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Characterizations of the Subdifferential of the Supremum of Convex Functions*

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Starting with some analysis of the support function of an arbitrary set, we obtain a formula for the subdifferential set of the supremum function of an arbitrary (possibly infinite) family of proper convex functions at each point of its effective domain, not necessarily at a continuity point. In this sense, our formula constitutes an extension of [14, Theorem A], and also allows us to derive a generalization of [2, p. 227]. Our approach is based on linearization via the Fenchel conjugate.

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1. Introduction

The supremum function arises in a variety of contexts, including duality, and this is why many authors contributed to the subject since 1965 ([3], [6], [7], [8], [10], [11], [12], [13], [14], [15], etc.). As it is stated in [5, p. 405], the most elaborated results are due to M. Valadier [13]. In [14] a new formula, making use of the concept of $\varepsilon-$subdifferential, is given. When one considers the supremum of affine functions both formulas in [13] and [14] are completely equivalent.

The main advantage of this approach, based on the use of approximate subdifferentials, is that it avoids qualification-type conditions as well as assumptions on the structure of the index set (the set of indices associated with the functions whose supremum is analyzed). In many cases it is not possible to express the subdifferential of the supremum function by means only of the exact subdifferentials of the involved functions. This fact is well known even for calculus rules dealing with finitely many functions (see, for instance, [6]). Here in this paper we show that this is also the case in the infinite setting with a general index set and where, consequently, the functions have no special property with respect to

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the indices. Nevertheless, when some structure is assumed for the model, our approach also yields expressions for the subdifferential set of the supremum function in terms of the exact subdifferential sets of the involved functions.

The summary of the paper is as follows. In Proposition 2.1 of Section 2 we give a new formula, (6), for the subdifferential mapping of the support function of an arbitrary set. This formula yields, via a homogenization process, the formula (21) in Proposition 3.1 of Section 3, which characterizes the subdifferential set of the supremum of an arbitrary family of affine functions. In Section 4 we use linearization via Fenchel conjugation to derive an extension of the formula (44) in [14, Theorem A] for the subdifferential set of the supremum of a family of lower semicontinuous proper convex functions. The extension comes from the fact that our formula (30) in Proposition 4.1 provides the subdifferential set at each point of the effective domain of the supremum function, not necessarily at a continuity point. In Proposition 4.3 of the same Section 4 our formula is extended to families of convex proper functions, not necessarily lower semicontinuous. Our Corollary 4.4 actually deduces (44) as a particular case of (30). In Section 5 we establish some conditions under which the subdifferential set of the supremum function is expressed in terms exclusively of the $\varepsilon-$subdifferentials of the original functions, by means of what we call here homogeneous formulae. Our Corollary 5.4 extends one result due to Brøndsted [2, p. 227] (see, also, [6, Theorem 2.1]). In the final section, Section 6, and for comparative purposes, we consider the continuous case (compact index set and some continuous behavior of the involved functions with respect to the indices). In this section, we make a short review of some classical formulae, as the Valadier's formula (see, for instance, [5, Theorem VI.4.4.8]) and the Ioffe-Tihkomirov's theorem (see, for instance, [16, Theorem 2.4.18]), and we give alternative proofs of them based on our main results (Proposition 4.1 and Lemma 6.2). Also in this last section our contribution is put in perspective and compared with some existing results in [11], [12], and [15].

Given non-empty sets $X \subset \mathbb{R}^p$ and $\Lambda \subset \mathbb{R}$, we shall use the notation $\Lambda X = \{\lambda x : \lambda \in \Lambda,\ x \in X\}$. If $X = \{x\}$, we simply write $\Lambda\{x\} = \Lambda x = \{\lambda x : \lambda \in \Lambda\}$. By $\operatorname{co} X$, $\overline{\operatorname{co}} X$, and $\operatorname{aff} X$ we denote the *convex hull*, the *closed convex hull*, and the *affine hull* of the set X, respectively. In the topological side, $\operatorname{int} X$, $\operatorname{cl} X$, and $\operatorname{rint} X$ represent the *interior set*, the *closure*, and the *relative interior* of X (i.e., the interior of X in the topology relative to $\operatorname{aff} X$). The following straightforward equality is applied very often: $\operatorname{cl}(X + Y) = \operatorname{cl}(X + \operatorname{cl} Y)$, where Y is another arbitrary set in $\mathbb{R}^p$. We represent by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the usual *scalar product* and the *Euclidean norm* in $\mathbb{R}^p$, respectively, whereas $\mathbb{B}_p$ is the associated *closed unit ball* centered at the origin 0_p. For the sake of simplicity we write the vectors in $\mathbb{R}^{p+1}$ in the form (x, x_{p+1}), with $x \in \mathbb{R}^p$. We also use the sets

$$X^\circ := \{y \in \mathbb{R}^p \mid \langle y, x\rangle \leq 0 \text{ for all } x \in X\},$$

and

$$X^\perp := \{y \in \mathbb{R}^p \mid \langle y, x\rangle = 0 \text{ for all } x \in X\};$$

i.e., the *dual cone* of X and the *orthogonal space* of X, respectively. If X is a non-empty convex set and $z \in X$, we define the *normal cone to X at z* as

$$\mathrm{N}_X(z) := (X - z)^\circ = \{y \in \mathbb{R}^p \mid \langle y, x - z\rangle \leq 0 \text{ for all } x \in X\},$$

and the set of *ε-normal directions to X at z*, with $\varepsilon > 0$, as

$$\mathrm{N}_X^\varepsilon(z) := \{y \in \mathbb{R}^p \mid \langle y, x - z\rangle \leq \varepsilon \text{ for all } x \in X\}.$$

If X is a closed convex set, X_∞ represents its *recession cone*

$$X_\infty := \{y \in \mathbb{R}^p : \ x + \lambda y \in X \text{ for some } x \in X \text{ and all } \lambda \geq 0\},$$

whereas

$$\operatorname{lin} X := X_\infty \cap (-X)_\infty$$

represents its *lineality space.*

Given a proper convex function $f : \mathbb{R}^p \longrightarrow \mathbb{R} \cup \{+\infty\}$, we consider the *effective domain*, the *graph*, and the *epigraph* of this function, which are the non-empty sets

$$\operatorname{dom} f := \{x \in \mathbb{R}^p : \ f(x) < +\infty\},$$
$$\operatorname{gph} f := \{(x, f(x)) : \ x \in \operatorname{dom} f\},$$

and

$$\operatorname{epi} f := \{(x, \alpha) : \ x \in \operatorname{dom} f \text{ and } f(x) \leq \alpha\},$$

respectively. For $z \in \operatorname{dom} f$, the *subdifferential set* of f at z is the (possibly empty) set

$$\partial f(z) := \{a \in \mathbb{R}^p : f(x) - f(z) \geq \langle a, x - z\rangle \text{ for all } x \in \mathbb{R}^p\}.$$

A point z is a global minimum of f if and only if $0_p \in \partial f(z)$.

For $\varepsilon > 0$ and $z \in \operatorname{dom} f$, the $\varepsilon-$*subdifferential* of f at z is given by

$$\partial_\varepsilon f(z) := \{a \in \mathbb{R}^p : f(x) - f(z) \geq \langle a, x - z\rangle - \varepsilon \text{ for all } x \in \mathbb{R}^p\}.$$

When f is further lower semicontinuous (lsc, in brief), this set is non-empty. Further, it is obvious that

$$\partial f(z) = \bigcap_{\varepsilon > 0} \partial_\varepsilon f(z). \tag{1}$$

For $\varepsilon = 0$ we simply write $\partial_0 f(z) = \partial f(z)$.

The *Fenchel conjugate* of f is the function $f^* : \mathbb{R}^p \longrightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$f^*(y) := \sup\{\langle x, y\rangle - f(x) : x \in \mathbb{R}^p\}.$$

If f is lower semicontinuous, $f^{**} = f$ [5, Corollary X.1.3.6] and f can be expressed as the supremum of an infinite family of affine functions

$$f(x) = \sup\{\langle y, x\rangle - f^*(y) : y \in \operatorname{dom} f^*\}.$$

This notion of conjugate function gives rise to the following characterizations of ∂f and $\partial_\varepsilon f$ [5, Proposition XI.1.2.1]: Given $z \in \operatorname{dom} f$,

$$\partial f(z) = \{a \in \mathbb{R}^p : f(z) + f^*(a) = \langle a, z\rangle\}, \tag{2}$$

and for $\varepsilon > 0$,

$$\partial_\varepsilon f(z) = \{a \in \mathbb{R}^p : f(z) + f^*(a) \leq \langle a, z\rangle + \varepsilon\}. \tag{3}$$

The *closure* of the convex proper function f is the *lower semi-continuous hull* of f denoted $\operatorname{cl} f : \mathbb{R}^p \longrightarrow \mathbb{R} \cup \{+\infty\}$ and given by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f).$$

The *indicator function* of the set $A \subset \mathbb{R}^p$ is defined as

$$\mathrm{I}_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{if } x \notin A, \end{cases}$$

whereas the *support function* of A is the function

$$\sigma_A(x) := \sup\{\langle a, x\rangle : a \in A\}. \tag{4}$$

The function σ_A is obviously a lsc sublinear function such that $\sigma_A = \sigma_{\overline{\mathrm{co}}A} = \mathrm{I}^*_{\overline{\mathrm{co}}A}$. Further,

$$\mathrm{cl}(\mathrm{dom}\,\sigma_A) = [(\overline{\mathrm{co}}A)_\infty]^\circ .$$

(See [5, Proposition V.2.2.4].)

Moreover, (2), (3) and the fact $\sigma_A^* = \mathrm{I}_{\overline{\mathrm{co}}A}$ yield, for every $z \in \mathrm{dom}\,\sigma_A$,

$$\begin{aligned} \partial\sigma_A(z) &= \{a \in \overline{\mathrm{co}}A : \sigma_A(z) = \langle a, z\rangle\} \\ \text{and} \quad \partial_\varepsilon\sigma_A(z) &= \{a \in \overline{\mathrm{co}}A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}. \end{aligned} \tag{5}$$

For a given $z \in \mathrm{dom}\,\sigma_A$, $z \neq 0_p$, the set $\partial\sigma_A(z)$ can be empty, but $\partial\sigma_A(0_p) = \overline{\mathrm{co}}A$.

The following lemma adds information about the relationship between some of these sets:

Lemma 1.1. *Let us consider a convex proper function* $f : \mathbb{R}^p \longrightarrow \mathbb{R} \cup \{+\infty\}$ *and let* $z \in \mathrm{dom}\, f$. *Then for every* $\varepsilon \geq 0$ *the following statements hold:*

(*i*) $\mathrm{N}_{\mathrm{dom}\, f}(z) = (\partial_\varepsilon f(z))_\infty$, *provided that* $\partial_\varepsilon f(z) \neq \emptyset$.
(*ii*) $\mathrm{N}_{\mathrm{dom}\, f}(z) = \left(\mathrm{N}^\varepsilon_{\mathrm{dom}\, f}(z)\right)_\infty$.

Remark (before the proof). See a proof of (*i*) for $\varepsilon = 0$ in [1, Proposition 2.5.4]).

Proof. (*i*) By definition, $\partial_\varepsilon f(z)$ is the closed convex set given by

$$\partial_\varepsilon f(z) = \cap_{y \in \mathrm{dom}\, f} C_y,$$

where C_y is the closed and convex set

$$C_y := \{u \in \mathbb{R}^p : \langle u, y - z\rangle \leq f(y) - f(z) + \varepsilon\}.$$

Then we obtain

$$\begin{aligned} (\partial_\varepsilon f(z))_\infty &= \left(\cap_{y \in \mathrm{dom}\, f} C_y\right)_\infty = \cap_{y \in \mathrm{dom}\, f} \left(C_y\right)_\infty \\ &= \cap_{y \in \mathrm{dom}\, f}\{u \in \mathbb{R}^p : \langle u, y - z\rangle \leq 0\} = \mathrm{N}_{\mathrm{dom}\, f}(z). \end{aligned}$$

(*ii*) We have

$$\begin{aligned} \left(\mathrm{N}^\varepsilon_{\mathrm{dom}\, f}(z)\right)_\infty &= \left(\cap_{y \in \mathrm{dom}\, f}\{u \in \mathbb{R}^p : \langle u, y - z\rangle \leq \varepsilon\}\right)_\infty \\ &= \cap_{y \in \mathrm{dom}\, f}\left(\{u \in \mathbb{R}^p : \langle u, y - z\rangle \leq \varepsilon\}\right)_\infty \\ &= \cap_{y \in \mathrm{dom}\, f}\{u \in \mathbb{R}^p : \langle u, y - z\rangle \leq 0\} \\ &= \mathrm{N}_{\mathrm{dom}\, f}(z). \end{aligned}$$

□

2. Subdifferential of the support function

In this section we establish an alternative characterization of $\partial\sigma_A(z)$ where A is a non-empty set (non necessarily convex), which turns out to be useful for different purposes.

Proposition 2.1. *Given a non-empty set $A \subset \mathbb{R}^p$ and the associated support function σ_A, for every $z \in \operatorname{dom}\sigma_A$ we have*

$$\partial\sigma_A(z) = \bigcap_{\varepsilon>0} \operatorname{cl}\left((\operatorname{co}\{a \in A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}) + A(z)\right), \tag{6}$$

where

$$A(z) := (\overline{\operatorname{co}}A)_\infty \cap \{z\}^\perp. \tag{7}$$

In particular,

$$z \in \operatorname{rint}(\operatorname{dom}\sigma_A) \Rightarrow \partial\sigma_A(z) = \bigcap_{\varepsilon>0} \operatorname{cl}\left((\operatorname{co}\{a \in A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}) + \operatorname{lin}(\overline{\operatorname{co}}A)\right),$$

and

$$z \in \operatorname{int}(\operatorname{dom}\sigma_A) \Rightarrow \partial\sigma_A(z) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\{a \in A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}.$$

Remark (before the proof). For every $z \in \operatorname{dom}\sigma_A$ we have

$$\mathrm{N}_{\operatorname{dom}\sigma_A}(z) = (\overline{\operatorname{co}}A)_\infty \cap \{z\}^\perp = A(z). \tag{8}$$

Indeed, since $\operatorname{dom}\sigma_A$ is a cone containing z, and applying Corollary 16.4.2 in [9], we write

$$\begin{aligned}
(\overline{\operatorname{co}}A)_\infty \cap \{z\}^\perp &= [\operatorname{cl}(\operatorname{dom}\sigma_A)]^\circ \cap (\mathbb{R}z)^\circ \\
&= (\operatorname{dom}\sigma_A)^\circ \cap (\mathbb{R}z)^\circ \\
&= ((\operatorname{dom}\sigma_A) + \mathbb{R}z)^\circ \\
&= ((\operatorname{dom}\sigma_A) - \mathbb{R}_+z)^\circ \\
&= [\mathbb{R}_+((\operatorname{dom}\sigma_A) - z)]^\circ \\
&= [(\operatorname{dom}\sigma_A) - z]^\circ = \mathrm{N}_{\operatorname{dom}\sigma_A}(z).
\end{aligned} \tag{9}$$

Proof. We shall decompose the proof in four steps in which, for the sake of simplicity, we denote

$$A_\varepsilon := \{a \in A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}. \tag{10}$$

Step 1. Fix $z \in \operatorname{dom}\sigma_A$. We use (8) and prove first the inclusion

$$\partial\sigma_A(z) \supset \bigcap_{\varepsilon>0} \operatorname{cl}\left((\operatorname{co}A_\varepsilon) + \mathrm{N}_{\operatorname{dom}\sigma_A}(z)\right).$$

This is obvious if the set in the right-hand side is empty. Otherwise, if u belongs to the right-hand side set, and for a fixed $\varepsilon > 0$, there will exist sequences

$$(u_k^\varepsilon)_{k=1}^\infty \subset \operatorname{co}A_\varepsilon \quad \text{and} \quad (v_k^\varepsilon)_{k=1}^\infty \subset \mathrm{N}_{\operatorname{dom}\sigma_A}(z),$$

such that

$$u = \lim_{k\to\infty} (u_k^\varepsilon + v_k^\varepsilon).$$

So we have

$$\langle u_k^\varepsilon, z\rangle \geq \sigma_A(z) - \varepsilon, \quad \text{for } k = 1, 2, ...,$$

and

$$\langle v_k^\varepsilon, x - z\rangle \leq 0, \quad \text{for } k = 1, 2, ..., \text{ and all } x \in \operatorname{dom} \sigma_A.$$

Moreover, for $x \in \mathbb{R}^p$,

$$\begin{aligned} \sigma_A(x) \geq &\ \langle u_k^\varepsilon, x\rangle \\ \geq &\ \langle u_k^\varepsilon, x\rangle + (-\langle u_k^\varepsilon, z\rangle + \sigma_A(z) - \varepsilon) + \langle v_k^\varepsilon, x - z\rangle \\ = &\ \langle u_k^\varepsilon + v_k^\varepsilon, x - z\rangle + \sigma_A(z) - \varepsilon. \end{aligned}$$

Taking limits for $k \to \infty$ one gets

$$\sigma_A(x) \geq \sigma_A(z) + \langle u, x - z\rangle - \varepsilon,$$

and $u \in \partial_\varepsilon \sigma_A(z)$, for every $\varepsilon > 0$. So, according to (1), $u \in \partial\sigma_A(z)$.

Step 2. We proceed by proving the reverse inclusion (recall that A_ε is defined in (10))

$$\partial\sigma_A(z) \subset \bigcap_{\varepsilon>0} \operatorname{cl}\left((\operatorname{co} A_\varepsilon) + A(z)\right). \tag{11}$$

This is obvious if $\partial\sigma_A(z)$ is empty. Otherwise, let $u^* \in \partial\sigma_A(z)$ and, reasoning by contradiction, assume that, for certain $\varepsilon > 0$,

$$u^* \notin \operatorname{cl}\left((\operatorname{co} A_\varepsilon) + A(z)\right).$$

Then, by the separation theorem, there will exist $v^* \in \mathbb{R}^p \setminus \{0_p\}$ and a scalar $\beta < 0$ such that

$$\langle a + u - u^*, v^*\rangle \leq 2\beta, \tag{12}$$

for all $a \in \operatorname{co} A_\varepsilon$ and all $u \in A(z)$. Because $A(z) := (\overline{\operatorname{co}} A)_\infty \cap \{z\}^\perp$ is a cone, we deduce

$$\langle u, v^*\rangle \leq 0,$$

for all $u \in A(z)$, and (9) yields

$$v^* \in \left[(\overline{\operatorname{co}} A)_\infty \cap \{z\}^\perp\right]^\circ = \operatorname{cl}\left(\operatorname{dom} \sigma_A + \mathbb{R}z\right).$$

Let us introduce the function $g : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g(x) := \sup\{\langle a + u - u^*, x\rangle : a \in A_\varepsilon,\ u \in A(z)\}.$$

We have, for all $x \in \operatorname{dom} \sigma_A \subset \operatorname{cl}(\operatorname{dom} \sigma_A) = [(\overline{\operatorname{co}} A)_\infty]^\circ$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned} g(x + \gamma z) = &\ \sup\{\langle a + u - u^*, x + \gamma z\rangle : a \in A_\varepsilon,\ u \in A(z)\} \\ \leq &\ \sup\{\langle a - u^*, x + \gamma z\rangle : a \in A_\varepsilon\} \\ \leq &\ \sup\{\gamma\langle a, z\rangle : a \in A_\varepsilon\} + \sigma_A(x) - \langle u^*, x + \gamma z\rangle \\ \leq &\ \max\{\gamma\sigma_A(z), \gamma\sigma_A(z) - \gamma\varepsilon\} + \sigma_A(x) - \langle u^*, x + \gamma z\rangle < +\infty, \end{aligned}$$

in other words $(\operatorname{dom}\sigma_A) + \mathbb{R}z \subset \operatorname{dom} g$.

Moreover, since $g(v^*) \leq 2\beta < \beta$ by (12), there exists, according to [9, Corollary 7.3.3], $w \in \operatorname{dom}\sigma_A$ and $\alpha \in \mathbb{R}$ such that for $w^* := w + \alpha z$ we have

$$\langle a + u - u^*, w^* \rangle \leq g(w^*) < \beta, \tag{13}$$

for all $a \in A_\varepsilon$ and all $u \in A(z)$.

Let $\rho > 0$ be such that $\sigma_A(w) \leq \rho$ and take $\gamma > 0$ small enough in order to make sure that $1 + \gamma\alpha > 0$ and

$$\alpha(\sigma_A(z) - \varepsilon) + \rho - \langle u^*, w^* \rangle - \beta < \frac{\varepsilon}{\gamma}.$$

Then, for all $a \in A \setminus A_\varepsilon$, we have

$$\begin{aligned}
\langle a, z + \gamma w^* \rangle &= \langle a, z + \gamma w + \gamma\alpha z \rangle \\
&= (1 + \gamma\alpha)\langle a, z \rangle + \gamma\langle a, w \rangle \\
&< (1 + \gamma\alpha)(\sigma_A(z) - \varepsilon) + \gamma\rho \\
&= \sigma_A(z) + \gamma\alpha(\sigma_A(z) - \varepsilon) + \gamma\rho - \varepsilon \\
&< \sigma_A(z) + \gamma\langle u^*, w^* \rangle + \gamma\beta,
\end{aligned} \tag{14}$$

whereas, for all $a \in A_\varepsilon$, we have

$$\begin{aligned}
\langle a, z + \gamma w^* \rangle &= \langle a, z \rangle + \gamma\langle a, w^* \rangle \\
&\leq \sigma_A(z) + \gamma\langle u^*, w^* \rangle + \gamma\beta,
\end{aligned}$$

the second inequality coming from (13) with $u = 0_p$. This inequality, together with (14), leads us to

$$\langle a, z + \gamma w^* \rangle \leq \sigma_A(z) + \gamma\langle u^*, w^* \rangle + \gamma\beta,$$

for all $a \in A$, so that, passing to the supremum over $a \in A$,

$$\sigma_A(z + \gamma w^*) \leq \sigma_A(z) + \gamma\langle u^*, w^* \rangle + \gamma\beta < \sigma_A(z) + \gamma\langle u^*, w^* \rangle,$$

and this contradicts $u^* \in \partial\sigma_A(z)$.

Step 3. Now assume $z \in \operatorname{rint}(\operatorname{dom}\sigma_A)$. In this case

$$\begin{aligned}
A(z) &= (\overline{\text{co}}A)_\infty \cap \{z\}^\perp = \mathrm{N}_{\operatorname{dom}\sigma_A}(z) = (\operatorname{dom}\sigma_A)^\perp \\
&= ([(\overline{\text{co}}A)_\infty]^\circ)^\perp = (\overline{\text{co}}A)_\infty \cap (-\overline{\text{co}}(A))_\infty = \operatorname{lin}(\overline{\text{co}}A).
\end{aligned} \tag{15}$$

Step 4. In the case $z \in \operatorname{int}(\operatorname{dom}\sigma_A)$, and using (15), we get

$$A(z) = \operatorname{lin}(\overline{\text{co}}A) = (\operatorname{dom}\sigma_A)^\perp = \{0_p\}.$$

□

Remark 2.2. Observe that the formula (6) does not require the closedness and convexity of the set A. If A is a closed convex set, then by (5) we have

$$\partial_\varepsilon\sigma_A(z) = A_\varepsilon := \{a \in A : \langle a, z \rangle \geq \sigma_A(z) - \varepsilon\}, \tag{16}$$

and so

$$\partial\sigma_A(z) = \bigcap_{\varepsilon > 0} A_\varepsilon. \tag{17}$$

When A is a general set, our formula (6) is not equivalent to (17) in the sense that the inclusion

$$\mathrm{cl}\left((\mathrm{co}\, A_\varepsilon) + (\overline{\mathrm{co}} A)_\infty \cap \{z\}^\perp\right) \subset \partial_\varepsilon \sigma_A(z),$$

established in Step 1 of the proof of Proposition 2.1, can be strict. This fact is shown in the following example:

Example 2.3. Let $A \subset \mathbb{R}^3$ be the set given by

$$A := \{(1, \alpha, \beta) : \ \alpha \geq 0, \ \beta \in \mathbb{R}\} \cup \{(0, \gamma, -\log \gamma) : \ 0 < \gamma \leq 1\},$$

and let us consider the support function σ_A. For $z := (-1, -1, 0)$ we have $\sigma_A(z) = 0$ and, with $\varepsilon_0 \leq 1$ fixed,

$$\begin{aligned} A_{\varepsilon_0} := \{a \in A : \langle a, z \rangle \geq \sigma_A(z) - \varepsilon_0\} &= \{a \in A : \langle a, z \rangle = -a_1 - a_2 \geq -\varepsilon_0\} \\ &= \{(0, \gamma, -\log \gamma) : \ 0 < \gamma \leq \varepsilon_0\}, \end{aligned}$$

and

$$\mathrm{co}\, A_{\varepsilon_0} = \{(0, \gamma, \delta) : \ 0 < \gamma \leq \varepsilon_0; \ -\log \varepsilon_0 \leq \delta \leq -\log \gamma\}.$$

Moreover

$$A(z) := (\overline{\mathrm{co}} A)_\infty \cap \{z\}^\perp = \mathbb{R}(0, 0, 1),$$

and

$$\begin{aligned} (\mathrm{co}\, A_{\varepsilon_0}) + A(z) &= (\overline{\mathrm{co}} A_{\varepsilon_0}) + (\overline{\mathrm{co}} A)_\infty \cap \{z\}^\perp \\ &= \{(0, \gamma, \delta) : \ 0 < \gamma \leq \varepsilon_0, \ \delta \in \mathbb{R}\}, \end{aligned}$$

which obviously is not closed.

At the same time, for every $\varepsilon > 0$, (5) yields

$$\begin{aligned} \partial_\varepsilon \sigma_A(z) &= \{a \in \overline{\mathrm{co}} A : \ \langle a, z \rangle \geq -\varepsilon\} \\ &= \{a \in \mathbb{R}^3 : \ 0 \leq a_1 \leq 1, \ a_2 \geq 0, \ a_1 + a_2 \leq \varepsilon\}. \end{aligned}$$

This shows that $\mathrm{cl}\left((\mathrm{co}\, A_{\varepsilon_0}) + (\overline{\mathrm{co}} A)_\infty \cap \{(-1, -1, 0)\}^\perp\right) \subsetneq \partial_\varepsilon \sigma_A(z)$, for every $\varepsilon > 0$.

Remark 2.4. This example shows that we cannot replace $\mathrm{cl}\left((\mathrm{co}\, A_\varepsilon) + (\overline{\mathrm{co}} A)_\infty \cap \{z\}^\perp\right)$ by $(\overline{\mathrm{co}} A_\varepsilon) + (\overline{\mathrm{co}} A)_\infty \cap \{z\}^\perp$ in (6) (recall that A_ε is defined in (16)); in other words, in general,

$$\partial \sigma_A(z) \supsetneq \bigcap_{\varepsilon > 0} \left(\overline{\mathrm{co}}\{a \in A : \langle a, z \rangle \geq \sigma_A(z) - \varepsilon\} + (\overline{\mathrm{co}} A)_\infty \cap \{z\}^\perp\right).$$

In the example, the right-hand side set is empty, meanwhile $\partial \sigma_A(z) = \mathbb{R}(0, 0, 1)$.

Proposition 2.5. *Given a non-empty convex set $A \subset \mathbb{R}^p$, for every $z \in \mathrm{dom}\, \sigma_A$ and every $\varepsilon > 0$ we have*

$$\partial_\varepsilon \sigma_A(z) = \mathrm{cl}\left(\{a \in A : \langle a, z \rangle \geq \sigma_A(z) - \varepsilon\} + A(z)\right) = \mathrm{cl}\{a \in A : \langle a, z \rangle \geq \sigma_A(z) - \varepsilon\},$$

where $A(z)$ is the set defined in (7).

Proof. In order to prove the first equality, we only need to check the inclusion

$$\partial_\varepsilon \sigma_A(z) \subset \mathrm{cl}\left(\{a \in A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\} + A(z)\right).$$

Indeed, according to (5),

$$\partial_\varepsilon \sigma_A(z) = \{a \in \mathrm{cl}\, A : \ \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\},$$

and so

$$\begin{aligned}
\partial_\varepsilon \sigma_A(z) &= \mathrm{cl}(A) \cap \{a \in \mathbb{R}^p : \ \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\} \\
&= \mathrm{cl}\left(A \cap \{a \in \mathbb{R}^p : \ \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}\right) \\
&= \mathrm{cl}\left(\{a \in A : \ \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\}\right) \\
&\subset \mathrm{cl}\left(\{a \in A : \langle a, z\rangle \geq \sigma_A(z) - \varepsilon\} + A(z)\right),
\end{aligned}$$

where the second equality above holds because it can easily be checked that

$$(\mathrm{rint}\, A) \cap \{a \in \mathbb{R}^p : \ \langle a, z\rangle > \sigma_A(z) - \varepsilon\} \neq \emptyset$$

and, then, [9, Theorem 6.5] applies. □

3. Subdifferential of the supremum of affine functions

Our goal in this section is to establish an affine counterpart of the main proposition in the previous section. To this aim, we recall the following result:

Let $f(x) := h(Ax + b)$, where h is a proper convex function on $\mathbb{R}^m$, A is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $b \in \mathbb{R}^m$. Then, for every $z \in \mathbb{R}^n$,

$$\partial f(z) \supset A^* \partial h(Az + b),$$

where $A^* : \mathbb{R}^m \to \mathbb{R}^n$ is the transpose of A. Moreover, if

$$(A(\mathbb{R}^n) + b) \cap \mathrm{rint}(\mathrm{dom}\, h) \neq \emptyset,$$

then one also has

$$\partial f(z) = A^* \partial h(Az + b). \tag{18}$$

Such result is established in [9, Theorem 23.9] in the case of linear transformations ($b = 0_m$). It is not difficult to check (by using [5, Theorem X 2.2.1]), that the proof in [9, Theorem 23.9] is still valid for the case of affine transformations.

By applying the previous result one can give the formula of the partial subdifferential of convex functions of two variables.
Let $h : \mathbb{R}^p \times \mathbb{R}^q :\to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Fixing $y = y_0 \in \mathbb{R}^q$ we get the function $f(x) := h(x, y_0)$, which can be written

$$f(x) = h(Ax + (0_p, y_0)),$$

where $A : \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^q$ is the linear transformation given by

$$Ax = (x, 0_q).$$

(The transpose $A^* : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ of A is given by $A^*(x,y) = x$.)

Then, under the condition

$$(\mathbb{R}^p \times \{y_0\}) \cap \operatorname{rint}(\operatorname{dom} h) \neq \emptyset, \tag{19}$$

and applying the formula (18) above with $b = (0_p, y_0)$, we have at $x = z$

$$\partial f(z) = A^* \partial h(z, y_0) = \{v \in \mathbb{R}^p : \exists\, u \in \mathbb{R}^q \text{ s.t. } (v,u) \in \partial h(z, y_0)\}. \tag{20}$$

Proposition 3.1. *Given a non-empty set* $\{(a_t, b_t) : \ t \in T\} \subset \mathbb{R}^{p+1}$, *and the supremum function* $f : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$

$$f(x) := \sup\{\langle a_t, x\rangle - b_t : t \in T\},$$

for every $z \in \operatorname{dom} f$ *we have*

$$\partial f(z) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + B(z)\right), \tag{21}$$

where

$$T_\varepsilon(z) := \{t \in T : \ \langle a_t, z\rangle - b_t \geq f(z) - \varepsilon\},$$

and

$$B(z) := \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}}\{\cup_{t\in T}\{(a_t, b_t)\}\})_\infty\}.$$

In particular, if $z \in \operatorname{rint}(\operatorname{dom} f)$ *one has*

$$\partial f(z) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in \operatorname{lin}(\overline{\operatorname{co}}\{\cup_{t\in T}\{(a_t, b_t)\}\})\}\right), \tag{22}$$

and if $z \in \operatorname{int}(\operatorname{dom} f)$

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\}. \tag{23}$$

Proof. We proceed again in four steps.

Step 1. Let us introduce the support function

$$\sigma_A(x,y) := \sup\{\langle (a,b), (x,y)\rangle : (a,b) \in A\} \text{ with } A := \cup_{t\in T}\{(a_t, b_t)\}. \tag{24}$$

Fix $z \in \operatorname{dom} f$. Then $(z, -1) \in \operatorname{dom} \sigma_A$ and, according to Proposition 3.1, the subdifferential of σ_A at $(z,-1)$ is given by

$$\partial \sigma_A(z,-1) = \bigcap_{\varepsilon>0} \operatorname{cl}\left((\operatorname{co} A_\varepsilon) + (\overline{\operatorname{co}} A)_\infty \cap \{(z,-1)\}^\perp\right),$$

where we denote

$$A_\varepsilon := \cup_{t\in T_\varepsilon(z)}\{(a_t, b_t)\}.$$

In order to apply (20) we check first that the condition (19) holds. By Theorem 6.1 in [9], there exists a sequence $((z_k, \lambda_k))_{k=1}^\infty \subset \operatorname{rint}(\operatorname{dom} \sigma_A)$ converging to $(z,-1)$ such that (w.l.o.g.) $\lambda_k < 0$, for $k = 1, 2, \cdots$. Since $\operatorname{rint}(\operatorname{dom}\sigma_A)$ is a cone, we also have $\left((|\lambda_k|^{-1} z_k, -1)\right)_{k=1}^{\infty} \subset \operatorname{rint}(\operatorname{dom}\sigma_A)$, and therefore

$$(\mathbb{R}^p \times \{-1\}) \cap \operatorname{rint}(\operatorname{dom}\sigma_A) \neq \emptyset.$$

Thus, applying (20) with $q = 1$ and $y_0 = -1$, we obtain

$$\partial f(z) = \left\{ u \in \mathbb{R}^p : \exists\, \lambda \in \mathbb{R} \text{ s.t. } (u, \lambda) \in \bigcap_{\varepsilon>0} \operatorname{cl}\left((\operatorname{co} A_\varepsilon) + (\overline{\operatorname{co}} A)_\infty \cap \{(z, -1)\}^\perp\right) \right\}. \quad (25)$$

We proceed now by showing the direct inclusion

$$\partial f(z) \subset \bigcap_{\varepsilon>0} \operatorname{cl}\left(\operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + B(z)\right). \quad (26)$$

This is obvious if $\partial f(z)$ is empty. Otherwise, let $u \in \partial f(z)$ and take $\varepsilon > 0$. The formula (25) above entails the existence of $\lambda \in \mathbb{R}$ such that

$$\begin{aligned}(u, \lambda) &\in \operatorname{cl}\left((\operatorname{co} A_\varepsilon) + (\overline{\operatorname{co}} A)_\infty \cap \{(z, -1)\}^\perp\right) \\ &= \bigcap_{\delta>0} \left((\operatorname{co} A_\varepsilon) + (\overline{\operatorname{co}} A)_\infty \cap \{(z, -1)\}^\perp + \delta\mathbb{B}_{p+1}\right).\end{aligned}$$

Thus, for every $\delta > 0$ fixed, there will exist

$$(a^\delta, b^\delta) \in \operatorname{co} A_\varepsilon \quad \text{and} \quad (v^\delta, \alpha^\delta) \in (\overline{\operatorname{co}} A)_\infty \cap \{(z, -1)\}^\perp,$$

such that

$$(u, \lambda) \in (a^\delta, b^\delta) + (v^\delta, \alpha^\delta) + \delta\mathbb{B}_{p+1}.$$

Since $(v^\delta, \alpha^\delta) \in \{(z, -1)\}^\perp$, we can write

$$(u, \lambda) \in (a^\delta, b^\delta) + \left(v^\delta, \langle v^\delta, z\rangle\right) + \delta\mathbb{B}_{p+1},$$

entailing

$$u \in a^\delta + v^\delta + \delta\mathbb{B}_p \subset \operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}} A)_\infty\} + \delta\mathbb{B}_p,$$

and so

$$\begin{aligned}u \in &\bigcap_{\delta>0} \left(\operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}} A)_\infty\} + \delta\mathbb{B}_p\right) \\ &= \operatorname{cl}\left(\operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}} A)_\infty\}\right),\end{aligned}$$

which yields (26).

Step 2. Now we prove the reverse inclusion in (26). If

$$u \in \bigcap_{\varepsilon>0} \operatorname{cl}\left(\operatorname{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} + \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}} A)_\infty\}\right)$$

(the set above is assumed non-empty, otherwise we are done), associated with each $\varepsilon > 0$ there will exist two sequences

$$(u_k^\varepsilon)_{k=1}^\infty \subset \mathrm{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\} \quad \text{and} \quad (v_k^\varepsilon)_{k=1}^\infty \subset \mathbb{R}^p \tag{27}$$

such that $(v_k^\varepsilon, \langle v_k^\varepsilon, z\rangle) \in (\overline{\mathrm{co}}A)_\infty$, for $k = 1, 2, \cdots$, and

$$u = \lim_{k\to\infty}(u_k^\varepsilon + v_k^\varepsilon).$$

Fix $k > 0$, and take $m \in \mathbb{N}$, $\lambda_1, \lambda_2, \cdots, \lambda_m \in]0, 1[$ with $\sum_{i=1}^m \lambda_i = 1$, and $t_1, t_2, \cdots, t_m \in T_\varepsilon(z)$ such that

$$u_k^\varepsilon = \sum_{i=1}^m \lambda_i a_{t_i}.$$

Since $u_k^\varepsilon \in \mathrm{co}\{\cup_{t\in T_\varepsilon(z)}\{a_t\}\}$ one has

$$\langle u_k^\varepsilon, z\rangle - \sum_{i=1}^m \lambda_i b_{t_i} \geq f(z) - \varepsilon.$$

Moreover, if $x \in \mathrm{dom}\, f$, that is $(x, -1) \in \mathrm{dom}\,\sigma_A \subset \mathrm{cl}(\mathrm{dom}\,\sigma_A) = [(\overline{\mathrm{co}}A)_\infty]^\circ$, one also has

$$\langle v_k^\varepsilon, x - z\rangle = \langle (v_k^\varepsilon, \langle v_k^\varepsilon, z\rangle), (x, -1)\rangle \leq 0,$$

and therefore

$$\begin{aligned} f(x) &\geq \langle u_k^\varepsilon, x\rangle - \sum_{i=1}^m \lambda_i b_{t_i} \\ &\geq \langle u_k^\varepsilon + v_k^\varepsilon, x\rangle - \langle v_k^\varepsilon, z\rangle - \sum_{i=1}^m \lambda_i b_{t_i} \\ &= \langle u_k^\varepsilon + v_k^\varepsilon, x - z\rangle + \langle u_k^\varepsilon, z\rangle - \sum_{i=1}^m \lambda_i b_{t_i} \\ &\geq \langle u_k^\varepsilon + v_k^\varepsilon, x - z\rangle + f(z) - \varepsilon. \end{aligned}$$

Taking limits for $k \to \infty$ one gets

$$f(x) \geq f(z) + \langle u, x - z\rangle - \varepsilon,$$

and $u \in \partial_\varepsilon f(z)$, for every $\varepsilon > 0$. Hence $u \in \partial f(z)$.

Step 3. If $z \in \mathrm{rint}(\mathrm{dom}\, f)$, it can be easily checked that $(z, -1) \in \mathrm{rint}(\mathrm{dom}\,\sigma_A)$ and we get, as in Step 3 in the proof of Proposition 2.1,

$$(\overline{\mathrm{co}}\{\cup_{t\in T}\{(a_t, b_t)\}\})_\infty \cap \{(z, -1)\}^\perp = (\mathrm{dom}\,\sigma_A)^\perp = \mathrm{lin}(\overline{\mathrm{co}}\{\cup_{t\in T}\{(a_t, b_t)\}\}).$$

Step 4. We have now $(\overline{\mathrm{co}}\{\cup_{t\in T}\{(a_t, b_t)\}\})_\infty \cap \{(z, -1)\}^\perp = \{0_n\}$. □

Remark 3.2. Let us observe that, actually, (21) is a consequence of (23). Given the function $f_E : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$

$$f_E(x) := f(P_E(x)) = \sup\{\langle P_E(a_t), x\rangle - b_t : t \in T\},$$

where E is the subspace parallel to $\operatorname{aff}(\operatorname{dom} f)$ and P_E is the orthogonal projection onto E, we have

$$\operatorname{int}(\operatorname{dom} f_E) = \operatorname{rint}(\operatorname{dom} f) + E^{\perp},$$

and

$$\partial f(z) = \partial f_E(z) + E^{\perp}, \text{ for every } z \in \operatorname{dom} f,$$

according to [1, Proposition 3.2.3] (see also [4]). Hence, it suffices to apply (23) to f_E in order to express $\partial f(z)$ at any given $z \in \operatorname{rint}(\operatorname{dom} f)$.

We close this section by giving the following example in order to illustrate Proposition 3.1.

Example 3.3. Consider the function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$f(x) := \sup\{a(t)x - b(t) : t > 0\},$$

where $a(\cdot), b(\cdot) :]0, +\infty[\to \mathbb{R}_+$ with $f(0) := \sup\{-b(t) : t > 0\} = 0$. Applying Proposition 3.1 to f, with $T \equiv]0, +\infty[$ and $(a_t, b_t) \equiv (a(t), b(t))$, for $t \in T$, we get

$$\partial f(0) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\operatorname{co}\{a(t) : b(t) \le \varepsilon\} + \{\lambda \in \mathbb{R} : (\lambda, 0) \in \left(\overline{\operatorname{co}}\{\cup_{t>0}\{(a(t), b(t))\}\}\right)_{\infty}\}\right). \quad (28)$$

In relation to the behavior of the functions $a(\cdot), b(\cdot)$, two cases may occur:

1) $(1, 0) \in \left(\overline{\operatorname{co}}\{\cup_{t>0}\{(a(t), b(t))\}\}\right)_{\infty}$. This leads us to

$$\mathbb{R}_+ \times \{0\} \subset \left(\overline{\operatorname{co}}\{\cup_{t>0}\{(a(t), b(t))\}\}\right)_{\infty} \subset \mathbb{R}_+ \times \mathbb{R}_+,$$

and so, by (28) we obtain

$$\partial f(0) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\operatorname{co}\{a(t) : b(t) \le \varepsilon\} + \mathbb{R}_+\right) = \bigcap_{\varepsilon > 0} \left[\inf_{b(t) \le \varepsilon} a(t), +\infty\right[.$$

2) $(1, 0) \notin \left(\overline{\operatorname{co}}\{\cup_{t>0}\{(a(t), b(t))\}\}\right)_{\infty}$. In this case

$$\{\lambda \in \mathbb{R} : (\lambda, 0) \in \left(\overline{\operatorname{co}}\{\cup_{t>0}\{(a(t), b(t))\}\}\right)_{\infty}\} = \{0\},$$

and now (28) entails

$$\partial f(0) = \bigcap_{\varepsilon > 0} \operatorname{cl}\left(\operatorname{co}\{a(t) : b(t) \le \varepsilon\}\right).$$

In this way we have given the exact formula for the subdifferential of f at 0. We shall discuss a pair of particular cases:

i) $a(t) \equiv t$ and $b(t) \equiv 1/t$. We have $\operatorname{dom} f =]-\infty, 0]$ so that $0 \in \operatorname{bd}(\operatorname{dom} f)$, and the formula in 1) entails

$$\partial f(0) = \bigcap_{\varepsilon>0} [1/\varepsilon, +\infty[\ = \emptyset;$$

i.e. f has no subgradient at 0. In fact, we have in this case $f(x) = -2\sqrt{-x}$, for $x \leq 0$.

ii) $a(t) \equiv t\,|\sin(t)|$ and $b(t) \equiv 1/t$. Again $\operatorname{dom} f =]-\infty, 0]$ and the formula in 1) also applies because

$$(1, 0) = \lim_{k\to\infty} \frac{2}{(2k+1)\pi} \left(\frac{(2k+1)\pi}{2} \left|\sin \frac{(2k+1)\pi}{2}\right|, \ \frac{2}{(2k+1)\pi} \right).$$

Thus

$$\partial f(0) = \bigcap_{\varepsilon>0} [0, +\infty[= \mathbb{R}_+.$$

4. Subdifferential of the supremum of convex functions

In this section we study the supremum function

$$f(x) := \sup\{f_t(x) : t \in T\}, \tag{29}$$

where $f_t : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$, $t \in T$, are convex proper functions. Assuming that f is finite somewhere, f is also a convex proper function whose subdifferential set is completely characterized in Propositions 4.1 and 4.3 below.

Proposition 4.1. *Given a non-empty family of lsc convex proper functions $\{f_t : t \in T\}$, consider the supremum function f defined in* (29). *Then, for every $z \in \operatorname{dom} f$ (assumed non-empty),*

$$\partial f(z) = \bigcap_{\varepsilon>0} \operatorname{cl}\left(\operatorname{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} + \mathrm{E}(z)\right), \tag{30}$$

where

$$T_\varepsilon(z) := \{t \in T : f_t(z) \geq f(z) - \varepsilon\}, \tag{31}$$

and

$$\begin{aligned}
\mathrm{E}(z) &= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}}\{\cup_{t\in T} \operatorname{gph} f_t^*\})_\infty\} && (32)\\
&= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\operatorname{co}}\{\cup_{t\in T} \operatorname{epi} f_t^*\})_\infty\} && (33)\\
&= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\operatorname{epi} f^*)_\infty\} && (34)\\
&= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in \operatorname{epi}(\sigma_{\operatorname{dom} f})\} && (35)\\
&= \mathrm{N}_{\operatorname{dom} f}(z). && (36)
\end{aligned}$$

Proof. First we verify that all the sets involved in the expressions from (32) to (36) coincide. In fact, according to [9, Theorem 16.5],

$$f^* = \operatorname{cl}\left(\operatorname{co}\{f_t^*, t \in T\}\right),$$

where $\operatorname{co}\{f_t^*, t \in T\}$ is the function defined as follows:

$$\left(\operatorname{co}\{f_t^*, t \in T\}\right)(x) := \inf\left(\mu : (x, \mu) \in \operatorname{co}\{\cup_{t\in T} \operatorname{epi} f_t^*\}\right),$$

which entails

$$\operatorname{epi} f^* = \overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{epi} f_t^*\},$$

and the equality in the expressions in (33) and (34). Moreover, [9, Theorem 13.3] establishes

$$(\operatorname{epi} f^*)_\infty = \operatorname{epi}(\sigma_{\operatorname{dom} f}),$$

hence the sets in (34) and (35) are the same, and elementary considerations provide the equality of the sets involved in (35) and (36). Finally, let us prove the equality between the sets appearing in (32), say $E_1(z)$, and (33), say $E_2(z)$.

Since, for $t \in T$,

$$\operatorname{epi} f_t^* = (\operatorname{gph} f_t^*) + \mathbb{R}_+(0_p, 1),$$

we have

$$\begin{aligned}
(\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{epi} f_t^*\})_\infty &= (\overline{\operatorname{co}} \{\cup_{t\in T} ((\operatorname{gph} f_t^*) + \mathbb{R}_+(0_p, 1))\})_\infty \\
&= (\operatorname{cl} \{\operatorname{co} (\cup_{t\in T} \operatorname{gph} f_t^*) + \mathbb{R}_+(0_p, 1)\})_\infty \\
&= (\operatorname{cl} \{\overline{\operatorname{co}} (\cup_{t\in T} \operatorname{gph} f_t^*) + \mathbb{R}_+(0_p, 1)\})_\infty.
\end{aligned}$$

Moreover, because

$$\overline{\operatorname{co}} (\cup_{t\in T} \operatorname{gph} f_t^*) \subset \overline{\operatorname{co}} (\cup_{t\in T} \operatorname{epi} f_t^*) = \operatorname{epi} f^*,$$

and f^* is proper, $(0_p, -1)$ cannot be a direction of recession of $\overline{\operatorname{co}} (\cup_{t\in T} \operatorname{gph} f_t^*)$. Consequently, [9, Corollary 9.1.2] yields

$$(\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{epi} f_t^*\})_\infty = (\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{gph} f_t^*\})_\infty + \mathbb{R}_+(0_p, 1). \tag{37}$$

Now let $v \in E_2(z)$, so that $(v, \langle v, z\rangle) \in (\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{epi} f_t^*\})_\infty$. Taking into account (37), let $(x, r) \in (\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{gph} f_t^*\})_\infty$ and $\lambda \geq 0$ be such that

$$(v, \langle v, z\rangle) = (x, r + \lambda).$$

Thus $v = x$. Multiplying by $(z, -1)$, and thanks to the equality

$$\operatorname{cl} \left(\mathbb{R}^+(\operatorname{dom} f \times \{-1\})\right) = [(\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{epi} f_t^*\})_\infty]^\circ,$$

one has

$$(z, -1) \in [(\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{epi} f_t^*\})_\infty]^\circ \subset [(\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{gph} f_t^*\})_\infty]^\circ,$$

and so,

$$0 \geq \langle (x, r), (z, -1)\rangle \equiv \langle (v, \langle v, z\rangle - \lambda), (z, -1)\rangle = \lambda.$$

This implies $\lambda = 0$, and so $r = \langle v, z\rangle$; i.e. $(v, \langle v, z\rangle) \in (\overline{\operatorname{co}} \{\cup_{t\in T} \operatorname{gph} f_t^*\})_\infty$ and $E_2(z) \subset E_1(z)$. Since the reverse inclusion is obvious, we get the conclusion.

Now we prove (30). Since all the involved functions are proper and lsc, for $x \in \mathbb{R}^p$,

$$f(x) = \sup\{\langle y, x\rangle - f_t^*(y) : y \in \operatorname{dom} f_t^*,\ t \in T\}. \tag{38}$$

For $z \in \operatorname{dom} f$ fixed, Proposition 3.1 applied to (38) allows us to write

$$\partial f(z) = \bigcap_{\varepsilon>0} \operatorname{cl}(\operatorname{co} \{y \in \operatorname{dom} f_t^* : \langle y, z\rangle - f_t^*(y) \geq f(z) - \varepsilon,\ t \in T\} + \mathrm{E}(z)).$$

Observe that, for every $y \in \mathbb{R}^p$ and $\varepsilon > 0$,

$$\{t \in T : \langle y, z\rangle - f_t^*(y) \geq f(z) - \varepsilon\} \subset \{t \in T : f_t(z) \geq f(z) - \varepsilon\} = T_\varepsilon(z),$$

and making use of (3), for $t \in T$,

$$\begin{aligned} &\{y \in \mathbb{R}^p : \langle y, z\rangle - f_t^*(y) \geq f(z) - \varepsilon\} \\ &\subset \{y \in \mathbb{R}^p : \langle y, z\rangle - f_t^*(y) \geq f_t(z) - \varepsilon\} = \partial_\varepsilon f_t(z). \end{aligned}$$

Thus we deduce

$$\partial f(z) \subset \bigcap_{\varepsilon>0} \mathrm{cl}(\mathrm{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} + \mathrm{E}(z)).$$

To finish the proof it is sufficient to prove

$$\begin{aligned} \partial f(z) \quad &\supset \quad \bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} + \mathrm{E}(z)\right) \\ &\equiv \quad \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left(\cup_{t\in T_\varepsilon(z)} (\partial_\varepsilon f_t(z) + \mathrm{E}(z))\right). \end{aligned} \tag{39}$$

For $x \in \mathrm{dom}\, f$ and $\varepsilon > 0$ fixed, we have for all $t \in T_\varepsilon(z)$, $u \in \partial_\varepsilon f_t(z)$, and $v \in \mathrm{E}(z) = \mathrm{N}_{\mathrm{dom}\, f}(z)$,

$$f(x) - f(z) \geq f_t(x) - f_t(z) - \varepsilon \geq \langle u, x - z\rangle - 2\varepsilon \quad \text{and} \quad \langle v, x - z\rangle \leq 0.$$

Hence

$$f(x) - f(z) \geq \langle u, x - z\rangle + \langle v, x - z\rangle - 2\varepsilon = \langle u + v, x - z\rangle - 2\varepsilon,$$

and this entails $u + v \in \partial_{2\varepsilon} f(z)$, which leads us to

$$\overline{\mathrm{co}}\left(\cup_{t\in T_\varepsilon(z)} (\partial_\varepsilon f_t(z) + \mathrm{E}(z))\right) \subset \overline{\mathrm{co}}(\partial_{2\varepsilon} f(z)) \equiv \partial_{2\varepsilon} f(z),$$

and so (39) holds, by passing to the intersection over $\varepsilon > 0$. The conclusion is then established. □

Remark 4.2. As shown in Example 2.3 and Remark 2.4, the closure operation in (30) cannot be omitted in general.

Next we extend Proposition 4.1 to general convex proper functions, not necessarily lsc.

Proposition 4.3. *Given a non-empty family of convex proper functions* $\{f_t : t \in T\}$, *we assume that the supremum function* f *is finite at least at a point common to every* $\mathrm{ri}\,(\mathrm{dom}\, f_t)$. *Then for every* $z \in \mathrm{dom}\, f$ *we have*

$$\partial f(z) = \bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} + \mathrm{E}(z)\right), \tag{40}$$

where $T_\varepsilon(z)$ *is the same as in* (31) *and* $\mathrm{E}(z)$ *is given by any of the expressions in* (32)-(36).

Proof. If $\partial f(z) = \emptyset$ then, from the general inclusion (see the proof of Proposition 4.1)

$$\bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_\varepsilon(z)}\partial_\varepsilon f_t(z)\right\} + \mathrm{E}(z)\right) \subset \partial f(z),$$

follows that (40) trivially holds in this case. Then, in the rest of the proof we suppose f subdifferentiable at $z \in \mathbb{R}^p$, which in particular implies $z \in \mathrm{dom}\, f$ and, accordingly to [9, Corollary 23.5.2],

$$\mathrm{cl}\, f(z) = f(z) \quad \text{ and } \quad \partial(\mathrm{cl}\, f)(z) = \partial f(z). \tag{41}$$

The proof consists of applying Proposition 4.1 to the new family of lower semicontinuous convex proper functions given by

$$\tilde{f}_t := \mathrm{cl}\, f_t$$

and the associated supremum function defined by

$$\tilde{f} := \sup_{t\in T} \tilde{f}_t.$$

But the current assumption on the relative interior sets implies ([9, Theorem 9.4])

$$\tilde{f} = \sup_{t\in T} \mathrm{cl}\, f_t = \mathrm{cl}\left(\sup_{t\in T} f_t\right) = \mathrm{cl}\, f.$$

So, taking into account (41), Proposition 4.1 yields

$$\partial f(z) = \partial(\mathrm{cl}\, f)(z) = \partial\tilde{f}(z) = \bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in\widetilde{T}_\varepsilon(z)}\partial_\varepsilon\tilde{f}_t(z)\right\} + \widetilde{\mathrm{E}}(z)\right), \tag{42}$$

where $\widetilde{T}_\varepsilon(z) := \{t \in T : \tilde{f}_t(z) \geq \tilde{f}(z) - \varepsilon\}$. Therefore

$$\widetilde{T}_\varepsilon(z) \subset \{t \in T : f_t(z) \geq \mathrm{cl}\, f(z) - \varepsilon\} = \{t \in T : f_t(z) \geq f(z) - \varepsilon\} = T_\varepsilon(z), \tag{43}$$

and

$$\begin{aligned}
\widetilde{\mathrm{E}}(z) &= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\mathrm{co}}\,\{\cup_{t\in T}\,\mathrm{gph}\,(\mathrm{cl}\, f_t)^*\})_\infty\} \\
&= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\overline{\mathrm{co}}\,\{\cup_{t\in T}\,\mathrm{epi}\,(\mathrm{cl}\, f_t)^*\})_\infty\} \\
&= \{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in (\mathrm{epi}\,(\mathrm{cl}\, f)^*)_\infty\} \\
&= \left\{v \in \mathbb{R}^p : (v, \langle v, z\rangle) \in \mathrm{epi}(\sigma_{\mathrm{dom}(\mathrm{cl}\, f)})\right\} \\
&= \mathrm{N}_{\mathrm{dom}(\mathrm{cl}\, f)}(z).
\end{aligned}$$

Hence $\widetilde{\mathrm{E}}(z)$ is nothing else but the set $\mathrm{E}(z)$ already given in Proposition 4.1 by the expressions (32)-(36).

On an other hand, if $u \in \partial_\varepsilon\tilde{f}_t(z)$, with $\varepsilon > 0$ and $t \in \widetilde{T}_\varepsilon(z)$, then for all $y \in \mathbb{R}^p$ we have

$$\begin{aligned}
f_t(y) \geq \mathrm{cl}\, f_t(y) &\geq \mathrm{cl}\, f_t(z) + \langle u, y - z\rangle - \varepsilon \\
&\geq \tilde{f}(z) + \langle u, y - z\rangle - 2\varepsilon \\
&= f(z) + \langle u, z - y\rangle - 2\varepsilon,
\end{aligned}$$

where we used $\tilde{f}(z) = f(z)$ and $f_t \geq \mathrm{cl}\, f_t$. Then $u \in \partial_{2\varepsilon} f_t(z)$ and so we can write

$$\mathrm{co}\left\{\cup_{t\in\widetilde{T}_\varepsilon(z)}\partial_\varepsilon \tilde{f}_t(z)\right\} \subset \mathrm{co}\left\{\cup_{t\in\widetilde{T}_\varepsilon(z)}\partial_{2\varepsilon} f_t(z)\right\} \subset \mathrm{co}\left\{\cup_{t\in T_{2\varepsilon}(z)}\partial_{2\varepsilon} f_t(z)\right\}.$$

At this time (42) leads us to

$$\partial f(z) \subset \bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_{2\varepsilon}(z)}\partial_{2\varepsilon} f_t(z)\right\} + \mathrm{E}(z)\right) = \bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_{\varepsilon}(z)}\partial_{\varepsilon} f_t(z)\right\} + \mathrm{E}(z)\right).$$

Since the reverse of the last inclusion always holds (see once again the proof of Proposition 4.1), we then obtain (40). □

The following corollary, originally established by M. Valadier [14, Théorème A], is a consequence of Proposition 4.3.

Corollary 4.4. *Given a non-empty family of convex functions $\{f_t : t \in T\}$, consider the supremum function f. Then, for every $z \in \mathrm{int}\,(\mathrm{dom}\, f)$ we have*

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\left\{\cup_{t\in T_\varepsilon(z)}\partial_\varepsilon f_t(z)\right\}, \tag{44}$$

where $T_\varepsilon(z)$ is defined in (31).

Proof. The current assumption on z entails the existence of $\delta > 0$ such that

$$z + \delta\mathbb{B} \subset \mathrm{int}\,(\mathrm{dom}\, f) \subset \mathrm{int}\,(\mathrm{dom}\, f_t) \subset \mathrm{ri}\,(\mathrm{dom}\, f_t) \quad \text{for all } t \in T,$$

which in particular implies, accordingly to [9, Theorem 7.2], that the functions f_t, $t \in T$, are proper. Obviously the assumption of Proposition 4.3 holds and so the aimed conclusion follows because $\mathrm{E}(z) = N_{\mathrm{dom}\, f}(z) = \{0_p\}$ in this case. □

5. Homogeneous formula for the subdifferential set

In this section we are interested in deriving formulae for the subdifferential set of the supremum function f, defined in (29), where the set $\mathrm{E}(z)$ in (30) can be removed. Such formulae are called homogeneous because they only appeal to the $\varepsilon-$subdifferentials of the nominal functions f_t.

Our main motivation in this section is to extend the result of Brøndsted [2] (see Corollary 5.5 below) to cover the case of infinitely many functions. We need some tools gathered in the following lemma.

Lemma 5.1. *Given a non-empty family of convex proper functions $\{f_t : t \in T\}$, the following statements hold:*

(*i*) *Assuming*

$$\cap_{t\in T}\mathbb{R}^+(\mathrm{dom}\, f_t - \{z\}) = \mathbb{R}^+ \cap_{t\in T} (\mathrm{dom}\, f_t - \{z\}),$$

we have

$$\mathrm{N}_{\cap_{t\in T} \mathrm{dom}\, f_t}(z) \subset \cap_{\varepsilon>0} \left(\overline{\mathrm{co}}\left\{\cup_{t\in T}\partial_\varepsilon f_t(z)\right\}\right)_\infty.$$

(*ii*) *If*

$$\mathrm{cl}\,(\mathrm{dom}\, f) = \cap_{t\in T}\,\mathrm{cl}\,(\mathrm{dom}\, f_t)\,,$$

then

$$(\mathrm{epi}\, f^*)_\infty = \overline{\mathrm{co}}\,\{\cup_{t\in T}\,(\mathrm{epi}\, f_t^*)_\infty\}\,.$$

Proof. (*i*) Fix $\varepsilon > 0$. Because both $\mathrm{N}_{\cap_{t\in T}\mathrm{dom}\, f_t}(z)$ and $(\overline{\mathrm{co}}\,\{\cup_{t\in T}\partial_\varepsilon f_t(z)\})_\infty$ are closed convex cones, it is sufficient to prove that

$$[(\overline{\mathrm{co}}\,\{\cup_{t\in T}\partial_\varepsilon f_t(z)\})_\infty]^\circ \subset \left[\mathrm{N}_{\cap_{t\in T}\mathrm{dom}\, f_t}(z)\right]^\circ.$$

We have

$$\begin{aligned}
[(\overline{\mathrm{co}}\,\{\cup_{t\in T}\partial_\varepsilon f_t(z)\})_\infty]^\circ &= \mathrm{cl}\left(\mathrm{dom}\,\sigma_{\overline{\mathrm{co}}\{\cup_{t\in T}\partial_\varepsilon f_t(z)\}}\right)\\
&= \mathrm{cl}(\mathrm{dom}(\sup_{t\in T}\sigma_{\partial_\varepsilon f_t(z)})) \qquad (45)\\
&\subset \mathrm{cl}\left(\cap_{t\in T}\,\mathrm{dom}\,\sigma_{\partial_\varepsilon f_t(z)}\right).
\end{aligned}$$

Now we shall show that

$$\cap_{t\in T}\,\mathrm{dom}\,\sigma_{\partial_\varepsilon f_t(z)} \subset \left[\mathrm{N}_{\cap_{t\in T}\mathrm{dom}\, f_t}(z)\right]^\circ. \qquad (46)$$

Indeed, for $x \in \cap_{t\in T}\,\mathrm{dom}\,\sigma_{\partial_\varepsilon f_t(z)}$, and for each $t\in T$, we have [5, XI Theorem 2.1.1]

$$(f_t)'_\varepsilon(z,x) = \inf_{s>0}\frac{f_t(z+sx) - f_t(z) + \varepsilon}{s} = \sigma_{\partial_\varepsilon f_t(z)}(x) < +\infty.$$

Hence, there exists $s_t > 0$ satisfying

$$\frac{f_t(z+s_t x) - f_t(z) + \varepsilon}{s_t} < +\infty.$$

In other words, $z + s_t x \in \mathrm{dom}\, f_t$ and so

$$\begin{aligned}
x \in \cap_{t\in T}\mathbb{R}^+(\mathrm{dom}\, f_t - \{z\}) &= \mathbb{R}^+ \cap_{t\in T}\,(\mathrm{dom}\, f_t - \{z\})\\
&= \mathbb{R}^+\,(\cap_{t\in T}\,\mathrm{dom}\, f_t - \{z\})\\
&\subset \mathrm{cl}\left(\mathbb{R}^+\,(\cap_{t\in T}\,\mathrm{dom}\, f_t - \{z\})\right)\\
&= \left[\mathrm{N}_{\cap_{t\in T}\mathrm{dom}\, f_t}(z)\right]^\circ.
\end{aligned}$$

Consequently, (46) holds, and then (45) entails

$$[\overline{\mathrm{co}}\,\{\cup_{t\in T}\partial_\varepsilon f_t(z)\}_\infty]^\circ \subset \left[\mathrm{N}_{\cap_{t\in T}\mathrm{dom}\, f_t}(z)\right]^\circ.$$

(*ii*) We write [9, Theorems 13.3 and 16.5]

$$\begin{aligned}
\overline{\mathrm{co}}\,\{\cup_{t\in T}\,(\mathrm{epi}\, f_t^*)_\infty\} &= \overline{\mathrm{co}}\,\{\cup_{t\in T}\,\mathrm{epi}\,(\sigma_{\mathrm{dom}\, f_t})\}\\
&= \overline{\mathrm{co}}\left\{\cup_{t\in T}\,\mathrm{epi}\left(\mathrm{I}^*_{\mathrm{dom}\, f_t}\right)\right\}\\
&= \mathrm{epi}\left(\sup_{t\in T}\mathrm{I}_{\mathrm{cl}(\mathrm{dom}\, f_t)}\right)^*\\
&= \mathrm{epi}\left(\mathrm{I}^*_{\cap_{t\in T}\mathrm{cl}(\mathrm{dom}\, f_t)}\right)\\
&= \mathrm{epi}\left(\sigma_{\mathrm{cl}(\mathrm{dom}\, f)}\right)\\
&= (\mathrm{epi}\, f^*)_\infty\,.
\end{aligned}$$

□

Proposition 5.2. *Given a non-empty family of lsc convex proper functions* $\{f_t : t \in T\}$, *consider the supremum function* f *defined in* (29). *Then, for every* $z \in \operatorname{dom} f$ *for which the following conditions are satisfied*

(*i*) $f_t(z) = f(z)$, *for all* $t \in T$,

(*ii*) $\operatorname{dom} f = \bigcap_{t\in T} \operatorname{dom} f_t$,

(*iii*) $\bigcap_{t\in T} \mathbb{R}^+(\operatorname{dom} f_t - \{z\}) = \mathbb{R}^+ \bigcap_{t\in T} (\operatorname{dom} f_t - \{z\})$,

we have

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\text{co}}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\}.$$

Proof. According to condition (*ii*) and Lemma 5.1(*i*) (because of condition (*iii*)), we have

$$N_{\operatorname{dom} f}(z) = \mathrm{N}_{\cap_{t\in T} \operatorname{dom} f_t}(z) \subset \bigcap_{\varepsilon>0} \left(\overline{\text{co}}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\}\right)_\infty.$$

Thus, observing that $T = T_\varepsilon(z)$, for every $\varepsilon > 0$, because of condition (*i*), and making use of Proposition 4.1, we obtain

$$\begin{aligned} \partial f(z) &= \bigcap_{\varepsilon>0} \operatorname{cl}\left(\text{co}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\} + N_{\operatorname{dom} f}(z)\right) \\ &\subset \bigcap_{\varepsilon>0} \operatorname{cl}\left(\overline{\text{co}}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\} + \left(\overline{\text{co}}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\}\right)_\infty\right) \\ &= \bigcap_{\varepsilon>0} \overline{\text{co}}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\}. \end{aligned}$$

The conclusion follows since the reverse inclusion always holds. □

The fact that the subdifferential of a convex function is a local notion motivates the following proposition.

Proposition 5.3. *Consider a non-empty family of proper lsc convex functions* $\{f_t : t \in T\}$, *the associated supremum function* f, *and* $z \in \operatorname{dom} f$. *Assume the existence of* $\delta > 0$ *such that the following statements hold*

(*i*) $f_t(z) = f(z)$, *for all* $t \in T$,

(*ii*) $(\operatorname{dom} f) \cap (z + \delta\mathbb{B}) = (\cap_{t\in T} \operatorname{dom} f_t) \cap (z + \delta\mathbb{B})$,

(*iii*) $\cap_{t\in T}\mathbb{R}^+((\operatorname{dom} f_t) \cap (z + \delta\mathbb{B}) - \{z\}) = \mathbb{R}^+ \cap_{t\in T} ((\operatorname{dom} f_t) \cap (z + \delta\mathbb{B}) - \{z\})$.

Then,

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\text{co}}\,\{\cup_{t\in T} \partial_\varepsilon f_t(z)\}.$$

Remark (before the proof). When z is a continuity point of f, so that assumptions (*ii*) and (*iii*) are trivially satisfied, this proposition is also a consequence of Valadier's formula (Corollary 4.4), observing that $T = T_\varepsilon(z)$, for every $\varepsilon > 0$, by (*i*).

Proof. Set, for $t \in T$,

$$\tilde{f}_t := f_t + \mathrm{I}_{z+\delta\mathbb{B}}$$

and

$$\tilde{f} := \sup_{t\in T} \tilde{f}_t = f + \mathrm{I}_{z+\delta\mathbb{B}}.$$

Then $\tilde{f}_t$, $t \in T$, and $\tilde{f}$ are also proper lsc convex functions, and by *(ii)* they satisfy

$$\begin{aligned} \operatorname{dom} \tilde{f} &= (\operatorname{dom} f) \cap (z + \delta\mathbb{B}) \\ &= (z + \delta\mathbb{B}) \cap (\cap_{t\in T} \operatorname{dom} f_t) \\ &= \cap_{t\in T} \operatorname{dom} \tilde{f}_t, \end{aligned} \tag{47}$$

and by *(iii)*

$$\begin{aligned} \cap_{t\in T}\mathbb{R}^+(\operatorname{dom} \tilde{f}_t - \{z\}) &= \cap_{t\in T} \mathbb{R}^+((\operatorname{dom} f_t) \cap (z + \delta\mathbb{B}) - \{z\}) \\ &= \mathbb{R}^+ \cap_{t\in T} ((\operatorname{dom} f_t) \cap (z + \delta\mathbb{B}) - \{z\}) \\ &= \mathbb{R}^+ \cap_{t\in T} \left(\operatorname{dom} \tilde{f}_t - \{z\}\right). \end{aligned} \tag{48}$$

Taking $0 < \varepsilon \leq \min\{\delta, 1\}$, and because

$$\operatorname{rint}(\operatorname{dom} f_t) \cap \operatorname{rint}(z + \delta\mathbb{B}) \neq \emptyset, \quad \text{for all } t \in T,$$

Theorem XI 3.1.1 in [5] applies and allows us to conclude the existence of $\varepsilon_1 \geq 0$ and $\varepsilon_2 \geq 0$ such that $\varepsilon_1 + \varepsilon_2 \leq \varepsilon^2$ and

$$\partial_{\varepsilon^2} \tilde{f}_t(z) \subset \partial_{\varepsilon_1} f_t(z) + \partial_{\varepsilon_2} \mathrm{I}_{z+\delta\mathbb{B}}(z) \subset \partial_{\varepsilon^2} f_t(z) + \partial_{\varepsilon^2} \mathrm{I}_{z+\delta\mathbb{B}}(z).$$

Moreover

$$\begin{aligned} u &\in \partial_{\varepsilon^2}\mathrm{I}_{z+\delta\mathbb{B}}(z) \Rightarrow \langle u, y - z\rangle \leq \varepsilon^2, \quad \text{for all } y \in z + \delta\mathbb{B} \\ &\Leftrightarrow \langle u, y - z\rangle \leq \varepsilon^2, \quad \text{for } y = z + \delta v \text{ and all } v \in \mathbb{B} \\ &\Leftrightarrow \delta \langle u, v\rangle \leq \varepsilon^2 \leq \varepsilon\delta, \quad \text{for all } v \in \mathbb{B} \\ &\Leftrightarrow \langle u, v\rangle \leq \varepsilon, \quad \text{for all } v \in \mathbb{B} \\ &\Rightarrow \|u\| \leq \varepsilon, \end{aligned}$$

and since $\varepsilon \leq 1$,

$$\partial_{\varepsilon^2} \tilde{f}_t(z) \subset \partial_{\varepsilon^2} f_t(z) + \partial_{\varepsilon^2}\mathrm{I}_{z+\delta\mathbb{B}}(z) \subset \partial_\varepsilon f_t(z) + \varepsilon\mathbb{B}.$$

Because we have $\partial f(z) = \partial\tilde{f}(z)$, $\tilde{f}_t(z) = \tilde{f}(z)$, for all $t \in T$, [47] and [48], applying Proposition 5.2 we obtain

$$\begin{aligned} \partial f(z) &= \partial\tilde{f}(z) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}} \left\{\cup_{t\in T}\partial_{\varepsilon^2} \tilde{f}_t(z)\right\} \\ &= \bigcap_{\min\{\delta,1\}\geq\varepsilon>0} \overline{\mathrm{co}} \left\{\cup_{t\in T}\partial_{\varepsilon^2} \tilde{f}_t(z)\right\} \\ &\subset \bigcap_{\min\{\delta,1\}\geq\varepsilon>0} \overline{\mathrm{co}} \left\{\cup_{t\in T}(\partial_\varepsilon f_t(z) + \varepsilon\mathbb{B})\right\} \\ &= \bigcap_{\varepsilon>0} \overline{\mathrm{co}} \left\{\cup_{t\in T}\partial_\varepsilon f_t(z)\right\}. \end{aligned}$$

The conclusion follows because the reverse inclusion also holds. □

Corollary 5.4. *Consider a non-empty family of lsc convex proper functions* $\{f_t : t \in T\}$ *and the associated supremum function* f*. Let* $z \in \operatorname{dom} f$ *and assume the existence of a non-empty finite subset* $S \subset T$ *verifying the following properties:*

(*i*) $f_t(z) = f(z)$, *for all* $t \in T$,
(*ii*) $\operatorname{dom} f = \cap_{t\in T} \operatorname{dom} f_t$,
(*iii*) *For every* $t \in T$, *there exists* $s_t \in S$ *such that*

$$\operatorname{dom} f_{s_t} \subset \operatorname{dom} f_t.$$

Then, we have

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \cup_{t\in T} \partial_\varepsilon f_t(z) \right\}.$$

Proof. According to Proposition 5.2 it is enough to show that

$$\cap_{t\in T}\mathbb{R}^+(\operatorname{dom} f_t - \{z\}) = \mathbb{R}^+ \cap_{t\in T} (\operatorname{dom} f_t - \{z\}). \tag{49}$$

Indeed, the inclusion $\mathbb{R}^+ \cap_{t\in T} (\operatorname{dom} f_t - \{z\}) \subset \cap_{t\in T}\mathbb{R}^+(\operatorname{dom} f_t - \{z\})$ always holds. To prove the reverse inclusion, take

$$x \in \cap_{t\in T}\mathbb{R}^+(\operatorname{dom} f_t - \{z\}).$$

If $x = 0$, then we are done. Otherwise, for each $t \in T$ there exist $\gamma_t > 0$ and $y_t \in \operatorname{dom} f_t - \{z\}$ such that

$$x = \gamma_t y_t.$$

Let $s \in S$ be such that $\gamma_s = \max\{\gamma_t : t \in S\}$ (this s exists because S is finite). Then we obtain, since $0_p \in \operatorname{dom} f_t - \{z\}$,

$$y_s = \frac{\gamma_t}{\gamma_s} y_t + (1 - \frac{\gamma_t}{\gamma_s})0_p \in \operatorname{dom} f_t - \{z\}, \text{ for every } t \in S,$$

that is,

$$\begin{aligned} x \in \gamma_s \cap_{t\in S} (\operatorname{dom} f_t - \{z\}) &\subset \mathbb{R}^+ \cap_{t\in S} (\operatorname{dom} f_t - \{z\}) \\ &= \mathbb{R}^+ \cap_{t\in T} (\operatorname{dom} f_t - \{z\}), \end{aligned}$$

and this implies (49). Hence Proposition 5.2 applies and leads us to the conclusion. □

In the case of finitely many functions Corollary 5.4 applies and yields the following result, originally due to Brøndsted [2].

Corollary 5.5. *Consider a non-empty finite family of proper lsc convex functions* $\{f_t : t \in T\}$ *and the associated supremum function* f*. Let* $z \in \operatorname{dom} f$ *and assume that*

$$f_t(z) = f(z), \quad \textit{for all } t \in T.$$

Then we have

$$\partial f(z) = \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \cup_{t\in T} \partial_\varepsilon f_t(z) \right\}.$$

We close this section with the following example which shows that, in general, the set $\mathrm{E}(z)$ in (30) cannot be removed.

Example 5.6. Let us consider the family of linear functions $f_t : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$, $t \in T := \{0\} \cup\ ]1, +\infty[$ given by

$$f_t(x, y) := \begin{cases} -y & \text{for } t = 0, \\ tx + \frac{y}{t-1} & \text{for } t > 1, \end{cases}$$

and the supremum function

$$f(x, y) := \max\{-y,\ \sup\{tx + y/(t-1) : t > 1\}\}$$

Observe that f is the support function of the set $\{(x, y) \in R^2 : x \geq 0, y \geq -1\}$. For $z := (-1, 0)$, we easily check that $f(z) = 0$, and

$$\partial f(z) = \{0\} \times [-1, +\infty[.$$

On the other hand we have, for all $\varepsilon < 1$,

$$T_\varepsilon(z) = \{t \in T : f_t(z) \geq -\varepsilon\} = \{0\},$$

and

$$\bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\}\right) = \bigcap_{\varepsilon\in]0,1[} \partial_\varepsilon f_0(z) = \partial f_0(z) = \{(0, -1)\}.$$

Thus

$$\bigcap_{\varepsilon>0} \mathrm{cl}\left(\mathrm{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\}\right) \subsetneqq \partial f(z).$$

6. Subdifferential set of the supremum function via exact subdifferentials

In this final section, and for comparative purposes, we make a short review of some classical formulae. Namely, applying our Propositions 4.1 and 4.3, and Lemma 6.2 below, we give alternative proofs of some well-known results expressing the subdifferential of the supremum function f defined in (29) by means of exact subdifferential sets of the nominal functions, f_t, $t \in T$. We begin with the first result called Valadier's formula (see, for instance, [5, Theorem VI.4.4.8]).

Proposition 6.1. *Consider a non-empty family of convex functions $\{f_t,\ t \in T\}$ and the associated supremum function f. Then, for every $z \in \mathrm{int}(\mathrm{dom}\, f)$, assumed to be non-empty, we have*

$$\partial f(z) = \cap_{\varepsilon>0} \overline{\mathrm{co}} \left\{\cup_{t\in T_\varepsilon(z),\ x\in z+\varepsilon\mathbb{B}} \partial f_t(x)\right\},$$

where $T_\varepsilon(z) := \{t \in T : f_t(z) \geq f(z) - \varepsilon\}$.

Proof. The proof of the inclusion "$\supset$" follows in the same way as in [5, Theorem VI.4.4.8], using the fact established in [5, Lemma VI.4.4.7] guarantying the existence of $L > 0$ such that for each $\varepsilon > 0$ one has

$$u \in \cup_{t\in T_\varepsilon(z),\ x\in z+\varepsilon\mathbb{B}} \partial f_t(x) \Longrightarrow f(y) \geq f(z) + \langle u, y - z\rangle - L\varepsilon \ \text{ for all } y \in \mathbb{R}^p.$$

Next we prove the other inclusion. By Brøndsted-Rockafellar's Theorem ([5, Theorem XI.4.2.1]) we have, for every $t \in T$ and $\varepsilon > 0$,

$$\partial_{\varepsilon^2} f_t(z) \subset \cup_{x\in z+\varepsilon\mathbb{B}} \partial f_t(x) + \varepsilon\mathbb{B},$$

so that Proposition 4.3 yields, observing that $E(z) = \{0_p\}$ by assumption,

$$\begin{aligned}
\partial f(z) = \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{\cup_{t\in T_{\varepsilon^2}(z)} \partial_{\varepsilon^2} f_t(z)\right\} \quad &\subset \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{\cup_{t\in T_{\varepsilon^2}(z), x\in z+\varepsilon\mathbb{B}} \partial f_t(x) + \varepsilon\mathbb{B}\right\} \\
&\subset \bigcap_{\varepsilon>0} \left(\overline{\text{co}} \left\{\cup_{t\in T_{\varepsilon^2}(z), x\in z+\varepsilon\mathbb{B}} \partial f_t(x)\right\} + \varepsilon\mathbb{B}\right) \\
&= \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{\cup_{t\in T_{\varepsilon^2}(z), x\in z+\varepsilon\mathbb{B}} \partial f_t(x)\right\} \\
&\subset \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{\cup_{t\in T_{\varepsilon}(z), x\in z+\varepsilon\mathbb{B}} \partial f_t(x)\right\},
\end{aligned}$$

where in the last inclusion above we used the evident fact that $T_{\varepsilon^2}(z) \subset T_\varepsilon(z)$ for every $\varepsilon > 0$ small enough. In this way the conclusion follows. □

Next, in addition to Proposition 6.1, we approach another case in which the subdifferential set is expressed by means exclusively of the subdifferential sets of the nominal functions f_t, $t \in T$. We shall need the following lemma.

Lemma 6.2. *Consider a non-empty family of convex functions $\{f_t,\, t \in T\}$ and the associated supremum function f, with T being a separated compact topological space and the mapping $t \to f_t(x)$ being upper semicontinuous for every $x \in \mathbb{R}^p$. Assume also that each function f_t, $t \in T$, is continuous at z, and let $\delta > 0$. Then, there exists $\eta > 0$ such that*

$$\cup_{t\in T_\eta(z)} \partial_\eta f_t(z) \subset \text{co} \left\{\cup_{t\in T(z)} \partial f_t(z)\right\} + \delta\mathbb{B},$$

where $T_\eta(z) := \{t \in T : f_t(z) \geq f(z) - \eta\}$ and $T(z) := \{t \in T : f(z) = f_t(z)\}$.

Proof. Standard arguments show, under the current continuity assumptions, that the set $T(z)$ is a non-empty compact subset of T.

Thanks to the compactness of T and the upper semicontinuity of the mappings $t \to f_t(x)$, for every $x \in \mathbb{R}^p$, we conclude that

$$\text{dom}\, f = \cap_{t\in T} \text{dom}\, f_t,$$

and

$$\mathbb{R}_+(\cap_{t\in T}(\text{dom}\, f_t - \{z\})) = \cap_{t\in T}\mathbb{R}_+(\text{dom}\, f_t - \{z\}).$$

Hence, since each f_t is continuous at z, we can write

$$\begin{aligned}
\mathbb{R}_+(\text{dom}\, f - \{z\}) &= \mathbb{R}_+((\cap_{t\in T} \text{dom}\, f_t) - \{z\}) \\
&= \mathbb{R}_+(\cap_{t\in T}(\text{dom}\, f_t - \{z\})) \\
&= \cap_{t\in T}\mathbb{R}_+(\text{dom}\, f_t - \{z\}) \\
&= \mathbb{R}^p,
\end{aligned}$$

and this entails $z \in \operatorname{int}(\operatorname{dom} f)$, i.e., there is some $\alpha > 0$ satisfying

$$z + \alpha\mathbb{B} \subset \operatorname{int}(\operatorname{dom} f) \subset \operatorname{int}(\operatorname{dom} f_t) \ \text{ for all } t \in T.$$

In particular, each f_t is finite and pointwise bounded on $z + \alpha\mathbb{B}$ by [9, Theorem 10.4], and so the family $\{f_t, t \in T\}$ is equi-Lipschitzian relative to $z + \frac{\alpha}{2}\mathbb{B}$ by [9, Theorem 10.6]. Letting $L > 0$ be one Lipschitz constant common to each f_t on $z + \frac{\alpha}{2}\mathbb{B}$, [5, Proposition XI.4.1.2] entails, for every $\varepsilon > 0$,

$$\partial_\varepsilon f_t(z) \subset \left(L + \frac{2\varepsilon}{\alpha}\right)\mathbb{B} \ \text{ for every } t \in T. \tag{50}$$

Now, letting $\delta > 0$ being fixed, the key point of the proof consists in showing that for each $t \in T(z)$ there exist a neighborhood of t, $V_t \subset T$, and a positive number ε_t such that

$$\partial_{\varepsilon_t} f_s(z) \subset \partial f_t(z) + \delta\mathbb{B} \ \text{ for all } s \in V_t \cap T_{\varepsilon_t}(z). \tag{51}$$

In fact, if this is not true then we will find $\bar{t} \in T(z)$, as well as w.l.o.g. a sequence (in general, a net) $t^r \in T_{1/r}(z)$, $r = 1, 2, \ldots,$ converging to $\bar{t}$, such that

$$\partial_{1/r} f_{t^r}(z) \not\subset \partial f_{\bar{t}}(z) + \delta\mathbb{B} \ \text{ for } r = 1, 2, \ldots; \tag{52}$$

that is, for each r, there is some

$$u_r \in \partial_{1/r} f_{t^r}(z) \setminus (\partial f_{\bar{t}}(z) + \delta\mathbb{B}).$$

In particular, and by definition, the vector u_r satisfies, recalling that $t^r \in T_{1/r}(z)$,

$$f_{t^r}(y) - f_{\bar{t}}(z) = f_{t^r}(y) - f(z) \geq f_{t^r}(y) - f_{t^r}(z) - 1/r \geq \langle u_r, y - z\rangle - 2/r, \ \text{ for all } y \in \mathbb{R}^p. \tag{53}$$

Further, since we have, for r large enough,

$$\partial_{1/r} f_t(z) \subset \left(L + \frac{2}{r\alpha}\right)\mathbb{B} \subset 2L\mathbb{B} \ \text{ for every } t \in T,$$

accordingly to (50), we may suppose w.l.o.g. that the sequence $\{u_r\}$ converges to some u. Moreover, by passing to the limit when r goes to ∞ in (53), with $y \in \mathbb{R}^p$ being fixed, we obtain

$$f_{\bar{t}}(y) - f_{\bar{t}}(z) \geq \limsup_{r\to\infty} f_{t^r}(y) - f(z) \geq \limsup_{r\to\infty}(\langle u_r, y - z\rangle - 2/r) = \langle u, y - z\rangle,$$

where, to establish the first inequality, we used the upper semi-continuity of the mapping $t \to f_t(y)$ for every $y \in \mathbb{R}^p$. In particular, this leads us to the contradiction $u \in \partial f_{\bar{t}}(z)$ because, at the same time, u is a cluster point of the sequence $(u^r)_{r=1}^\infty \subset \mathbb{R}^p \setminus (\partial f_{\bar{t}}(z) + \delta\mathbb{B})$ and, so, a fortiori we must have $u \notin \partial f_{\bar{t}}(z) + \frac{\delta}{2}\mathbb{B}$.

In this way we have shown that, for each $t \in T(z)$, there exist a neighborhood of t, $V_t \subset T$, and a positive number ε_t satisfying (51). Since that $T(z)$ is compact we then can find a finite number of indices $\{t_1, \ldots, t_k\} \subset T(z)$ such that $T(z) \subset \cup_{1\leq i\leq k} V_{t_i}$, and let us consider the associated positive numbers $\varepsilon_{t_1}, \ldots, \varepsilon_{t_k}$ as in (51).

Additionally, by the continuity assumptions on T there also exists $\varepsilon_0 > 0$ such that

$$T_{\varepsilon_0}(z) \subset \cup_{1\leq i\leq k} V_{t_i}. \tag{54}$$

This is because, otherwise, there would exist a sequence of elements satisfying $t_r \in T_{1/r}(z)$ with $t_r \notin \cup_{1\leq i\leq k} V_{t_i}$. In such a case, we may suppose w.l.o.g. that t_r converges to some $\bar{t} \in T(z) \subset \cup_{1\leq i\leq k} V_{t_i}$ by the upper semicontinuity assumption; thus we get a contradiction.

Finally, we prove that the aimed conclusion holds with $\eta := \min\{\varepsilon_0, \varepsilon_{t_1}, \ldots, \varepsilon_{t_k}\} > 0$. Indeed, for $s \in T_\eta(z) \subset T_{\varepsilon_0}(z) \subset \cup_{1\leq i\leq k} V_{t_i}$ (recall (54)) we find t_j with $1 \leq j \leq k$ such that $s \in V_{t_j} \cap T_\eta(z) \subset V_{t_j} \cap T_{\varepsilon_{t_j}}(z)$ for some $1 \leq j \leq k$. Thus (51) implies

$$\partial_\eta f_s(z) \subset \partial_{\varepsilon_j} f_s(z) \subset \partial f_{t_j}(z) + \delta\mathbb{B} \subset \operatorname{co}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\} + \delta\mathbb{B}.$$

□

The following proposition is the so-called Ioffe-Tihkomirov's theorem (see, for instance, Theorem 2.4.18 in [16]).

Proposition 6.3. *Consider a non-empty family of convex functions $\{f_t,\ t \in T\}$ and the associated supremum function f, with T being a separated compact topological space, and the mapping $t \to f_t(x)$ being upper semicontinuous for every $x \in \mathbb{R}^p$. If every function f_t is continuous at z, then*

$$\partial f(z) = \operatorname{co}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\},$$

where $T(z) := \{t \in T : f(z) = f_t(z)\}$.

Proof. As it was said above, in the proof of Lemma 6.2, the set $T(z)$ is a non-empty compact subset of T, and the function f is continuous at z. On an other hand, from Corollary 4.4 we obtain the inclusion "$\supset$", since that $\partial f_t(z) \subset \partial_\varepsilon f_t(z)$ and $T(z) \subset T_\varepsilon(z)$ for every $t \in T$, and $\varepsilon > 0$,

$$\overline{\operatorname{co}}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\} \subset \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} = \partial f(z). \tag{55}$$

For the reverse inclusion we use Lemma 6.2. In fact, for every given $\delta > 0$ there exists $\eta > 0$ such that

$$\cup_{t\in T_\eta(z)} \partial_\eta f_t(z) \subset \operatorname{co}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\} + \delta\mathbb{B},$$

and so

$$\cap_{\varepsilon>0} \operatorname{co}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} \subset \operatorname{co}\left\{\cup_{t\in T_\eta(z)} \partial_\eta f_t(z)\right\} \subset \operatorname{co}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\} + \delta\mathbb{B}.$$

Thus, applying once again Corollary 4.4 and observing that $\delta > 0$ was arbitrarily chosen, we obtain

$$\begin{aligned} \partial f(z) = \bigcap_{\varepsilon>0} \overline{\operatorname{co}}\left\{\cup_{t\in T_\varepsilon(z)} \partial_\varepsilon f_t(z)\right\} &\subset \bigcap_{\delta>0} \left(\operatorname{co}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\} + 2\delta\mathbb{B}\right) \\ &= \overline{\operatorname{co}}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\}. \end{aligned}$$

Finally, it is not difficult to see that the set $\cup_{t\in T(z)} \partial f_t(z)$ is a no-empty compact set and, so, $\operatorname{co}\left\{\cup_{t\in T(z)} \partial f_t(z)\right\}$ is also a compact set, entailing the aimed formula for $\partial f(z)$. □

Remark 6.4. It should be noted that the formula in Proposition 6.3 can be adapted when the same continuity properties are assumed for T and for the mappings $t \to f_t(x)$, $x \in \mathbb{R}^p$, but the original point z is not necessarily a continuity point of the functions f_t. Assuming in Proposition 6.3 the existence of a common point for all $\operatorname{ri}(\operatorname{dom} f_t)$, it is shown in [11] (see also [12]) that

$$\partial f(z) = \operatorname{co}\left\{\cup_{t\in T(z)}\partial f_t(z)\right\} + \mathrm{N}_{\operatorname{dom} f}(z). \tag{56}$$

This result extends to this compact setting the results of [15] established for the supremum function of finitely many proper convex functions.

The following example shows that (56) can fail when we relax the continuity assumptions, even for functions defined in $\mathbb{R}$. In this example the index set T is compact but the parametrized mappings $t \to f_t(x)$ are not all upper semi-continuous.

Example 6.5. Consider the functions $f_t : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$, $t \in [0,1]$, defined for $t \in \,]0,1]$ by

$$f_t(x) := \begin{cases} x & \text{if } |x| \le t, \\ -t & \text{if } x \le -t, \\ +\infty & \text{if } x > t, \end{cases}$$

and for $t = 0$ by

$$f_0(x) := x.$$

Each f_t is continuous at $z = 0$, whereas the index set $T := [0,1]$ is obviously compact. For every $t \in T(0) = \{t \in [0,1] : f_t(0) = f(0) = 0\} = [0,1]$ we have

$$\partial f_t(0) = \{1\}.$$

The supremum functions $f := \sup_{t\in[0,1]} f_t$ is given in this case by

$$f(x) = \begin{cases} 0, & \text{if } x \le 0, \\ +\infty, & \text{if } x > 0, \end{cases}$$

so that

$$\partial f(0) = [0, +\infty[.$$

Thus

$$\operatorname{co}\{\cup_{t\in[0,1]}\partial f_t(0)\} + N_{\operatorname{dom} f}(0) = \{1\} + [0,+\infty[\;\subsetneqq [0,+\infty[\;= \partial f(0),$$

and so (56) is not valid in this situation.

Even more, Proposition 6.3 can also fail as it is shown in the following example.

Example 6.6. Consider the functions $f_t : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$, $t \in [0,1]$, defined for $t \in \,]0,1]$ by

$$f_t(x) := \begin{cases} -t, & \text{if } x \le -t, \\ x, & \text{if } x > -t, \end{cases}$$

and for $t = 0$ by

$$f_0(x) := x.$$

Also now each f_t is continuous at z, and the supremum functions $f := \sup_{t\in[0,1]} f_t$ is

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x > 0. \end{cases}$$

So we get $\partial f(0) = [0,1]$, but $\partial f_t(0) = \{1\}$, for all $t \in T$, and Proposition 6.3 does not apply here.

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