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Optimal Shapes in Force Fields

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We study a variational problem involving material shapes, modeled as Radon measures on a given ambient space, in a force vector field. We prove some properties of absolutely continuity of the optimal shapes in dependence on the distributions of the zeros of the force field. More regularity properties of the optimal shapes, as well as simple geometrical characterizations, are established in the case of central force fields, in dependence on the regularity of the field and on the choice of the ambient space.

Introduction

Many problems arising in materials science and structural optimization ask for the determination of geometries, representing material regions, which minimize certain energy functionals. In this area, a classical problem is the equilibrium shape of a crystal in presence of an external field and it relies in minimizing a functional involving bulk and interfacial energies. If one considers sufficiently small grains of crystal, the bulk contribution is negligible and then the problem reduces to minimize a surface energy functional for a given volume of material whose solution is known as *Wulff shape*. Then the shape of the crystal is obtained as an equilibrium configuration of minimum energy. Typically, the surface energy to be minimized takes the form

$$\mathcal{F}(E) = \int_{\partial E} \Gamma(\nu_E(x))\, d\mathcal{H}^{N-1}, \tag{1}$$

where ν_E is the outward unit normal to the boundary of the set $E \subset \mathbb{R}^N$ with given volume, and Γ denotes the anisotropic free energy density per unit area. When Γ is constant we reduce to the classical isovolumetric problem (see [1]). There is a wide literature on the subject and we refer the reader to the works [5], [10], [13], [14], [17], [19], [20] and to the references therein contained. We just observe that a standard assumption in the Wulff problem is Γ *continuous and bounded away from zero*, i.e. for every $\nu \in S^{N-1}$

$$\Gamma(\nu) > \alpha > 0.$$

On the other hand, these assumptions ensure compactness among the sets of finite perimeter.

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Though the Wulff problem is largely studied, less is known for the problem involving both the energetic contributions, under the assumption of non-uniform external fields. Indeed, in presence of an external field ∇g we have in addition a bulk contribution, so that we have to consider a total energy as

$$\mathcal{F}(E) = \int_E g(x)\, d\mathcal{H}^N + \int_{\partial E} \Gamma(\nu_E(x))\, d\mathcal{H}^{N-1} \tag{2}$$

and thus the shape of the crystal is obtained by minimizing $\mathcal{F}$ among the sets $E \subset \mathbb{R}^N$ with fixed volume contained in a bounded region. The interaction between surface and bulk energies makes this variational problem more complicated than the Wulff problem. Moreover, in this case, the regularity properties of the minimizers are more involved (see [3, 4] and the references therein).

In the perspective of structural optimization (see, for instance, the works [6], [7], [8], [9], [11]), the problem of optimal geometries consists in designing the shape of a material body to the aim of optimizing its mechanical efficiency which is generally represented by an integral functional. In this context, usually, the material body is assumed to enjoy some constitutive properties and to be in equilibrium under given loads. In such a case the designing process can be carried on determining the best arrangement of two different materials, or on the global shape of the body.

In this paper we study a variational problem which seems to fit the above physical questions at a very general level. Indeed, we deal with a minimization problem for a material region in which the functional to minimize becomes like (2) or, in particular, like (1), under certain regularity conditions. More precisely, given a force field $F : \mathbb{R}^N \to \mathbb{R}^N$, we model material regions as Radon measures σ in a given ambient space $\overline{\Omega} \subset \mathbb{R}^N$ and we relate σ and F through the notion of one-dimensional current or, equivalently, with the vector measure $F\sigma$, which represents the idea of *a force acting on a matter distribution.* The energy functional is given by the mass of the boundary of the current or, in terms of measures, by the total variation of the measure $\mathrm{div}(F\sigma)$. Therefore, the energy to be minimized is given by

$$\mathbf{E}(\sigma) = \sup \left\{ \int_{\overline{\Omega}} \langle df, F \rangle d\sigma \mid \|f\|_\infty \leq 1 \right\}. \tag{3}$$

We observe that arguing in terms of currents emphasizes the duality structure connecting forces and matter and the geometric aspects of the problem. Finally, in our opinion this formulation of the problem of optimal geometries could help to get insights in the regularity properties of the minimizers of more particular choices (dictated by specific physical problems) of the energies.

Now we are going to outline the plan of the paper. In Section 1, after establishing some notation and terminology, we state the main variational problem and give the proof of the existence of minimizers. Moreover, in this section, we show how the problem reduces to that of the equilibrium shapes of crystals and that this reduction is a matter of regularity. Of course, in this formulation, the bulk and surface energies are not independent but they come from the same force field (see Section 1.1). Actually, these regularity questions are not immediate. Indeed, by the expression of the energy $\mathbf{E}(\sigma)$ in (3) we see that *a priori* the measure σ cannot be controlled in the orthogonal direction to the force field

F (see Example 2.4). In particular, through a one-dimensional reduction, we can find minimizers which have the form $\sigma = a(x)\mathcal{L}^N \llcorner \Omega$ with $a \in BV(\Omega)$. Moreover, in some special cases, this kind of regularity allows to produce explicit computations. In Section 2 we prove some properties of absolute continuity of the optimal shapes in dependence of the distributions of the zeros of the force field (see Theorem 2.7). Section 3 is devoted to the study of the minimizers under the assumption of a central force field. This kind of force, apart its physical relevance, deserves attention since several regularity properties of the optimal shapes, as well as simple geometrical characterizations, are established in dependence of the regularity of the field and of the choice of the ambient space.

1. Formulation of the variational problem and existence of optimal shapes

Let $\Omega \subset \mathbb{R}^N$ be any given bounded open subset, in the following we shall refer to $\overline{\Omega}$, as *ambient space*. We shall consider a force field as a given vector field

$$F : \mathbb{R}^N \to \mathbb{R}^N$$

for which we also use the notation $F = (F_1, \dots, F_N)$, where F_i, $i = 1, \dots, N$ are the components of F with respect to the canonical basis of $\mathbb{R}^N$. Roughly speaking, we say that the essential aim of the problem we are going to formulate relies in determining a *material shape*, inside the ambient space $\overline{\Omega}$, which turns out to be *optimal* in some sense to be specified below, among the *material shapes carried by F*. To this aim we first proceed in giving precise definitions of the italicized terms. By *material shape* in $\overline{\Omega}$ we mean any positive Radon measure σ with finite total mass on $\overline{\Omega}$ (i.e., by normalization, probability measures). We shall denote by $\mathcal{P}(\overline{\Omega})$ the set of all probability measures on $\overline{\Omega}$, thus a material shape is any $\sigma \in \mathcal{P}(\overline{\Omega})$. In order to relate material shapes and force fields we use the natural duality of Geometric Measure Theory which relies on the concept of *current*. Thus, for any $\sigma \in \mathcal{P}(\overline{\Omega})$, let

$$T^F_\sigma := \sigma \wedge F$$

be the one-dimensional current defined by

$$T^F_\sigma(\omega) = \int_{\mathbb{R}^N} \langle \omega, F \rangle d\sigma \quad \forall \omega \in \mathcal{D}^1(\mathbb{R}^N),$$

where $\mathcal{D}^1(\mathbb{R}^N)$ denotes the space of all infinitely differentiable and compactly supported $1-$forms and $\langle\ ,\ \rangle$ denotes the pairing for vectors and covectors. The space of linear and continuous functionals on $\mathcal{D}^1(\mathbb{R}^N)$ is the space of 1-dimensional currents (1-currents) in $\mathbb{R}^N$, which is denoted by $\mathcal{D}_1(\mathbb{R}^N)$. Then any material shape σ carried by F in $\mathbb{R}^N$ determines the 1-current T^F_σ just defined. Since, in the sequel, we shall consider the force field F as prescribed, we use the symbol T_σ, in place of T^F_σ. The boundary of a 1-current $T \in \mathcal{D}_1(\mathbb{R}^N)$ is defined through the formula

$$\partial T(f) = T(df) \quad \forall f \in \mathcal{C}^\infty_0(\mathbb{R}^N).$$

The mass of $T \in \mathcal{D}_1(\mathbb{R}^N)$ is defined by

$$\mathbf{M}(T) = \sup\{T(\omega) \mid \omega \in \mathcal{D}^1(\mathbb{R}^N),\ \|\omega\|_\infty \le 1\},$$

where $\|\omega\|_\infty := \sup\{|\omega(x)| \mid x \in \mathbb{R}^N\}$.

Then, with the above notation, we have

$$\begin{aligned}\mathbf{M}(\partial T_\sigma) &= \sup\{T_\sigma(df) \mid f \in \mathcal{C}_0^\infty(\mathbb{R}^N),\ \|f\|_\infty \leq 1\}\\ &= \sup\left\{\int_{\mathbb{R}^N} \langle df, F\rangle d\sigma \mid\ f \in \mathcal{C}_0^\infty(\mathbb{R}^N),\ \|f\|_\infty \leq 1\right\}.\end{aligned}$$

Notice that $\mathbf{M}(T_\sigma) = \int_\Omega |F(x)|d\sigma$ and $\mathbf{M}(T_\sigma) < +\infty$ for any $\sigma \in \mathcal{P}(\overline{\Omega})$. We recall (see p. 127 of [16]) that 1-currents of finite mass can be uniquely extended to forms with bounded Borel coefficients, moreover, since

$$T_\sigma(\omega) = \int_{\mathbb{R}^N} \langle \omega, F\rangle d\sigma = \int_{\overline{\Omega}} \langle \omega, F\rangle d\sigma$$

for every $\omega \in \mathcal{D}^1(\mathbb{R}^N)$, we can regard T_σ acting on infinitely differentiable forms, or forms with Borel bounded coefficients, defined on $\overline{\Omega}$.

For every $\sigma \in \mathcal{P}(\overline{\Omega})$ we set $\mathbf{E}(\sigma) = \mathbf{M}(\partial T_\sigma)$. We shall call *optimal shape carried by F inside $\overline{\Omega}$* any solution of the following variational problem:

$$\mathbb{P}(F, \overline{\Omega}) \qquad\qquad \text{Minimize } \left\{\mathbf{E}(\sigma) \mid \sigma \in \mathcal{P}(\overline{\Omega})\right\}.$$

In the sequel we will refer to $\mathbf{E}(\sigma)$ as to the *energy* of the shape σ and we set $m(F, \overline{\Omega}) := \inf_\sigma \mathbf{E}(\sigma)$.

1.1. Prescribed curvature and Wulff problems

In this section we see how problem $\mathbb{P}(F, \overline{\Omega})$ is related with some classical problems such as *prescribed curvature* and Wulff problems (see [1]). To this aim, suppose that the force field F is regular, say $\mathcal{C}^1$, and let $E \subset \overline{\Omega}$ be a set of finite perimeter with ν_E denoting the outward normal unit vector, and take $\sigma_E = \frac{1}{|E|}\mathcal{H}^N \llcorner E$. By Gauss-Green Theorem we have:

$$\begin{aligned}\mathbf{E}(\sigma_E) &= \sup_{\|f\|_\infty \leq 1} \frac{1}{|E|} \int_E \langle df, F\rangle dx \qquad (4)\\ &= \sup_{\|f\|_\infty \leq 1} \frac{1}{|E|} \left(-\int_E f \mathrm{div} F dx + \int_{\partial E} f F \cdot \nu_E \ d\mathcal{H}^{N-1}\right)\\ &= \frac{1}{|E|}\left(\int_E |\mathrm{div} F| dx + \int_{\partial E} |F \cdot \nu_E| \ d\mathcal{H}^{N-1}\right).\end{aligned}$$

Therefore, $\mathbb{P}(F, \overline{\Omega})$ can be viewed as a relaxation of a normalized prescribed curvature type problem, or a general equilibrium shape problem for crystals as treated in [19]. Observe that in our case the surface and bulk energies depend both on the vector field F. Furthermore, there is no convexity assumption here.

Finally, we observe that equation (4) always ensures that there exists a measure $\sigma_E \in \mathcal{P}(\overline{\Omega})$ such that $\mathbf{E}(\sigma_E) < +\infty$.

If F is a solenoidal field, that is $\mathrm{div} F = 0$, the energy takes the form

$$\mathbf{E}(\sigma_E) = \frac{1}{|E|} \int_{\partial E} |F \cdot \nu_E| \ d\mathcal{H}^{N-1},$$

which corresponds to a Wulff energy. Obviously, if $F = \nabla u$ for a smooth potential $u : \mathbb{R}^N \to \mathbb{R}$, the energy of σ_E can be written as

$$\mathbf{E}(\sigma_E) = \frac{1}{|E|}\left(\int_E |\Delta u|\, dx + \int_{\partial E} |\nabla u \cdot \nu_E|\, d\mathcal{H}^{N-1}\right). \tag{5}$$

In this framework, due to the generality of the force field F, we do not have compactness among sets of finite perimeter. This constitutes a motivation for defining the problem $\mathbb{P}(F,\overline{\Omega})$ on probability measures. Furthermore, more regularity properties of optimal shapes, such as measures whose densities are characteristic functions of sets of finite perimeter, could be useful in the study of these classical problems. In this perspective, in the next sections, we will focus our attention on some regularity properties of the optimal shapes.

The next existence result guarantees that $\mathbb{P}(F,\overline{\Omega})$ has solutions.

Proposition 1.1. *Let F be a given continuous force field. Then $\mathbb{P}(F,\overline{\Omega})$ always admits minimizers.*

Proof. Let $(\sigma_n)_{n\in\mathbb{N}} \subset \mathcal{P}(\overline{\Omega})$ be any energy minimizing sequence. By weak* compactness we can suppose that $\sigma_n \overset{*}{\rightharpoonup} \sigma \in \mathcal{P}(\overline{\Omega})$ for a subsequence not relabelled. For every $f \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ with $\|f\|_\infty \le 1$, by continuity of F we have

$$\int_{\overline{\Omega}} \langle df, F\rangle d\sigma = \lim_{n\to+\infty} \int_{\overline{\Omega}} \langle df, F\rangle d\sigma_n \le \liminf_{n\to+\infty} \mathbf{E}(\sigma_n).$$

Finally, taking the supremum with respect to $f \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ such that $\|f\|_\infty \le 1$ in the last inequality, we infer

$$\mathbf{E}(\sigma) \le \liminf_{n\to+\infty} \mathbf{E}(\sigma_n).$$

□

2. General properties of the minimizers

In order to have a non trivial problem, we have to characterize the measures σ such that $\mathbf{E}(\sigma) < +\infty$. Then we are going to state some general properties regarding such measures, which are defined as shapes with bounded energy. In this direction we have the following result.

Lemma 2.1. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous vector field. Let $\bar{x} \in \overline{\Omega}$ be given. If $F(\bar{x}) \neq 0$ then*

$$\mathbf{E}(\sigma) < +\infty \Rightarrow \sigma(\{\bar{x}\}) = 0.$$

Proof. Let us suppose for instance that $F_1(\bar{x}) > 0$. By continuity we can assume that $F_1(x) > 0$ on $B_r := B(\bar{x}, r) \cap \overline{\Omega}$ for some $r > 0$ small enough. Consider the function

$$f_r(x) = \frac{1}{r}\chi_{B_r}(x)(x_1 - \bar{x}_1),$$

where χ_A denotes the characteristic function of the set A. Observe that $|f_r(x)| \leq \frac{1}{r}|x_1 - \bar{x}_1| \leq \frac{1}{r}|x - \bar{x}| \leq 1$ for every $x \in B_r$, whence $\|f_r\|_\infty \leq 1$. Therefore, since $\mathbf{E}(\sigma) < +\infty$, we get the following estimate

$$\frac{1}{r}\int_{B_r} F_1(x)d\sigma = \int_{B_r} \langle df_r, F\rangle d\sigma \leq \mathbf{E}(\sigma) \leq C, \tag{6}$$

for some constant $C < +\infty$. By (6) we obtain

$$F_1(\bar{x})\sigma(\{\bar{x}\}) = \lim_{r\to 0^+}\int_{B_r} F_1(x)d\sigma = 0$$

and this implies $\sigma(\{\bar{x}\}) = 0$. □

By Lemma 2.1 we can trivially deduce the following regularity property enjoyed by the optimal shapes.

Corollary 2.2. *If $F(x) \neq 0$ for every $x \in \overline{\Omega}$, then every minimizer of $\mathbb{P}(F, \overline{\Omega})$ is a nonatomic measure.*

Although the following considerations are not strictly required for the results of this paper, we think it is useful to point out some link with the notion of tangent space to a measure, as developed in [6, 8, 9, 15]. Indeed, we have the following

$$\mathbf{E}(\sigma) < +\infty \Leftrightarrow F(x) \in \mathbb{T}_\sigma(x) \quad \sigma - a.e.$$

where $\mathbb{T}_\sigma(x)$ is the tangent space to the measure σ at the point x. For convenience we recall here the definition of tangent space to a measure given in [15]. If σ is a positive Borel regular measure one can consider the following set of tangent fields

$$X_\sigma := \{\Phi \in (L^1_\sigma)^N \mid \mathrm{div}(\Phi\sigma) \in \mathcal{M}\},$$

where $(L^1_\sigma)^N := L^1_\sigma(\mathbb{R}^N, \mathbb{R}^N)$, the divergence operator is intended in the distributional sense and $\mathcal{M}$ denotes the set of signed Borel measures with finite total variation on $\mathbb{R}^N$. Then, for σ-a.e. $x \in \mathbb{R}^N$, the tangent space to σ is defined as

$$\mathbb{T}_\sigma(x) := \sigma - ess\bigcup\{\Phi(x) \mid \Phi \in X_\sigma\}. \tag{7}$$

The above equation (7) means that $\mathbb{T}_\sigma(x)$ is the unique σ-measurable, closed valued multifunction on $\mathbb{R}^N$ characterized by:

1. $\Phi \in X_\sigma \Rightarrow \Phi(x) \in \mathbb{T}_\sigma(x)$ for σ-a.e. $x \in \mathbb{R}^N$;
2. $\mathbb{T}_\sigma$ is the minimal σ-measurable multifunction satisfying (1), namely for every σ-measurable closed valued multifunction Σ on $\mathbb{R}^N$ such that $\Phi \in X_\sigma \Rightarrow \Phi(x) \in \Sigma(x)$ for σ-a.e. $x \in \mathbb{R}^N$, it results $\mathbb{T}_\sigma(x) \subset \Sigma(x)$.

One can check that $\mathbb{T}_\sigma(x)$ individuates a linear subspace of $\mathbb{R}^N$ whose dimension may depend on x.

The following nice property of tangent spaces was proved in [15] for $N = 1$.

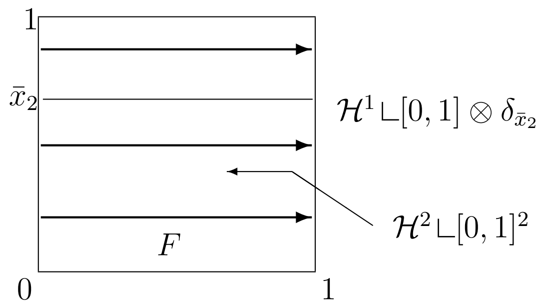

Figure 2.1: Two optimal measures

Lemma 2.3 (Fragalá, Mantegazza). *Let σ be a positive Radon measure on $\mathbb{R}$, and let σ^s be the singular part of σ with respect to the Lebesgue measure. Then*

$$\mathbb{T}_\sigma(x) = \{0\} \quad \sigma^s - a.e.$$

Actually, the basic idea is that $\mathbf{E}(\sigma) < +\infty$ implies that the distributional derivative of $F\sigma$ is a measure, i.e. $F\sigma$ is a function of bounded variation. Therefore, the restriction of σ to the set of $x \in \Omega$ such that $F(x) \neq 0$ is absolutely continuous. In fact, we will use this argument in Section 3 where we deal with radial energies.

Therefore, by Lemma 2.3, in dimension one, if $\mathbf{E}(\sigma) < +\infty$, the optimal shapes are absolutely continuous measures where the field is non-vanishing. This is not the case in higher dimension, as we are going to see in the next example.

Example 2.4. Let $\overline{\Omega} = [0,1]^2$, we consider the constant vector field $F(x) = e_1 = (1,0)$. Let us fix $\bar{x} \in \overline{\Omega}$, $\bar{x} = (0, \bar{x}_2)$, then the measure $\tilde{\sigma} = \mathcal{H}^1 \llcorner [0,1] \otimes \delta_{\bar{x}_2}$ is of bounded energy since, for every f, we have

$$\begin{aligned}\int_{\overline{\Omega}} \langle df, e_1 \rangle d\tilde{\sigma} &= \int_{\overline{\Omega}} \frac{\partial f}{\partial x_1}(x_1, x_2)\; d\left(\mathcal{H}^1 \llcorner [0,1] \otimes \delta_{\bar{x}_2}\right)(x_1, x_2) \\ &= \int_0^1 \frac{\partial f}{\partial x_1}(x_1, \bar{x}_2)\; dx = f(1, \bar{x}_2) - f(0, \bar{x}_2) \leq 2.\end{aligned}$$

In particular, we find that $\mathbf{E}(\tilde{\sigma}) = 2$. Now, let us introduce the 1-dimensional energy $\mathbf{E}_1$, which is the energy obtained by considering test functions depending on the first coordinate only. Therefore $\mathbf{E}_1 \leq \mathbf{E}$. For any measure $\mu \in \mathcal{P}(\overline{\Omega})$, we take $\sigma = (\pi_1)_\# \mu \otimes \mathcal{H}^1 \llcorner [0,1]$, where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection on the first axis and $(\pi_1)_\# \mu$ is the image measure of μ through π_1, i.e. $(\pi_1)_\# \mu(A) = \mu(\pi_1^{-1}(A))$ for every Borel set $A \subset \mathbb{R}$. If $f = f(x_1)$ is a test function for $\mathbf{E}_1$, we have

$$\int_{\overline{\Omega}} \langle df, e_1 \rangle d\sigma = \int_{\overline{\Omega}} f'(x_1) d\mu(x_1, x_2),$$

which implies $\mathbf{E}_1(\sigma) = \mathbf{E}_1(\mu)$. Therefore, without loss of generality, in computing $\mathbf{E}_1$ we can restrict ourselves to consider measures having the form $\sigma = \nu \otimes \mathcal{H}^1 \llcorner [0,1]$, where ν is a probability measure on $[0,1]$. Observe that

$$\mathbf{E}_1(\sigma) = \sup_{\|f\|_\infty \leq 1} \int_0^1 f'(x_1) d\nu(x_1). \tag{8}$$

By virtue of Lemma 2.3 we infer that if $\mathbf{E}_1(\sigma) < +\infty$ then $\nu \ll \mathcal{H}^1 \llcorner [0,1]$. Moreover, if $a \in L^1([0,1])$ is the density of ν, by (8) we get $a \in BV((0,1))$ and, after choosing a *good representative* of a (see [2, Theorem 3.28]) and integrating by parts, we deduce the following formula:

$$\begin{aligned}\mathbf{E}_1(\sigma) &= \sup_{\|f\|_\infty \leq 1} \left\{ -\int_0^1 f d(Da) + f(1)a(1_+) - f(0)a(0_-) \right\} \\ &= \|Da\|(0,1) + a(1_+) + a(0_-).\end{aligned} \tag{9}$$

If we take $a(x) = 1$, then by the above expression of the energy we see that $\bar{\sigma} = \mathcal{H}^2 \llcorner [0,1]^2$ is an optimal shape for $\mathbf{E}_1$. Observe that

$$\mathbf{E}_1(\bar{\sigma}) = 2. \tag{10}$$

In order to relate $\mathbf{E}_1$ with $\mathbf{E}$ we have to check that if

$$\mu = a(x_1)\mathcal{H}^1 \llcorner [0,1] \otimes \mathcal{H}^1 \llcorner [0,1],$$

then $\mathbf{E}_1(\mu) = \mathbf{E}(\mu)$. To this aim let $f(x_1, x_2)$ be any test function, we set

$$f_*(x_1) = \int_0^1 f(x_1, x_2) dx_2$$

and so we have

$$\begin{aligned}\int_0^1 f_*'(x_1)a(x_1)dx_1 &= \int_0^1 a(x_1) \left(\int_0^1 \partial_{x_1} f(x_1, x_2) dx_2 \right) dx_1 \\ &= \int_\Omega \partial_{x_1} f(x_1, x_2) a(x_1) dx_1 dx_2 = \int_\Omega \partial_{x_1} f(x_1, x_2) d\mu.\end{aligned}$$

Hence we deduce

$$\mathbf{E}(\mu) \leq \mathbf{E}_1(\mu),$$

and then the claim follows. Thus, for every $\sigma \in \mathcal{P}(\overline{\Omega})$ we obtain

$$\mathbf{E}(\bar{\sigma}) = \mathbf{E}_1(\bar{\sigma}) \leq \mathbf{E}_1(\sigma) \leq \mathbf{E}(\sigma)$$

and therefore $\bar{\sigma}$ is an optimal shape for $\mathbf{E}$. By summarizing, we can say that, in general, the optimal shapes can exhibit very different regularity properties, indeed we have seen that $\tilde{\sigma}$ and $\bar{\sigma}$ are both optimal shapes for $\mathbb{P}(e_1, [0,1]^2)$. We point out that $\bar{\sigma}$ has been obtained through a dimensional reduction argument and in Section 3 we will see how a similar 1-dimensional reduction for central force fields works in order to find regular minimizers.

We emphasize that by (9) we have that any optimal shape σ for $\mathbb{P}(e_1, [0,1]^2)$ has to satisfy the condition

$$(\pi_1)_\# \sigma = \mathcal{H}^1 \llcorner [0,1]. \tag{11}$$

Indeed, let $a \in BV((0,1))$ be a *good representative* of the density of $\sigma_1 := (\pi_1)_\# \sigma$, we suppose for instance that $a(0_-) < 1$. Since $\int_0^1 a(x)dx = 1$, we find a point $t \in [0,1]$ such that $a(t) = 2 - a(0_-)$. In order to compute the total variation of a we can equivalently consider the pointwise variation

$$\mathrm{p}V(a,(0,1)) := \sup \left\{ \sum_{i=1}^{n-1} |a(t_{i+1}) - a(t_i)| \;\Big|\; n \geq 2,\ 0 < t_1 < \cdots < t_n < 1 \right\}.$$

Therefore we obtain

$$\mathrm{p}V(a,(0,t]) \geq 2 - 2a(0_-).$$

Now, if $a(1_+) \geq 1$ we have

$$\begin{aligned} \mathbf{E}_1(\sigma) &= \mathbf{E}_1(\sigma_1 \otimes \mathcal{H}^1 \llcorner [0,1]) \\ &\geq \mathrm{p}V(a,(0,t]) + a(1_+) + a(0_-) \geq 2 - a(0_-) + a(1_+) > 2. \end{aligned}$$

If $a(1_+) < 1$, we find as above a point $s \in [0,1]$, $t < s$, such that $a(s) = 2 - 2a(1_+)$. Hence

$$\mathbf{E}_1(\sigma) = \mathrm{p}V(a,(0,1)) + a(1_+) + a(0_-) \geq 2 - a(0_-) + 2 - a(1_+) > 2.$$

Therefore, the best choice is $a(x) \equiv 1$. Since $\mathbf{E}_1(\sigma) \leq \mathbf{E}(\sigma)$, (11) holds.

We remark that if $F(\bar{x}) = 0$ at some point $\bar{x} \in \overline{\Omega}$, then the Dirac measure $\delta_{\bar{x}}$ is an optimal shape for $\mathbb{P}(F, \overline{\Omega})$ and $\mathbf{E}(\delta_{\bar{x}}) = 0$. Nevertheless, the non-vanishing of the vector field F is strictly related to the absolute continuity of the *optimal shapes* with respect to $\mathcal{H}^1$ as stated in the following statement.

Lemma 2.5. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous force field such that $F(x) \neq 0$ for every $x \in \overline{\Omega}$. If $\sigma \in \mathcal{P}(\overline{\Omega})$ has bounded energy, i.e. $\boldsymbol{E}(\sigma) < +\infty$, then $\sigma \ll \mathcal{H}^1 \llcorner \overline{\Omega}$.*

Proof. As in the proof of Lemma 2.1, assume $F_1 > 0$ in $B_r = B(\bar{x}, r) \cap \overline{\Omega}$ for some $r > 0$ small enough and consider the estimate (6)

$$\frac{1}{r} \int_{B_r} F_1(x) d\sigma \leq \mathbf{E}(\sigma) \leq C.$$

Hence we infer

$$\sigma(B_r) \leq \frac{Cr}{\min_{B_r} F_1(x)} \ .$$

By the above inequality we can estimate the upper 1-dimensional density of σ at $\bar{x}$, that is

$$\theta_1^*(\sigma, \bar{x}) := \limsup_{r \to 0^+} \frac{\sigma(\overline{\Omega} \cap B(\bar{x}, r))}{2r} \leq \frac{C}{2F_1(\bar{x})} \ . \tag{12}$$

For every $n \geq 1$, by taking the compact sets

$$\mathcal{C}_n = \left\{ x \in \overline{\Omega} \;\Big|\; \exists i = 1, \ldots, N \ s.t.\ \frac{1}{n} \leq |F_i(x)| \right\},$$

by virtue of (12), we obtain that for every $\bar{x} \in \mathcal{C}_n$ the following estimate holds true

$$\theta_1^*(\sigma, \bar{x}) \leq \frac{1}{2} nC. \tag{13}$$

Therefore the upper density $\theta_1^*(\sigma, \bar{x})$ is uniformly bounded on $\mathcal{C}_n$ and this implies (see for instance Theorem 2.56 in [2]) that, for every $n \geq 1$, $\sigma \llcorner \mathcal{C}_n \ll \mathcal{H}^1 \llcorner \mathcal{C}_n$.

Finally, if $A \subset \overline{\Omega}$ is a Borel subset such that $\mathcal{H}^1(A) = 0$, we obtain

$$\sigma(A) = \sigma\left(\bigcup_{n\geq 1} A \cap \mathcal{C}_n\right) \leq \sum_{n=1}^{+\infty} \sigma(A \cap \mathcal{C}_n) = 0.$$

□

The discussion in Example 2.4 suggests to introduce a non-vanishing assumption on the force field F in all directions in order to obtain the absolute continuity of the measures of finite energy with respect to $\mathcal{H}^N$.

Lemma 2.6. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a given continuous force field. Let us suppose that for every $x \in \overline{\Omega}$ and for every $i = 1, \dots, N$ it results:*

$$F_i(x) \neq 0. \tag{14}$$

If $\sigma \in \mathcal{P}(\overline{\Omega})$ has bounded energy, i.e. $\boldsymbol{E}(\sigma) < +\infty$, then $\sigma \ll \mathcal{H}^N \llcorner \overline{\Omega}$.

Proof. Consider the case $N = 2$. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first factor, that is $\pi_1 : (x, y) \mapsto x$, and take $\sigma_1 := (\pi_1)_\# \sigma$. By the Disintegration Theorem (see Theorem 2.28 in [2]) we find probability measures $\sigma_x \in \mathcal{P}(\mathbb{R})$ such that $\sigma = \sigma_1 \otimes \sigma_x$, which means that for every Borel set $A \subset \mathbb{R}^2$ we have

$$\sigma(A) = \sigma_1 \otimes \sigma_x(A) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_A(x, y) d\sigma_x(y)\right) d\sigma_1(x). \tag{15}$$

If we consider test functions f depending on the first coordinate only we get

$$\int_{\overline{\Omega}} \langle df, F\rangle d\sigma = \int_{\overline{\Omega}} f'(x) F_1(x, y) d\sigma = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f'(x) F_1(x, y) d\sigma_x(y)\right) d\sigma_1(x).$$

Since $\mathbf{E}(\sigma) \leq C$ and recalling that $\sigma_x \in \mathcal{P}(\mathbb{R})$, we infer

$$\int_{\mathbb{R}} f'(x) \left(\min_{\overline{\Omega}} F_1(x, y)\right) d\sigma_1(x) \leq C, \tag{16}$$

for every $f \in \mathcal{C}_0^\infty(\mathbb{R})$ such that $\|f\|_\infty \leq 1$ and $f' \geq 0$. Let $F_1 := \min_{\overline{\Omega}} F_1(x, y)$ (here we assume $F_1(x, y) > 0$, otherwise we take $F_1 := -\max_{\overline{\Omega}} F_1(x, y)$) and let I_1 be the support of the measure σ_1. By (16) we deduce that the measure σ_1 is of finite energy for the problem $\mathbb{P}(F_1, I_1)$. Therefore Lemma 2.5 yields $\sigma_1 \ll \mathcal{H}^1 \llcorner I_1$.

Now, choosing test functions g depending on the second coordinate only, we get

$$\int_{\overline{\Omega}} \langle dg, F\rangle d\sigma = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} g'(y) F_2(x, y) d\sigma_x(y)\right) d\sigma_1(x). \tag{17}$$

For every x, let I_x be the support of σ_x and let $F_2(x) = F(x, \cdot)$. Since σ is of finite energy, (17) implies that for $\mathcal{H}^1 - a.e.\ x \in I_1$ the measure σ_x is of finite energy for the problem $\mathbb{P}(F_2(x), I_x)$. Therefore, by applying again Lemma 2.5, for $\mathcal{H}^1 - a.e.\ x \in I_1$ we get $\sigma_x \ll \mathcal{H}^1 \llcorner I_x$.

Finally, let $A \subset \overline{\Omega}$ be any Borel set such that $\mathcal{H}^2(A) = 0$. By the definition of product measures, for $\mathcal{H}^1 - a.e.\ x \in \mathbb{R}$ it results $\mathcal{H}^1(A_x) = 0$, where $A_x := \{y \in \mathbb{R} : (x, y) \in A\}$. By using the absolute continuity properties of σ_1 and σ_x, we get

$$\sigma(A) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_A(x, y) d\sigma_x(y) \right) d\sigma_1(x) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \chi_{A_x}(y) d\sigma_x(y) \right) d\sigma_1(x) = 0.$$

By iterating the previous disintegration argument, we get the thesis for every N. □

By the previous Lemma we infer the following regularity property of the optimal shapes.

Theorem 2.7. *Let $F : \mathbb{R}^N \to \mathbb{R}^N$ be a given continuous force field. Let us suppose that for every $x \in \overline{\Omega}$ and for every $i = 1, \ldots, N$ it results*

$$F_i(x) \neq 0. \tag{18}$$

Then every optimal shape σ is absolutely continuous.

Let us observe that if $F(x) = 0$, then the optimal shape is δ_x. We remark that Theorem 2.7 applies, for instance, to constant vector fields $F = v$ with $v_i \neq 0$ for every $i = 1, \ldots, N$, ensuring that the optimal shapes are absolutely continuous measures.

Example 2.8. Let $v = (1/\sqrt{2}, 1/\sqrt{2})$, $F(x) = v$ and let $\overline{\Omega} \subset \mathbb{R}^2$ be a unit square oriented with one edge direction equal to v. Let $Q \in SO_2(\mathbb{R})$ be a rotation such that $Qv = e_1$. By Theorem 2.7 we know that the optimal shapes of $\mathbb{P}(F, \overline{\Omega})$ are absolutely continuous. In order to get an estimate on the minimum energy, let us begin by noticing that, if $a \in L^1(\Omega)$, we have

$$\begin{aligned} \int_\Omega \langle d(f \circ Q), v \rangle a(x) dx &= \int_\Omega \langle Q^T df(Qx), v \rangle a(x) dx \qquad (19) \\ = \int_\Omega \langle df(Qx), e_1 \rangle a(x) dx &= \int_{Q(\Omega)} \langle df(y), e_1 \rangle a(Q^T y) dy. \end{aligned}$$

We put $\bar{\sigma} = \mathcal{H}^2 \llcorner \Omega$ and, for any $a \in L^1(\Omega)$ with $\int_\Omega a(x)dx = 1$, we put $\sigma_Q = a \circ Q^T \mathcal{H}^2 \llcorner \Omega$, $\sigma = a(x) \mathcal{H}^2 \llcorner \Omega$. Taking into account (19) and Example 2.4 we can state the following relations

$$\mathbf{E}(\bar{\sigma}) = 2 = m(e_1, [0, 1]^2) \leq \mathbf{E}(\sigma_Q) \leq \mathbf{E}(\sigma)$$

and so we have that $\bar{\sigma}$ is an optimal shape for $\mathbb{P}(F, \overline{\Omega})$. Now we address the question of changing the ambient space $\overline{\Omega}$. Consider for example $\overline{\Omega} = [0, 1]^2$. As before, the optimal shapes are absolutely continuous measures σ with density $a \in L^1(\Omega)$. Since we can inscribe $\overline{\Omega}$ in a square Σ with edge of length $\sqrt{2}$ and parallel to v, see Figure 2.2, by the previous discussion we have

$$\mathbf{E}(\sigma) \geq \sqrt{2}.$$

In fact we have that $m(F, \overline{\Omega}) = \sqrt{2}$ by taking rectangles with one edge parallel to v which shrink to the diagonal of $[0, 1]^2$. We claim that for every set E of finite perimeter in $\overline{\Omega}$, by setting $\sigma_E = \mathcal{H}^2 \llcorner E$ and $\sigma_{QE} = \chi_E \circ Q^T \mathcal{H}^2 \llcorner \Omega$, it results

$$\mathbf{E}(\sigma_E) > \sqrt{2}$$

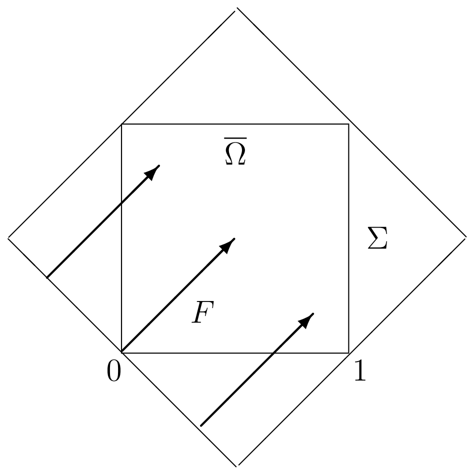

Figure 2.2.

and then the optimal shapes are not reached by sets of finite perimeter. Indeed, otherwise we would have $\mathbf{E}(\sigma_E) = m(F, \Sigma)$, and by (19) we obtain

$$\mathbf{E}(\sigma_{QE}) = m(e_1, Q(\Sigma)).$$

Therefore, by condition (11) we infer

$$(\pi_1)_\#(\sigma_{QE}) = \frac{1}{\sqrt{2}}\mathcal{H}^2 \llcorner \Sigma.$$

However, the above condition yields

$$\int_0^{\sqrt{2}} \chi_E(Q^T(x,y))dy = \frac{1}{\sqrt{2}} \quad \forall x \in [0, \sqrt{2}],$$

but this is a contradiction since, for example,

$$\int_0^{\sqrt{2}} \chi_E(Q^T(0,y))dy = 0.$$

The Examples 2.4 and 2.8 show that the *geometry* of the force field influences the geometry of the optimal shapes. Moreover, a right choice of the ambient space in correspondence of the force field can simplify the description of the optimal shapes.

2.1. Invariance properties

Definition 2.9 (Invariance). We say that the vector field $F : \mathbb{R}^N \to \mathbb{R}^N$ is invariant with respect to $Q \in GL_N(\mathbb{R})$ if

$$F(Qx) = QF(x).$$

Proposition 2.10. *Let F be an invariant vector field with respect to $Q \in GL_N(\mathbb{R})$ and suppose that $Q^T(\overline{\Omega}) = \overline{\Omega}$. If σ is an optimal shape for $\mathbb{P}(F, \overline{\Omega})$ then $Q_\#\sigma$ is an optimal shape too.*

Proof. Let $\|f\|_\infty \leq 1$. We have

$$\begin{aligned}\partial T_{Q_\# \sigma}(f) &= \int_{\overline{\Omega}} \langle df, F\rangle d(Q_\#\sigma) = \int_{\overline{\Omega}} \langle df(Qx), F(Qx)\rangle d\sigma \\ &= \int_{\overline{\Omega}} \langle df(Qx), QF(x)\rangle d\sigma = \int_{\overline{\Omega}} \langle Q^T df(Qx), F(x)\rangle d\sigma \\ &= \int_{\overline{\Omega}} \langle d(f\circ Q), F\rangle d\sigma = \partial T_\sigma(f\circ Q) \leq \mathbf{E}(\sigma).\end{aligned} \tag{20}$$

Taking the supremum with respect to $\|f\|_\infty \leq 1$ we obtain

$$\mathbf{E}(Q_\#\sigma) \leq \mathbf{E}(\sigma).$$

Since σ is optimal the statement follows. □

Proposition 2.11. *Let F be an invariant vector field with respect to $Q \in SO_N(\mathbb{R})$ and suppose that $\sigma = a(|x|)\mathcal{H}^N \llcorner \Omega$, with $Q^T(\Omega) = \Omega$. Then*

$$Q_\#(\partial T_\sigma) = \partial T_\sigma = \partial T_{Q_\#\sigma}.$$

Proof. We have

$$\begin{aligned}Q_\#(\partial T_\sigma)(f) = \partial T_\sigma(f\circ Q) &= \int_\Omega \langle d(f\circ Q), F\rangle a(|x|)dx \\ &= \int_\Omega \langle Q^T df(Qx), F(x)\rangle a(|x|)dx = \int_\Omega \langle df(Qx), F(Qx)\rangle a(|Qx|)dx \\ &= \int_\Omega \langle df(y), F(y)\rangle a(|y|)dy = \partial T_\sigma(f).\end{aligned}$$

Comparing to (20) we obtain the other equality. □

Proposition 2.10 and Proposition 2.11 suggest the natural question: If the vector field F is invariant with respect to $Q \in SO_N(\mathbb{R})$ is it true that the optimal shapes are radially symmetric? We address this question in the next section.

3. Central fields

Let $\overline{\Omega} = \bar{B}(0,R)$ and let $x \mapsto \varphi(|x|)$ be a continuous map defined on $\overline{\Omega}\setminus\{0\}$ with $\varphi(|x|) > 0$ for $x \neq 0$. We define the central force vector field as

$$F(x) = \varphi(|x|)\tfrac{x}{|x|}, \quad x \neq 0. \tag{21}$$

3.1. Radial minimizers

In order to investigate existence and regularity properties of minimizers, we are going to use a dimensional reduction, as made in Example 2.4, and to this aim we introduce the radial energy $\mathbf{E}_r(\sigma)$, obtained by considering radially symmetric test functions only. Let $a \in L^1(\Omega)$, with $\int_\Omega a(x) = 1$, we define the function

$$a^*(r) := \fint_{\partial B(0,r)} a(x)\ d\mathcal{H}^{N-1} \quad \text{for } \mathcal{H}^1 - a.e.\ r, \tag{22}$$

which is well defined because of Coarea Formula (see Sect. 3.4.3 of [12]). By using polar coordinates, we have

$$\int_\Omega a(x)dx = \int_0^R \left(\int_{\partial B(0,r)} a(x)\; d\mathcal{H}^{N-1}\right) dr = N\omega_N \int_0^R a^*(r) r^{N-1} dr = \int_\Omega a^*(|x|)dx,$$

where ω_N is the Lebesgue measure of the unit ball of $\mathbb{R}^N$. Hence, if $\sigma = a(x)\mathcal{H}^N \llcorner \Omega \in \mathcal{P}(\overline{\Omega})$, then $\sigma^* = a^*(|x|)\mathcal{H}^n \llcorner \Omega \in \mathcal{P}(\overline{\Omega})$ and so it is admissible for $\mathbb{P}(F, \overline{\Omega})$. By keeping this notation for σ and σ^*, in the next proposition we prove some properties regarding radial energies and radial minimizers.

Proposition 3.1. *Let $a \in L^1(\Omega)$ be such that $\sigma = a(x)\mathcal{H}^N \llcorner \Omega \in \mathcal{P}(\overline{\Omega})$. Then the following statements hold true:*

(1) $\boldsymbol{E}_r(\sigma) \le \boldsymbol{E}(\sigma)$;

(2) $\boldsymbol{E}_r(\sigma^*) = \boldsymbol{E}_r(\sigma)$;

(3) *If a is radially symmetric then $\boldsymbol{E}_r(\sigma) = \boldsymbol{E}(\sigma)$;*

(4) *If σ minimizes $\boldsymbol{E}_r$ then σ^* minimizes $\boldsymbol{E}$;*

(5) *If σ minimizes $\boldsymbol{E}$ then σ also minimizes $\boldsymbol{E}_r$;*

(6) *If σ minimizes $\boldsymbol{E}$ then σ^* minimizes $\boldsymbol{E}$.*

Proof. The first claim is trivial. To prove (*2*), let $f(|x|)$ be a test function for the energy $\mathbf{E}_r$, we have

$$\int_\Omega \langle df, F\rangle a(x)dx = \int_0^R \left(\int_{\partial B(0,r)} f'(|x|)\varphi(|x|)a(x)\; d\mathcal{H}^{N-1}\right) dr$$
$$= \int_0^R f'(r)\varphi(r) \left(\int_{\partial B(0,r)} a(x)\; d\mathcal{H}^{N-1}\right) dr = N\omega_N \int_0^R f'(r)\varphi(r) r^{N-1} a^*(r)dr.$$

On the other hand

$$\int_\Omega \langle df, F\rangle a^*(|x|)dx = \int_\Omega f'(|x|)\varphi(|x|)a^*(|x|)dx = N\omega_N \int_0^R f'(r)\varphi(r) r^{N-1} a^*(r)dr.$$

By taking the supremum with respect to $f(|x|)$ we find that σ and σ^* have the same radial energy and so the claim is proved.

To prove (*3*) let a be radially symmetric and let f be a test function for the energy $\mathbf{E}_r$, we define

$$f_*(r) = \frac{1}{N\omega_N}\int_{S^{N-1}} f(r,\theta)d\theta.$$

By using polar coordinates, we get

$$\int_\Omega \langle df, F\rangle a(x)dx = \int_0^R \left(\int_{S^{N-1}} \partial_r f(r,\theta)\varphi(r)a(r)r^{N-1}d\theta\right) dr$$
$$= \int_0^R \varphi(r)a(r)r^{N-1}\left(\int_{S^{N-1}} \partial_r f(r,\theta)d\theta\right) dr.$$

On the other hand we evaluate

$$\int_\Omega \langle df_*, F\rangle a(x)dx = N\omega_N \int_0^R f_*'(r)\varphi(r)a(r)r^{N-1}dr$$
$$= \int_0^R \varphi(r)a(r)r^{N-1}\left(\int_{S^{N-1}} \partial_r f(r,\theta)d\theta\right) dr.$$

Therefore we have

$$\int_\Omega \langle df, F\rangle a(x)dx = \int_\Omega \langle df_*, F\rangle a(x)dx \leq \mathbf{E}_r(\sigma).$$

By taking the supremum with respect to f it follows that $\mathbf{E}(\sigma) \leq \mathbf{E}_r(\sigma)$ and so the statement is proved.

To prove (*4*), let us suppose that σ minimizes the radial energy. For every $\nu = b(x)\mathcal{H}^N \llcorner \Omega$, $b \in L^1(\Omega)$, $\int_\Omega b(x)dx = 1$, we have

$$\mathbf{E}(\sigma^*) = \mathbf{E}_r(\sigma^*) = \mathbf{E}_r(\sigma) \leq \mathbf{E}_r(\nu) \leq \mathbf{E}(\nu).$$

To prove (*5*), let us suppose now that σ minimizes the energy $\mathbf{E}$. We have

$$\mathbf{E}_r(\sigma) \leq \mathbf{E}(\sigma) \leq \mathbf{E}(\nu^*) = \mathbf{E}_r(\nu^*) = \mathbf{E}_r(\nu).$$

Finally, (*6*) follows by (*4*) and (*5*). □

Although the force field F is in general not regular at the origin, we have the following

Theorem 3.2. *Let F be the vector field defined by* (21) *and assume that*

$$\frac{1}{r^{N-1}\varphi(r)} \in L^1(0,R). \tag{23}$$

Then the problem $\mathbb{P}(F,\Omega)$ admits minimizers among absolutely continuous measures.

Proof. Let $\sigma_n = a_n(|x|)\mathcal{H}^N \llcorner \Omega$ be a minimizing sequence for the radial energy $\mathbf{E}_r$. We can assume that, by passing to a subsequence, there exists a constant C such that, for every n, $\mathbf{E}_r(\sigma_n) \leq C$. Fix $\bar{x} \in B(0,R)$, $\bar{x} \neq 0$. For $\delta > 0$ small enough let us take the test functions

$$f_\delta(x) = \frac{1}{\delta}\chi_{\mathcal{C}_\delta}\left(|x| - |\bar{x}|\right),$$

where $\mathcal{C}_\delta = \{x \in \Omega \mid |\bar{x}| - \delta \leq |x| \leq |\bar{x}| + \delta\}$. Since $\mathbf{E}_r(\sigma_n) \leq C$, for every n we have

$$\frac{1}{\delta}\int_{\mathcal{C}_\delta} \varphi(|x|)a_n(|x|)dx = \int_{\mathcal{C}_\delta} \langle df_\delta, F\rangle d\sigma_n \leq C. \tag{24}$$

Let $t = |\bar{x}|$, by (24) we deduce:

$$\frac{1}{2\delta}\int_{t-\delta}^{t+\delta} \varphi(r)a_n(r)r^{N-1}dr \leq \frac{C}{2N\omega_N}\ . \tag{25}$$

By applying Lebesgue Differentiation Theorem we get

$$a_n(t) \le \frac{C}{2N\omega_N t^{N-1}\varphi(t)} \quad \mathcal{H}^1 - a.e.\ t. \tag{26}$$

Thanks to the condition (23), the inequality (26) is actually an equiintegrability condition for the sequence $a_n(r)$. Therefore there exists a function $a \in L^1(0,R)$ such that $a_n \rightharpoonup a$ in L^1. Let $\sigma = a(|x|)\mathcal{H}^N \llcorner \Omega \in \mathcal{P}(\overline{\Omega})$, we observe that $\sigma_n \overset{*}{\rightharpoonup} \sigma$ in the sense of measures. Fix a radial test function f such that $\|f\|_\infty \le 1$. By standard properties of the weak* convergence of measures (see for instance Proposition 1.62 of [2]), we get

$$\begin{aligned}\int_\Omega \langle df, F\rangle d\sigma &\le \int_\Omega |\langle df, F\rangle| a(|x|)dx \\ &\le \liminf_{n\to+\infty} \int_\Omega |\langle df, F\rangle| a_n(|x|)dx \le \liminf_{n\to+\infty} \mathbf{E}_r(\sigma_n).\end{aligned}$$

Taking the supremum with respect to f we find that σ is an optimal shape for the radial energy. The result follows by Proposition 3.1. □

Observe that in the case $\varphi(|x|) = |x|^p$ it may happen that $F(0) = 0$. In such a case of course the measure δ_0 is the only optimal shape. However the equiintegrability condition (26) ensures that there are also cases with minimizers among absolutely continuous measures. This is the case for example of $\varphi(|x|) = |x|^p$ with $p < 3-N$. Under the hypotheses of Theorem 3.2, the problem $\mathbb{P}(F, \overline{\Omega})$ has at least one radially symmetric solution, since we have used merely radially symmetric test functions in proving it. Furthermore, since F has only the radial component, we can certainly conclude that $\varphi(|x|)a(|x|) \in BV(\Omega)$. Therefore, if $0 < C < \varphi(|x|)$ or the function $\frac{1}{\varphi}$ is Lipschitz we have that $a(|x|) \in BV(\Omega)$.

However, the radial solution is not the only one. Indeed, one can take a function $b(\theta) \in L^1(\Omega)$ such that

$$\int_{S^{N-1}} b(\theta)d\theta = 1,$$

and consider the function $c(r,\theta) = N\omega_N a^*(r)b(\theta)$. Let $f(|x|)$ be any radially symmetric test function, by using polar coordinates we have

$$\begin{aligned}\int_\Omega \langle df, F\rangle c(x)dx &= N\omega_N \int_0^R f'(r)\varphi(r)a^*(r)r^{N-1}\left(\int_{S^{N-1}} b(\theta)d\theta\right)dr \\ &= N\omega_N \int_0^R f'(r)\varphi(r)a^*(r)r^{N-1}dr = \int_\Omega \langle df, F\rangle a^*(|x|)dx.\end{aligned}$$

Therefore $\sigma = c(x)\mathcal{H}^N \llcorner \Omega$ is also a minimizer for the radial energy $\mathbf{E}_r$. Now, we can observe that if the energy $\mathbf{E}$ admits only radial minimizers, then problem $\mathbb{P}(F, \overline{\Omega})$ is equivalent to minimize the radial energy $\mathbf{E}_r$, but does the energy $\mathbf{E}$ admit non radially symmetric minimizers?

Finally, we also notice that the radial behavior is crucial. Indeed, if a does not depend on the radial coordinate, then the function

$$a^*(r) = \fint_{S^{N-1}} a(\theta)d\theta$$

is in fact constant.

3.2. p-growth fields

In this section we deal with vector fields of the type $F = \nabla u$, with $u(x) = |x|^p$. In such a case the condition (23) implies $p < 3 - N$ and with this choice there exists a minimum for $\mathbb{P}(F, \overline{\Omega})$ among all absolutely continuous measures. We have

$$F(x) = p|x|^{p-2}x, \qquad \mathrm{div}F(x) = p(p+N-2)|x|^{p-2}.$$

For $p > 2 - N$ and for every $r > 0$, let us consider the measures

$$\sigma_r = \frac{1}{|B(0,r)|}\mathcal{H}^N \llcorner B(0,r).$$

By (4) we infer

$$\begin{aligned} \mathbf{E}(\sigma_r) &= \frac{|p|(p+N-2)}{\omega_N r^N}\int_{B(0,r)}|x|^{p-2}dx + \frac{|p|}{\omega_N r^N}\int_{\partial B(0,r)}|x|^{p-2}|x\cdot\nu|\; d\mathcal{H}^{N-1} \\ &= \frac{|p|(p+N-2)}{\omega_N r^N}N\omega_N\int_0^r \rho^{p-2}\rho^{N-1}d\rho + \frac{|p|}{\omega_N r^N}\omega_N N r^{p-1}r^{N-1} \\ &= \frac{N|p|r^{p+N-2}}{r^N} + N|p|r^{p-2} = 2N|p|r^{p-2}. \end{aligned} \tag{27}$$

Therefore, if $p > 2 > 2 - N$ we have that $m(F, \overline{\Omega}) = 0$ and then there is no absolutely continuous minimizing measure. In particular, this circumstance justifies the integrability assumption (23). On the other hand, we observe that if $p < 2 - N$ then $\mathbf{E}(\sigma_r) = +\infty$. Nevertheless, in this case, Theorem 3.2 ensures that there exists at least one absolutely continuous minimum. Notice that for $p = 2$ the energy of the ball does not depend on the radius and we are going to study the details of such a situation in the next subsection.

3.3. The case $p = 2$

Let us take $p = 2$, then $F(x) = 2x$, while $\mathrm{div}F(x) = 2N$. Of course, the measure δ_0 is an optimal shape. By putting a discontinuity at the origin, by Lemma 2.6 we know that the optimal shapes, if they exist, are absolutely continuous measures in $\overline{\Omega}\setminus\{0\}$. Therefore we need to check that, by defining $F(0) = v$, $\;v \neq 0$, it results $\sigma(\{0\}) = 0$. Indeed, let $\mathbf{E}(\sigma) \le C$, and suppose for instance that $v_1 > 0$. If $B_r = B(0,r)\cap\{x_1 \ge 0\}$, by taking the test functions

$$f_r(x) = \frac{1}{r}\chi_{B_r}(x)x_1,$$

we can evaluate:

$$\frac{1}{r}v_1\sigma(\{0\}) + \frac{2}{r}\int_{B(0,r)\cap\{x_1>0\}} x_1 d\sigma = \int_{B(0,r)\cap\{x_1\ge 0\}}\langle df_r, F\rangle d\sigma \le \mathbf{E}(\sigma) \le C.$$

Hence we infer that

$$v_1\sigma(\{0\}) \le Cr.$$

Letting $r \to 0^+$ we conclude that $\sigma(\{0\}) = 0$ and therefore σ is absolutely continuous and so $\mathbb{P}(F, \overline{\Omega})$ has absolutely continuous solutions.

For measures concentrated on sets of finite perimeter by (4) we have:

$$\mathbf{E}(\sigma) = \frac{1}{|E|}\left(\int_E |\mathrm{div}F(x)|\,dx + \int_{\partial E} |F\cdot\nu_E|\,d\mathcal{H}^{N-1}\right) = 2N + \frac{2}{|E|}\int_{\partial E}|x\cdot\nu_E|\,d\mathcal{H}^{N-1}.$$

Hence, if we put the volume constraint $|E| = 1$, the problem of determining the optimal shapes carried by F inside $\overline{\Omega}$ becomes the following one:

$$m_{|E|=1}(F,\overline{\Omega}) := \min_{|E|=1}\int_{\partial E}|x\cdot\nu_E|\,d\mathcal{H}^{N-1}, \tag{28}$$

which is a Wulff problem for the energy density $\Gamma(x,\nu_E(x)) = |x\cdot\nu_E(x)|$. Let us remark that the energy Γ does not satisfy the usual assumptions asked for the classical Wulff problem as in [10, 13, 20]. However, in the present situation, one can make a direct computation. Indeed, by using the Gauss-Green formula we obtain:

$$\int_{\partial E}|x\cdot\nu_E|\,d\mathcal{H}^{N-1} \geq \int_{\partial E} x\cdot\nu_E\,d\mathcal{H}^{N-1} = \int_E \mathrm{div}(x)\,dx = N.$$

Now, let $B_1 := B(0,r)$ be the ball of volume 1. Then

$$\int_{\partial B_1}|x\cdot\nu_{B_1}|\,d\mathcal{H}^{N-1} = \int_{\partial B_1}|x|\,d\mathcal{H}^{N-1} = rN\omega_N r^{N-1} = N.$$

Therefore the ball B_1 solves problem $\mathbb{P}_{|E|=1}(F,\overline{\Omega})$. Moreover, since the energy of the ball does not depend on the radius, any other measure like $\sigma = \frac{1}{|B(0,r)|}\mathcal{H}^N \llcorner B(0,r)$ minimizes $\mathbb{P}_{|E|=1}(F,\overline{\Omega})$ as well. However, we claim that

$$m(F,\overline{\Omega}) < m_{|E|=1}(F,\overline{\Omega}). \tag{29}$$

To prove (29) let us consider the radial function $a(|x|) = \frac{C}{|x|}$ and the measure $\sigma = a(|x|)\mathcal{H}^N \llcorner \Omega$. By Proposition 3.1, taking radial test functions f and since $F(x) = 2x$ we have

$$\begin{aligned}\mathbf{E}(\sigma) &= \sup_{\|f\|_\infty\leq 1} 2N\omega_N\int_0^R f'(r)a(r)r^N dr = \sup_{\|f\|_\infty\leq 1} 2NC\omega_N\int_0^R f'(r)r^{N-1}dr\\ &= \sup_{\|f\|_\infty\leq 1} 2NC\omega_N\left\{f(R)R^{N-1} - \int_0^R f(r)(N-1)r^{N-2}dr\right\}.\end{aligned}$$

Taking the supremum we obtain

$$\begin{aligned}\mathbf{E}(\sigma) &= 2NC\omega_N R^{N-1} + 2NC\omega_N(N-1)\int_0^R r^{N-2}dr\\ &= 2NC\omega_N R^{N-1} + 2NC\omega_N R^{N-1} = 4CN\omega_N R^{N-1}.\end{aligned}$$

Let us now choose the constant C in order to have a probability measure, then we get

$$\int_\Omega a(|x|)dx = 1 \Rightarrow CN\omega_N\int_0^R r^{N-2}dr = 1 \Rightarrow C = \frac{N-1}{N\omega_N R^{N-1}}\ .$$

Therefore

$$\mathbf{E}(\sigma) = 4(N-1) < 4N = \mathbf{E}(\sigma_R),$$

where

$$\sigma_R = \frac{1}{|B(0,R)|}\mathcal{H}^N \llcorner B(0,R).$$

Finally, we observe that if one restricts to consider BV functions and requires the following generalized flow condition

$$\int_{\partial\Omega} aF \cdot \nu \; d\mathcal{H}^{N-1} = 2N, \tag{30}$$

recalling that $\int_\Omega a \; dx = 1$ and that $\mathrm{div}F = 2N$, then condition (30) implies

$$\int_\Omega \nabla a \cdot F \; dx + \int_\Omega F \cdot d(D_s a) = -\int_\Omega a \; \mathrm{div}F \; dx + \int_{\partial\Omega} aF \cdot \nu \; d\mathcal{H}^{N-1} = 0.$$

Hence, for every $\sigma = a(x)\mathcal{H}^N \llcorner \Omega$ we have the estimate

$$\begin{aligned}\mathbf{E}(\sigma) = &\int_\Omega |a \; \mathrm{div}F + \nabla a \cdot F| \; dx + \|F \cdot D_s a\| + \int_{\partial\Omega} a|F \cdot \nu| \; d\mathcal{H}^{N-1} \\ &\geq 2N + \int_{\partial\Omega} aF \cdot \nu \; d\mathcal{H}^{N-1} = 4N.\end{aligned}$$

Therefore we can conclude that the ball is optimal among all BV functions with the same fixed flow through Ω.

Let us remark that σ_R is not in general an optimal measure. To see this let us consider the case

$$2 - N < p \leq 1.$$

By taking the radial function $a(|x|) = C|x|^\alpha$, with $\alpha = -(p+N-2)$, for $\sigma = a(|x|)\mathcal{H}^N \llcorner \Omega$ and radial test functions f, recalling that $F(x) = p|x|^{p-2}x$, we obtain

$$\begin{aligned}\mathbf{E}(\sigma) &= \sup_{\|f\|_\infty \leq 1} N\omega_N Cp \int_0^R f'(r) r^{p-1} r^\alpha r^{N-1} dr \\ &= \sup_{\|f\|_\infty \leq 1} N\omega_N Cp \int_0^R f'(r) dr = \sup_{\|f\|_\infty \leq 1} N\omega_N Cp \left(f(R) - f(0)\right).\end{aligned}$$

Taking the supremum we get

$$\mathbf{E}(\sigma) = 2N\omega_N C|p|.$$

Let us now choose the constant C in order to have a probability measure.

$$\int_\Omega a(|x|) dx = 1 \Rightarrow CN\omega_N \int_0^R r^{-(p+N-2)+N-1} dr = 1 \Rightarrow C = \frac{2-p}{N\omega_N} R^{p-2}.$$

Finally, by taking into account (27), we can conclude that

$$\mathbf{E}(\sigma) = 2|p|(2-p)R^{p-2} < 2|p|NR^{p-2} = \mathbf{E}(\sigma_R).$$

3.4. The solenoidal case

Let us now come to consider here the case of energies with only the surface term. According to (4), this is obtained, in this framework, by assuming that the vector field F is solenoidal, that is div$F = 0$. Let $u(x) = \log|x|$ when $N = 2$, $u(x) = \frac{1}{|x|^{N-2}}$ when $N \geq 3$, so that div$F = \Delta u = 0$. Among the sets of finite perimeter, the problem of optimal shapes takes the form:

$$\text{Minimize } \left\{ \frac{1}{|E|} \int_{\partial E} \frac{\varphi(|x|)}{|x|} |x \cdot \nu_E| \; d\mathcal{H}^{N-1} \mid E \subset \overline{\Omega},\ E \text{ of finite perimeter} \right\}.$$

Since we have no bulk term, by (27) we get

$$\mathbf{E}(\sigma_R) = 2N|p|R^{p-2} = \frac{2N(N-2)}{R^N}. \tag{31}$$

In the case $N = 2$ we have

$$\mathbf{E}(\sigma_R) = \frac{2}{R^2}.$$

By virtue of the arguments discussed in Section 3, we have a radial solution which turns out to be a BV function. If $a \in BV(\Omega)$ is a radial function and $\sigma = a\mathcal{H}^N \llcorner \Omega$, taking a radial test function f, since $\varphi(r) = (2-N)r^{1-N}$, we have

$$\begin{aligned} \int_\Omega \langle df, F \rangle a(x) dx &= N(2-N)\omega_N \int_0^R f'(r)\varphi(r) r^{N-1} a(r) dr \\ &= N(2-N)\omega_N \int_0^R f'(r) a(r) dr. \end{aligned}$$

Finally, integrating by parts, we obtain

$$\begin{aligned} \mathbf{E}_r(\sigma) &= \sup_{\|f\|_\infty \leq 1} N(2-N)\omega_N \left\{ -\int_0^R f d(Da) + f(R)a(R_+) - f(0)a(0_-) \right\} \\ &= N(N-2)\omega_N \left\{ \|Da\|(0,R) + a(R_+) + a(0_-) \right\}. \end{aligned}$$

Therefore, by the above expression of the energy and taking into account Proposition 3.1, we can conclude that σ_R is an optimal shape.

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