatorname{ri}\left(0^+ B\right)^\circ$ it holds

$$\sigma_A(y^*) \overset{(1)}{\leq} \limsup_{n\to\infty} \; \limsup_{w\to x_n} \sigma_{f(w)}(y^*) = \limsup_{n\to\infty} \sigma_{f(x_n)}(y^*) \overset{\text{Lem. 2.5}}{=} \sigma_B(y^*).$$

Lemma 2.4 yields that $A \subset B$. □

The *upper $\mathcal{C}$-semicontinuous hull* of a function $f : X \to \mathcal{C}$ is defined by

$$(\operatorname{usc} f) : X \to \mathcal{C}, \;\; (\operatorname{usc} f)(x) := \limsup_{x'\to x} f(x').$$

The *hypograph* of a function $f : X \to \mathcal{C}$ is the set $\operatorname{hyp} f := \{(x, A) \in X \times \mathcal{C} |\ A \subset f(x)\}$. A subset $\mathcal{A} \subset X \times \mathcal{C}$ is said to be (sequentially) closed if for every sequence $((x_n, A_n))$ in $\mathcal{A}$ with $(x_n) \to \bar{x} \in X$ and $(A_n) \xrightarrow{\mathcal{C}} \bar{A} \in \mathcal{C}$ one has $(\bar{x}, \bar{A}) \in \mathcal{A}$. The (sequential) $\mathcal{C}$-*closure* of a set $\mathcal{A} \subset X \times \mathcal{C}$, denoted by $\operatorname{cl} \mathcal{A}$, is the set of all such limits $(\bar{x}, \bar{A}) \in X \times \mathcal{C}$ of sequences $((x_n, A_n))$ in $\mathcal{A}$. For the notion of $\mathcal{C}$-closure compare the remarks in Section 6.

Let us collect some properties of the upper $\mathcal{C}$-semicontinuous hull.

**Proposition 3.7.** *For $f : X \to \mathcal{C}$ the following statements hold true:*

($i$) $\operatorname{gr}(\operatorname{usc} f) \supset \operatorname{cl}(\operatorname{gr} f)$,
($ii$) $\operatorname{hyp}(\operatorname{usc} f) \supset \operatorname{cl}(\operatorname{hyp} f)$,
($iii$) $(\operatorname{usc} f)$ *is $\mathcal{C}$-usc,*
($iv$) $\forall x \in X,\ (\operatorname{usc} f)(x) \supset f(x)$,
($v$) *$f$ is $\mathcal{C}$-usc at $\bar{x} \in X$* $\quad\Leftrightarrow\quad (\operatorname{usc} f)(\bar{x}) = f(\bar{x})$,
($vi$) $\operatorname{gr}(\operatorname{usc} f)$ *is $\mathcal{C}$-closed,*
($vii$) $\operatorname{hyp}(\operatorname{usc} f)$ *is $\mathcal{C}$-closed.*

**Proof.** ($i$) Let $(\bar{x}, \bar{y}) \in \operatorname{cl}(\operatorname{gr} f)$. Then there exists a sequence $((x_n, y_n)) \to (\bar{x}, \bar{y})$ in $\operatorname{gr} f$. For all $n \in \mathbb{N}$, we have $\{y_n\} \subset f(x_n)$. Hence

$$\{\bar{y}\} = \lim_{n\to\infty} \{y_n\} = \limsup_{n\to\infty} \{y_n\} \subset \limsup_{n\to\infty} f(x_n) \subset \limsup_{x\to\bar{x}} f(x) = (\operatorname{usc} f)(\bar{x}),$$

i.e., $(\bar{x}, \bar{y}) \in \operatorname{gr}(\operatorname{usc} f)$.

The proof of ($ii$) is similar.

Statement ($iii$) follows from Proposition 3.6.

The proofs of ($iv$) and ($v$) are analogous to those of Proposition 2.6 ($iv$) and ($v$).

($vi$) Let $((x_n, y_n)) \to (\bar{x}, \bar{y}) \in X \times Y$ be sequence in $\operatorname{gr}(\operatorname{usc} f)$. Proceeding as in ($i$), but replacing $f$ by $(\operatorname{usc} f)$, we obtain $\{\bar{y}\} \subset (\operatorname{usc}(\operatorname{usc} f))(\bar{x})$. From ($iii$) and ($v$) we conclude that $(\operatorname{usc}(\operatorname{usc} f))(\bar{x}) = (\operatorname{usc} f)(\bar{x})$. Hence $(\bar{x}, \bar{y}) \in \operatorname{gr}(\operatorname{usc} f)$.

The proof of ($vii$) is similar to that of ($iv$). □

The next example shows that neither $\mathcal{C}$-closedness of $\operatorname{hyp} f$ nor closedness of $\operatorname{gr} f$ implies that $f$ is $\mathcal{C}$-usc.

**Example 3.8.** Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be defined by $f(x) := \{1/x\}$ if $x \neq 0$ and $f(0) := \emptyset$. Then, $\operatorname{gr} f \subset \mathbb{R} \times \mathbb{R}$ is closed and $\operatorname{hyp} f \subset \mathbb{R} \times \mathcal{C}(\mathbb{R})$ is $\mathcal{C}$-closed, but $f$ is not $\mathcal{C}$-usc.

In Remark 2.8 (due to [18]) we noticed that the lower $\mathcal{F}$-semicontinuous hull that is defined analogously to the upper $\mathcal{F}$-semicontinuous hull is not necessarily lower $\mathcal{F}$-semicontinuous. There are analogous problems with the lower $\mathcal{C}$-semicontinuous hull. This is due to the fact that there is no assertion analogous to Proposition 3.6 for lower $\mathcal{C}$-limits, as the following example shows.

**Example 3.9.** For functions $f : X \to \mathcal{C}$, in general, we have

$$\liminf_{x\to\bar{x}} f(x) \neq \liminf_{x\to\bar{x}} \ \liminf_{w\to x} f(w).$$

Indeed, consider the function $f : \mathbb{R}^2 \to \mathcal{C}(\mathbb{R})$ defined by

$$f(x) := \begin{cases} \{\|x\|\} & \text{if } x_1 \geq 0 \\ \{-\|x\|\} & \text{if } x_1 < 0. \end{cases}$$

Then it holds

$$\liminf_{w \to x} f(w) := \begin{cases} \{\|x\|\} & \text{if } x_1 > 0 \\ \{-\|x\|\} & \text{if } x_1 < 0 \\ \emptyset & \text{if } x_1 = 0 \text{ and } x_2 \neq 0 \\ \{0\} & \text{if } x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

Hence we obtain $\{0\} = \liminf_{x \to 0} f(x) \neq \liminf_{x \to 0} \liminf_{w \to x} f(w) = \emptyset$.

## 4. A generalization of local boundedness

The concept of local boundedness of a set-valued map plays an important role in Variational Analysis, see [18]. It turns out that a generalization of this concept provides a sufficient condition for the coincidence of $\mathcal{F}$-limits and $\mathcal{C}$-limits.

A function $f : X \to \mathcal{C}$ is said to be *recessively constant* on $U \subset \operatorname{dom} f$ if $x \longmapsto 0^+ f(x)$ is constant on $U$. A function $f : X \to \mathcal{C}$ is called *recessively rigid* at $\bar{x} \in \operatorname{dom} f$ if there exists a neighborhood $V$ of $\bar{x}$ such that $f$ is recessively constant on $V \cap \operatorname{dom} f$ and

$$0^+ \sup_{x \in V} f(x) = 0^+ f(\bar{x}). \tag{4}$$

If $f$ is recessively constant (rigid) on its whole domain, we just say $f$ is recessively constant (rigid). In the case where $\emptyset \neq f(\bar{x}) \subset Y$ is bounded, $f$ is *recessively rigid* at $\bar{x}$ if and only if $f$ is locally bounded at $\bar{x}$. In this case, the condition that $f$ is recessively constant around $\bar{x}$ is superfluous, because it follows from (4).

**Theorem 4.1.** *Let $f : X \to \mathcal{C}$ be recessively rigid at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\limsup_{x \to \bar{x}} f(x) = \operatorname{cl} \operatorname{conv} \operatorname*{Lim\,sup}_{x \to \bar{x}} f(x).$$

**Proof.** Clearly we have $\limsup_{x \to \bar{x}} f(x) \supset \operatorname{cl} \operatorname{conv} \operatorname{Lim\,sup}_{x \to \bar{x}} f(x)$. To show the opposite inclusion let $y \in \limsup_{x \to \bar{x}} f(x)$ be given. Then there exists a sequence $(x_n) \to \bar{x}$ such that $y \in \limsup_{n \to \infty} f(x_n)$. Without loss of generality we can assume that $(x_n)$ is a sequence in $\operatorname{dom} f$. The function $f$ being recessively rigid at $\bar{x}$, we find some $n_0 \in \mathbb{N}$ such that $f(x_n) \in \mathcal{C}_K$ for all $n \geq n_0$ and $\sup_{n \geq n_0} f(x_n) \in \mathcal{C}_K$, where $K := 0^+ f(\bar{x})$. Theorem 2.2 yields $y \in \operatorname{cl} \operatorname{conv} \operatorname{Lim\,sup}_{n \to \infty} f(x_n) \subset \operatorname{cl} \operatorname{conv} \operatorname{Lim\,sup}_{x \to \bar{x}} f(x)$. □

The following examples show that the assertion of the preceding theorem may fail if one of the two conditions on $f$ is not satisfied.

**Example 4.2.** Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R})$ be defined by $f(x) := \left\{\frac{1}{x}\right\}$ if $x \neq 0$ and $f(0) := \{0\}$, i.e., $f$ is recessively constant on $\mathbb{R}$, but not recessively rigid at 0, because (4) is violated. Then, $\mathbb{R} = \limsup_{x \to 0} f(x) \neq \operatorname{cl} \operatorname{conv} \operatorname{Lim\,sup}_{x \to 0} f(x) = \{0\}$.

**Example 4.3.** Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$ be defined by $f(x) = \left\{y \in \mathbb{R}^2 |\, y_1 = \frac{1}{x},\, y_2 = 1\right\}$ if $x \neq 0$ and $f(0) := \{y \in \mathbb{R}^2 |\, y_2 = 0\}$, i.e., (4) is satisfied, but $f$ is not recessively constant around 0. It is easy to check that $\{y \in \mathbb{R}^2 |\, 0 \leq y_2 \leq 1\} = \limsup_{x\to 0} f(x) \neq \operatorname{cl}\operatorname{conv} \operatorname{Lim\,sup}_{x\to 0} f(x) = \{y \in \mathbb{R}^2 |\, y_2 = 0\}$.

The assumption that $f : X \to \mathcal{C}$ being recessively rigid at $\bar{x} \in \operatorname{dom} f$ also implies that the lower $\mathcal{F}$-limit and lower $\mathcal{C}$-limit coincide (see Corollary 4.6 below). As shown in the following theorem, a weaker assumption is already sufficient.

**Theorem 4.4.** *Let $f : X \to \mathcal{C}$ and $\bar{x} \in \operatorname{dom} f$ be such that for all sequences $(x_n) \to \bar{x}$ in $X$ there exists a subsequence $(x_{k_n})$ and a nonempty closed convex cone $K \subset Y$ with $f(x_{k_n}) \in \mathcal{C}_K$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} f(x_{k_n}) \in \mathcal{C}_K$. Then it holds $\liminf_{x\to\bar{x}} f(x) = \operatorname{Lim\,inf}_{x\to\bar{x}} f(x)$.*

**Proof.** Of course, $\liminf_{x\to\bar{x}} f(x) \supset \operatorname{Lim\,inf}_{x\to\bar{x}} f(x)$. In order to show the opposite inclusion let $y \in Y \setminus \operatorname{Lim\,inf}_{x\to\bar{x}} f(x)$ be given (the case $\operatorname{Lim\,inf}_{x\to\bar{x}} f(x) = Y$ is obvious). Hence there exists a sequence $(x_n) \to \bar{x}$ such that $y \notin \operatorname{Lim\,inf}_{n\to\infty} f(x_n)$. Our assumption ensures that Theorem 2.3 is applicable. Hence $y \notin \liminf_{n\to\infty} f(x_n)$, and so $y \in Y \setminus \liminf_{x\to\bar{x}} f(x)$. □

In the next example the conclusion of the previous theorem fails, because the assumption is not satisfied.

**Example 4.5.** Let $f : \mathbb{R} \to \mathcal{C}(\mathbb{R}^2)$ be defined by $f(x) := \operatorname{conv}\left\{\left(-\frac{1}{x}, -1\right), \left(\frac{1}{x}, 1\right)\right\}$ if $x > 0$ and $f(x) := \mathbb{R}^2$ if $x \leq 0$, i.e., the condition in the previous theorem is not satisfied. Then we have $\{y \in \mathbb{R}^2 |\, -1 \leq y_2 \leq 1\} = \liminf_{x\to 0} f(x) \neq \operatorname{Lim\,inf}_{x\to 0} f(x) = \{y \in \mathbb{R}^2 |\, y_2 = 0\}$.

**Corollary 4.6.** *Let $f : X \to \mathcal{C}$ be recessively rigid at $\bar{x} \in \operatorname{dom} f$. Then,*

$$\liminf_{x\to\bar{x}} f(x) = \operatorname*{Lim\,inf}_{x\to\bar{x}} f(x).$$

**Proof.** We can assume that $\bar{x} \in \operatorname{int}\operatorname{dom} f$, otherwise both lower limits equal the empty set. Since $f$ is recessively rigid at $\bar{x}$, for every sequence $(x_n) \to \bar{x}$ there exists some $n_0 \in \mathbb{N}$ such that $f(x_n) \in \mathcal{C}_K$ for all $n \geq n_0$ and $\sup_{n\geq n_0} f(x_n) \in \mathcal{C}_K$, where $K := 0^+ f(\bar{x})$. Theorem 4.4 yields the desired assertion. □

**Corollary 4.7.** *Suppose $f : X \to \mathcal{C}$ is recessively rigid. Then it holds*

$$\operatorname{hyp} f \text{ is } \mathcal{C}\text{-closed} \quad\Leftrightarrow\quad f \text{ is } \mathcal{C}\text{-usc} \quad\Leftrightarrow\quad \operatorname{gr} f \subset X \times Y \text{ is closed.}$$

**Proof.** By Theorem 4.1 and Corollary 4.6, $f$ is $\mathcal{F}$-usc if and only if $f$ is $\mathcal{C}$-usc and $\mathcal{F}$-convergence coincides with $\mathcal{C}$-convergence. Hence the result follows from (2). □

## 5. Graph-convex set-valued maps

Let $(X, \|\cdot\|)$ be a normed space. It is well-known that the graph of a function $f : X \to \mathcal{C}$ is convex if and only if

$$\forall\lambda \in [0,1],\, \forall x_1, x_2 \in X,\ f(\lambda \cdot x_1 + (1-\lambda)\cdot x_2) \supset \lambda\, f(x_1) + (1-\lambda)\, f(x_2).$$

Since the relation $\supset$ has the meaning of "greater or equal" throughout the paper, a function $f : X \to \mathcal{C}$ satisfying the latter condition is said to be *concave*, even though the term "convex" is mostly used in the literature. The reason is that the usage of the term "concave" entails more analogies between the following results and the corresponding results for real-valued functions. For instance, it is easy to see that a function $f : X \to \mathcal{C}$ is concave if and only if $\operatorname{hyp} f \subset X \times \mathcal{C}$ is convex.

We show that $\mathcal{F}$-semicontinuity and $\mathcal{C}$-semicontinuity coincide in the concave case (at least for $X = \mathbb{R}^d$, see Corollary 5.9 and the remark in Section 6).

**Theorem 5.1.** *Let $f : X \to \mathcal{C}$ be concave. Then, for all $\bar{x} \in X$ it holds*

$$\limsup_{x\to\bar{x}} f(x) = \operatorname*{Lim\,sup}_{x\to\bar{x}} f(x).$$

**Proof.** By Proposition 3.1 we have $\limsup_{x\to\bar{x}} f(x) = \bigcap_{\delta>0} \operatorname{cl}\operatorname{conv}\bigcup_{x\in B_\delta(\bar{x})} f(x)$. Since $f$ is concave, the sets $\bigcup_{x\in B_\delta(\bar{x})} f(x)$ are convex. The statement now follows from (3). □

**Corollary 5.2.** *Let $f : X \to \mathcal{C}$ be concave. Then the following statements hold true:*

($i$) $(\operatorname{usc} f) = (\operatorname{Usc} f)$,
($ii$) $(\operatorname{usc} f)$ *is concave.*

**Proof.** ($i$) follows from Theorem 5.1 and ($ii$) follows from ($i$) and Proposition 2.6 ($vi$). □

**Corollary 5.3.** *For any concave function $f : X \to \mathcal{C}$ the following assertion holds true:*

$$\operatorname{hyp} f \text{ is } \mathcal{C}\text{-closed} \quad\Leftrightarrow\quad f \text{ is } \mathcal{C}\text{-usc} \quad\Leftrightarrow\quad \operatorname{gr} f \subset X \times Y \text{ is closed.}$$

**Proof.** The second equivalence follows from (2) and the fact that $f$ is $\mathcal{F}$-usc if and only if $f$ is $\mathcal{C}$-usc. If $\operatorname{hyp} f$ is $\mathcal{C}$-closed, then $\operatorname{gr} f$ is obviously closed. On the other hand, if $f$ is $\mathcal{C}$-usc, Proposition 3.7 ($v$), ($vii$) yields that $\operatorname{hyp} f$ is closed. □

In the next statement, $\operatorname{ri} U$ denotes the relative algebraic interior of a subset $U$ of $X$. If $U$ is convex, it holds (see [21, (1.1)])

$$\bar{x} \in \operatorname{ri} U \iff \forall x \in U,\ \exists\lambda > 0 :\ (1+\lambda)\bar{x} - \lambda x \in U.$$

If $X = \mathbb{R}^d$ this concept coincides with the relative interior as defined in [17].

**Proposition 5.4.** *Any concave function $f : X \to \mathcal{C}$ is recessively constant on* $\operatorname{ri}(\operatorname{dom} f)$.

**Proof.** Of course, $\operatorname{dom} f$ is convex. Letting $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$ and $x \in \operatorname{ri}(\operatorname{dom} f)$, there exists $\lambda > 0$ such that $\hat{x} := (1+\lambda)\bar{x} - \lambda x \in \operatorname{dom} f$. The concavity of $f$ yields $f(\bar{x}) \supset \frac{1}{1+\lambda} f(\hat{x}) + \frac{\lambda}{1+\lambda} f(x)$. Since $\hat{x} \in \operatorname{dom} f$, it follows $0^+ f(\bar{x}) \supset 0^+ f(x)$. Analogously, we get the opposite inclusion. □

**Proposition 5.5.** *Let $f : X \to \mathcal{C}$ concave. Then,* $(\operatorname{usc} f) : X \to \mathcal{C}$ *is recessively constant.*

**Proof.** Since $g := (\operatorname{usc} f)$ is $\mathcal{F}$-usc and concave, its graph is closed and convex. For any closed convex set $A$ of a topological vector space it holds

$$\forall a \in A : \quad 0^+ A = \bigcap_{t>0} t(A - a), \tag{5}$$

see e.g. [21, (1.3)]. Consequently, for arbitrary $x \in \operatorname{dom} g$ we have

$$\begin{aligned} k \in 0^+ g(x) &\iff \forall y \in g(x), \forall t > 0: \; k \in t(g(x) - y) \\ &\iff \forall y \in g(x), \forall t > 0: \; (0,k) \in t(\operatorname{gr} g - (x,y)) \\ &\iff (0,k) \in 0^+ \operatorname{gr} g \end{aligned}$$

This proves that $x \longmapsto 0^+ g(x)$ is constant on $\operatorname{dom} g$. □

**Theorem 5.6.** *Let $f : X \to \mathcal{C}$. Then the following statements are equivalent:*

(*i*) *$f$ is concave and $\mathcal{C}$-usc,*
(*ii*) *$f$ is recessively constant with recession cone $K$ and for all $y^* \in \operatorname{ri} K^\circ$ the function $\sigma_{f(\cdot)}(y^*) : X \to \overline{\mathbb{R}}$ is concave and usc.*

**Proof.** Follows from Theorem 3.3 and Proposition 5.5. □

For finite dimensional spaces we obtain the following continuity properties for concave functions. Since in the present case $\mathcal{F}$-semicontinuity and $\mathcal{C}$-semicontinuity coincide (see also Corollary 5.9 below), the following statements are just reformulations of well-known results, see e.g. [18].

**Corollary 5.7.** *Let $f : \mathbb{R}^d \to \mathcal{C}$ be concave. Then the following assertions hold true:*

(*i*) *$f$ is $\mathcal{C}$-usc on* $\operatorname{ri}(\operatorname{dom} f)$*,*
(*ii*) *$f$ is $\mathcal{C}$-continuous on* $\operatorname{int}(\operatorname{dom} f)$*.*

**Proof.** (*i*) Let $\bar{x} \in \operatorname{ri}(\operatorname{dom} f)$ be given and let $K := 0^+ f(\bar{x})$. By Theorem 3.3, it remains to show that for all $y^* \in \operatorname{ri} K^\circ$, $\sigma_{f(\cdot)}(y^*)$ is usc at $\bar{x}$. By Proposition 5.4, $f$ is recessively constant on $\operatorname{ri}(\operatorname{dom} f)$. Hence for all $y^* \in \operatorname{ri} K^\circ$ it is true that $\bar{x} \in \operatorname{ri}\big(\operatorname{dom} \sigma_{f(\cdot)}(y^*)\big)$. Thus (see e.g. [17, Theorem 7.4]), $\sigma_{f(\cdot)}(y^*)$ is usc at $\bar{x}$.

(*ii*) By [18, Theorem 5.9 (b)], $f$ is $\mathcal{F}$-lsc at $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$. Hence $f$ is $\mathcal{C}$-lsc at $\bar{x}$. Now the assertion follows from (*i*). □

Next we show that a concave function is recessively rigid on the relative interior of its domain, in case of being $\mathcal{C}$-usc even on the whole domain. These assertions can be regarded as generalization of well-known statements concerning local boundedness of both real-valued functions and set-valued maps.

**Theorem 5.8.** *Any concave function $f : \mathbb{R}^d \to \mathcal{C}$ is recessively rigid on* $\operatorname{ri}(\operatorname{dom} f)$*. The function $f$ being additionally $\mathcal{C}$-usc is recessively rigid (on its whole domain).*

**Proof.** Set $g := (\operatorname{usc} f)$. Let $\bar{x} \in \operatorname{dom} g$, $V := \big\{x \in \mathbb{R}^d \,|\; \|x - \bar{x}\| \le 1\big\}$ and $K := 0^+ g(\bar{x})$. By Proposition 5.5, $g$ is recessively constant.

We next show that $0^+ \sup_{x \in V} g(x) = K$. Indeed, let $k \in 0^+ \sup_{x \in V} g(x)$ be given. Since $V$ is convex and compact and $\operatorname{gr} g$ is convex and closed, the set $A := \bigcup_{x \in V} g(x)$ is convex and closed; thus $k \in 0^+ A$. By [17, Theorem 8.2], $k$ is the limit of a sequence $(\lambda_n y_n)$ where $(\lambda_n) \downarrow 0$ and $y_n \in A$. Clearly, for all $n \in \mathbb{N}$ there exists $x_n \in V$ such that $y_n \in g(x_n)$. Since $V$ is bounded, we have $((\lambda_n x_n, \lambda_n y_n)) \to (0, k)$. Applying [17, Theorem 8.2] to the set $\operatorname{gr} g \subset \mathbb{R}^d \times Y$, we obtain $(0,k) \in 0^+ \operatorname{gr} g$. As in the proof of Proposition 5.5 we obtain $k \in 0^+ g(\bar{x}) = K$. Thus we have $0^+ \sup_{x \in V} g(x) \subset K$. Since $K = 0^+ g(\bar{x})$ we have equality in the latter inclusion. We have shown that $g$ is recessively rigid. If $f$ is $\mathcal{C}$-usc, we have $f = g$, i.e., $f$ is recessively rigid.

The first assertion now follows from Corollary 5.7 $(i)$. □

**Corollary 5.9.** *Let $f : \mathbb{R}^d \to \mathcal{C}$ be concave. Then for all $\bar{x} \in X$ it holds*

$$\liminf_{x\to\bar{x}} f(x) = \operatorname{Lim\,inf}_{x\to\bar{x}} f(x).$$

**Proof.** If $\bar{x} \in \operatorname{int}(\operatorname{dom} f)$, this follows from Theorem 5.8 and Corollary 4.6. Otherwise, we have $\liminf_{x\to\bar{x}} f(x) = \operatorname{Lim\,inf}_{x\to\bar{x}} f(x) = \emptyset$. □

## 6. Some remarks and open questions

In [14] is is shown that $\mathcal{C}_0 = \mathcal{C} \setminus \{\emptyset\}$ equipped with (sequential) $\mathcal{C}$-convergence is an $L^*$-space (see e.g. [11, 12]). As a consequence (see Kisynski [11]), $\mathcal{C}$-convergence in $\mathcal{C}_0$ is induced by a $T_1$-topology $\tau$. This means, each sequentially open subset of $\mathcal{C}_0$ is $\tau$-open (A subset $\mathcal{A}$ of $\mathcal{C}_0$ is *sequentially open* iff each sequence in $\mathcal{C}_0$, $\mathcal{C}$-converging to a point in $\mathcal{A}$ is eventually in $\mathcal{A}$). Such spaces are called *sequential spaces* in [8]. Franklin [8] has shown that there are sequential spaces that are neither first countable nor Fréchet-spaces (A topological space $\mathcal{X}$ is called a *Fréchet-space* iff the closure of any subset $\mathcal{A}$ of $\mathcal{X}$ is the set of limits of sequences in $\mathcal{A}$). It is an open question whether $(\mathcal{C}_0, \tau)$ is a Fréchet-space or even first countable. We thus do not know whether the (sequential) $\mathcal{C}$-closure as defined in Section 3 coincides with the $\tau$-closure.

Penot and Thera [16] defined the following very general semicontinuity notion for functions from a topological space $(\mathcal{X}, \tau_{\mathcal{X}})$ into a preordered topological space $(\mathcal{Y}, \tau_{\mathcal{Y}}, \leq)$. A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be upper semicontinuous at $\bar{x} \in \mathcal{X}$ if for any $\tau_{\mathcal{Y}}$-neighborhood $\mathcal{V}$ of $f(\bar{x})$, there exists a $\tau_{\mathcal{X}}$-neighborhood $U$ of $\bar{x}$ with

$$f(U) \subset \{Y \in \mathcal{Y} |\ \exists V \in \mathcal{V} : V \geq Y\}.$$

In our setting, namely for functions from a metric space $(X, d)$ into the ordered topological space $(\mathcal{C}, \tau, \subset)$, a function $f : X \to \mathcal{C}$ is upper semicontinuous at $\bar{x} \in \operatorname{dom} f$ in the sense of Penot and Thera (shortly PT-usc) if

$$\forall \mathcal{V} \in \mathcal{N}(f(\bar{x})),\ \exists\, \delta > 0, \forall x \in B_\delta(\bar{x}), \exists\, V \in \mathcal{V} :\ f(x) \subset V, \tag{6}$$

where $\mathcal{N}(f(\bar{x}))$ is the family of all $\tau$-neighborhoods of some $f(\bar{x})$. Let us show that $f$ being upper $\mathcal{C}$-usc implies that $f$ is PT-usc.

**Proposition 6.1.** *Let $f : X \to C$ be $\mathcal{C}$-usc at $\bar{x} \in \operatorname{dom} f$. Then $f$ is PT-usc at $\bar{x}$.*

**Proof.** We have $f(\bar{x}) = \bigcap_{\delta>0} \operatorname{cl\,conv} \bigcup_{x\in B_\delta(\bar{x})} f(x)$. The sequence $(A_n)$ defined by $A_n := \operatorname{cl\,conv} \bigcup_{x\in B_{1/n}(\bar{x})} f(x) \neq \emptyset$ is monotone decreasing with respect to set inclusion. Thus $(A_n)$ $\mathcal{C}$-converges to $f(\bar{x})$, see e.g. [14, Proposition 2.3]. Consequently, for all $\mathcal{V} \in \mathcal{N}(f(\bar{x}))$ there exists $n \in \mathbb{N}$ such that $A_n \in \mathcal{V}$. Hence for all $\mathcal{V} \in \mathcal{N}(f(\bar{x}))$ there exists $\delta := 1/n > 0$ such that for all $x \in B_\delta(\bar{x})$ there is $V \in \mathcal{V}$ with $f(x) \subset V$. □

We didn't succeed in showing that $f$ being PT-usc at $\bar{x} \in \operatorname{dom} f$ implies that $f$ is $\mathcal{C}$-usc at $\bar{x}$. In case we would know that $(\mathcal{C}_0, \tau)$ is first countable (and hence a Fréchet-space), this could be proved as follows:

Let $f$ be PT-usc at $\bar{x} \in \operatorname{dom} f$, i.e., (6) is satisfied. For an arbitrary sequence $(x_n) \to \bar{x}$ and a countable neighborhood base $(\mathcal{V}_n)_{n\in\mathbb{N}}$ of $f(\bar{x})$ we obtain

$$\forall n \in \mathbb{N},\, \exists\, k_n \geq n,\, \exists V_n \in \mathcal{V}_n :\ f(x_{k_n}) \subset V_n.$$

Since $(x_{k_n}) \to \bar{x}$ and $(V_n) \xrightarrow{\mathcal{C}} f(\bar{x})$, it follows $(\bar{x}, f(\bar{x})) \in \operatorname{cl}(\operatorname{hyp} f) \subset \operatorname{hyp}(\operatorname{usc} f)$, i.e., $f$ is $\mathcal{C}$-usc at $\bar{x}$.

Another open questions is whether Corollary 5.9 remains valid if the origin space of $f$ is not supposed to be finite dimensional.

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