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Singularities for a Class of Non-Convex Sets and Functions, and Viscosity Solutions of some Hamilton-Jacobi Equations

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We study nondifferentiability points for a class of continuous functions $f : \mathbb{R}^N \to \mathbb{R}$ whose epigraph satisfies a kind of external sphere condition with uniform radius (called φ-convexity or proximal smoothness). The functions belonging to this class are not necessarily Lipschitz. However, they enjoy some properties analogous to semiconvex functions; in particular they are twice $\mathcal{L}^N$-a.e. differentiable (see [11]). In partial analogy with the study of singularities of semiconcave functions (see [7]), under suitable conditions we give estimates from below of the nondifferentiability set, which consists of points where the subdifferential is not a singleton, as well as (differently from semiconvex functions) of points where it is empty. Furthermore, we show that if a function in this class is an a.e. solution of a Hamilton-Jacobi equation, then under suitable assumptions it is actually a viscosity solution. Methods of nonsmooth analysis and geometric measure theory are used, including a representation of Clarke's generalized gradient as the closed convex hull of limits of Fréchet derivatives.

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1. Introduction

Let $g(\rho) = \operatorname{sign}(\rho - 1) \cdot \sqrt{|\rho - 1|}$, $\rho \in \mathbb{R}$, and set

$$f(x, y) = g(|x| + |y|). \tag{1}$$

The function f is nonsmooth, because: a) there are points where the epigraph $\operatorname{epi}(f)$ has corners (the set $\{xy = 0\}$); and b) there are points where $\operatorname{epi}(f)$ has vertical tangent plane (the set $\{|x| + |y| = 1, xy \neq 0\}$), i.e., both its subdifferential and superdifferential are empty. Consider also the following minimum time problem: reach the target

$$S = \{(x, y) \in \mathbb{R}^2 \, : \, y \geq \operatorname{sign}(x)x^2\},$$

subject to the dynamics

$$\begin{cases} \dot{x} = u \in [-1, 1] \\ \dot{y} = 0. \end{cases}$$

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The minimum time function, for $(x,y) \notin S$, is

$$T(x,y) = x - \operatorname{sign}(y)\sqrt{|y|}, \tag{2}$$

which is not differentiable at all points of the positive x-axis, where both subdifferential and superdifferential are empty.

The functions f and T in the above example feature a common property. Indeed, their epigraphs satisfy a kind of uniform external sphere condition (see Definition 3.1 below), which is known in the literature under different names: *positive reach* (see [15, §4]), *proximal smoothness* (see [10] and references therein), or *φ-convexity* (see, e.g., [11] and references therein). Convex and semiconvex functions satisfy this condition, but, as the above examples show, there exist other cases as well, which may be relevant in applications. In particular, it was shown in [12] that the minimum time function, for a class of linear control problems with a convex target, has φ-convex epigraph.

This property is close to semiconvexity. In particular – among locally Lipschitz functions – semiconvexity is characterized exactly by the φ-convexity of the epigraph. However, as the above examples show, functions with φ-convex epigraph need not be locally Lipschitz. Nevertheless, it turns out that such functions enjoy rather strong regularity properties, including a.e. differentiability (see [11]) and second order Taylor expansion around a.e. point of the domain. In this paper we perform a study of singularities (i.e., of nondifferentiability points) of functions with φ-convex epigraph, in analogy to previous results valid for semiconvexity/semiconcavity which are described, e.g., in Chapter 4 of the book [7] (see also [1]). In particular, we are concerned with *propagation* of singularities, namely with finding sufficient conditions in order that at least a Lipschitz curve all made of singularities emanates from a point of nondifferentiability. Differently from the semiconvex case, due to the lack of Lipschitz continuity, singularities may be of two different natures, i.e., either corners or "vertical spots" (i.e., points where both the sub and the superdifferential are empty) in the epigraph. Therefore, both the type of results and the techniques appearing in [7] need adaptations. In particular, we prove new representation formulas for the normal cone and the Clarke generalized gradient as convex hull of limiting cones, in analogy with the well known equality between the generalized gradient of a Lipschitz function and the convex hull of limits of Fréchet derivatives. This result requires the assumption that a normal cone contains no lines, and the necessity of this requirement is discussed with an example. The representation formula is not entirely straightforward, because, due to the lack of Lipschitz continuity, sequences of Fréchet derivatives may be unbounded. As an application, using these representations we prove that a continuous function with φ-convex hypograph and no cusps, which satisfies a.e. a Hamilton-Jacobi equation of the type

$$H(x,u,Du) = 0,$$

is actually a viscosity solution, provided H is convex in Du and continuous. As a consequence, a generalization of the classical uniqueness result for semiconcave solutions by Kruzhkov (see [18]) is obtained. The propagation of singularities is divided into two parts. In the first one we deal with points where the normal cone has dimension > 1, and the result is similar to the case of semiconcave functions (see [7, §4.2]). The second one is concerned with points of the epigraph where the tangent cone is vertical. We study the propagation for the particular case of a $\mathcal{C}^{1,1}$-graph, and then we apply our analysis to general functions with φ-convex epigraph, exploiting the fact that r-level sets of the distance from the epigraph are locally the graph of a $\mathcal{C}^{1,1}$-function, provided r is sufficiently small.

Sects. 2 and 3 are concerned with a brief survey of some concepts in nonsmooth analysis and of properties of φ-convexity. Sect. 4 contains results of nonsmooth analysis, including the representation formula for the normal cone, while Sect. 5 is concerned with viscosity solutions. The last section contains results on the propagation of both types of singularities, i.e., corners and vertical spots.

2. Preliminaries

Throughout this paper our main tools are some concepts from nonsmooth analysis.

Let $K \subseteq \mathbb{R}^N$ be closed. We denote by $\operatorname{bdry} K$ the topological boundary of K, and, for $x \in \mathbb{R}^N$,

$$\begin{aligned} d_K(x) &= \min\{|y-x| \,:\, y \in K\} && \text{(the distance of } x \text{ from } K)\\ \pi_K(x) &= \{y \in K \,:\, |y-x| = d_K(x)\} && \text{(the projections of } x \text{ onto } K). \end{aligned}$$

Moreover, we set

$$\begin{aligned} \operatorname{Unp}(K) &= \{x \in \mathbb{R}^N \,:\, \pi_K(x) \text{ is a singleton}\}\\ K_\rho &= \{x \in \mathbb{R}^N \,:\, d_K(x) \le \rho\} \end{aligned}$$

For a convex set C, its relative boundary is denoted by $\operatorname{relbd} C$. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at x, we denote by $Df(x)$ its gradient at x.
The following concepts of normals and tangents will be used (see [9, Ch. 1] and [23, Ch. 6]). Let $x \in K$ and $v \in \mathbb{R}^N$. We say that:

1. v is a *proximal normal* to K at x (and will be denoted by $v \in N_K^P(x)$) if there exists $\sigma = \sigma(v,x) \ge 0$ such that:
$$\langle v, y-x\rangle \le \sigma|y-x|^2 \quad \text{for all } y \in K; \tag{3}$$
equivalently $v \in N_K^P(x)$ iff there exists $\lambda > 0$ such that $\pi_K(x+\lambda v) = \{x\}$;
2. v is a *Fréchet normal* (or Bouligand normal) to K at x ($v \in N_K^F(x)$) if
$$\limsup_{K \ni y \to x} \langle v, \frac{y-x}{|y-x|}\rangle \le 0;$$
3. v is a *limiting normal* to K at x ($v \in N_K^L(x)$) if
$$v \in \{w :\, w = \lim w_n,\, w_n \in N_K^P(x_n),\, x_n \to x\}$$
and is a *Clarke normal* ($v \in N_K^C(x)$) if $v \in \overline{co} N_K^L(x)$;
4. v is a *Fréchet tangent* (or Bouligand tangent) to K at x ($v \in T_K^F(x)$) if
$$\liminf_{h \to 0^+} \frac{d_K(x+hv)}{h} = 0;$$
equivalently, $0 \ne v \in T_K^F(x)$ iff there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset K$ such that
$$\lim_{n\to\infty} \frac{y_n - x}{|y_n - x|} = \frac{v}{|v|}.$$

It can be proved (see [2, Prop. 4.4.1]) that

$$N_K^F(x) = \{v \in \mathbb{R}^N : \langle v, w\rangle \le 0 \text{ for all } w \in T_K^F(x)\} := (T_K^F(x))^0. \tag{4}$$

Let $f : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. By using $\mathrm{epi}(f) := \{(x,\xi) : \xi \ge f(x)\}$ and $\mathrm{hypo}(f) := \{(x,\xi) : \xi \le f(x)\}$ one can define generalized gradient concepts for f at $x \in \mathrm{dom}(f) = \{x \in \mathbb{R}^N : f(x) < +\infty\}$. Let $x \in \mathrm{dom}(f)$, $v \in \mathbb{R}^N$. We say that:

1. v is a *proximal subgradient* of f at x ($v \in \partial_P f(x)$) if $(v,-1) \in N^P_{\mathrm{epi}(f)}(x, f(x))$; equivalently (see [9, Theorem 1.2.5]), $v \in \partial_P f(x)$ iff there exist σ, $\eta > 0$ such that
$$f(y) \ge f(x) + \langle v, y-x\rangle - \sigma|y-x|^2 \quad \text{for all } y \in B(x,\eta) \cap \mathrm{dom}\,(f); \tag{5}$$
2. v is a *proximal supergradient* of f at x ($v \in \partial^P f(x)$) if $(-v,1) \in N^P_{\mathrm{hypo}(f)}(x, f(x))$; equivalently $v \in \partial^P f(x)$ iff $-v \in \partial_P(-f)(x)$, i.e., iff there exist σ, $\eta > 0$ such that
$$f(y) \le f(x) + \langle v, y-x\rangle + \sigma|y-x|^2 \quad \text{for all } y \in B(x,\eta) \cap \mathrm{dom}\,(f); \tag{6}$$
3. v is a *Fréchet subgradient* of f at x ($v \in \partial_F f(x)$) if $(v,-1) \in N^F_{\mathrm{epi}(f)}(x, f(x))$, i.e.,
$$\liminf_{y\to x} \frac{f(y) - f(x) - \langle v, y-x\rangle}{|y-x|} \ge 0;$$
4. v is a *Fréchet supergradient* of f at x ($v \in \partial^F f(x)$) if $(-v,1) \in N^F_{\mathrm{hypo}(f)}(x, f(x))$;
5. v is a *limiting subgradient* of f at x ($v \in \partial_L f(x)$) if $(v,-1) \in N^L_{\mathrm{epi}(f)}(x, f(x))$.
6. v is a *limiting supergradient* of f at x ($v \in \partial^L f(x)$) if $(-v,1) \in N^L_{\mathrm{hypo}(f)}(x, f(x))$.
7. v is a *Clarke generalized gradient* of f at x ($v \in \partial f(x)$) if $(v,-1) \in N^C_{\mathrm{epi}(f)}(x, f(x))$. We recall that *if f is Lipschitz continuous in a neighborhood of x*, then $v \in \partial f(x)$ if and only if $v \in \overline{\mathrm{co}}\{\zeta : \zeta = \lim Df(x_i), x_i \in \mathrm{dom}(Df), x_i \to x\}$ (see [9, Theorem 8.1]).

3. Review of φ-convex sets and functions

We introduce now the class of sets which is analyzed in the present paper (see also [11] and references therein).

Definition 3.1. Let $K \subset \mathbb{R}^N$ be closed and let $\varphi : K \to [0,+\infty)$ be continuous. We say that K is φ-convex if for all $x, y \in K$, $v \in N_K^P(x)$, the inequality

$$\langle v, y-x\rangle \le \varphi(x)|v|\,|y-x|^2 \tag{7}$$

holds. By φ_0-convexity we mean φ-convexity with $\varphi \equiv \varphi_0$, a constant.

Of course, the case $\varphi_0 = 0$ is equivalent to convexity.

The distance from a φ-convex set K and the metric projection onto K enjoy remarkable properties, which are fundamental for our analysis.

Theorem 3.2. *Let $K \subset \mathbb{R}^N$ be a φ-convex set. Then there exists an open set $U \supset K$ such that*

(*1*) *$d_K \in \mathcal{C}^{1,1}(U \setminus K)$ and $Dd_K(y) = \frac{y-\pi_K(y)}{d_K(y)}$ for every $y \in U \setminus K$;*

(*2*) *$Unp(K) \supset U$ and $\pi_K : U \to K$ is locally Lipschitz. Moreover, $\pi_K : U \setminus K \to \operatorname{bdry} K$ is onto.*

In particular, if K is φ_0-convex (with $\varphi_0 > 0$), then $U \supset K_{\frac{1}{4\varphi_0}}$ and $\pi_K : K_{\frac{1}{4\varphi_0}} \to K$ is Lipschitz with Lipschitz ratio 2.

Proof. The proof can be found in [6, Proposition 2.6, 2.9, Remark 2.10] or in [15, §4]. $\square$

Remark 3.3. Conditions (*1*) and (*2*) in Theorem 3.2 are actually equivalent to φ-convexity, as it is proved, e.g., in [15, §4].

Corollary 3.4. *Let $K \subset \mathbb{R}^N$ be φ-convex. Let*

$$K_\varphi = \{x \,:\, 4d_K(x)\varphi(\pi_K(x)) < 1\}.$$

Assume that $\varphi(x) > 0$ for all $x \in K$. Then the set

$$\operatorname{bdry} K_\varphi = \{x \in \mathbb{R}^N \,:\, 4d_K(x)\varphi(\pi_K(x)) = 1\}$$

is a $\mathcal{C}^{1,1}$-manifold. Moreover, for all $x \in \operatorname{bdry} K_\varphi$,

$$N^P_{K_\varphi}(x) = \mathbb{R}^+(x - \pi_K(x)) \subseteq N^P_K(\pi_K(x)). \tag{8}$$

Proof. $\operatorname{bdry} K_\varphi$ is a $\mathcal{C}^{1,1}$-manifold because $|Dd_K| \equiv 1$ on $\operatorname{bdry} K_\varphi$. Formula (8) is Corollary 4.15(2) in [10]. $\square$

The concept introduced above has other consequences, among which we mention:

Proposition 3.5. *Let $K \subset \mathbb{R}^N$ be φ-convex. Then:*

(*1*) *for all $x \in K$, $N^P_K(x) = N^F_K(x) = N^C_K(x)$ (and therefore it never reduces to $\{0\}$) and the set-valued map N^P_K from K into $\mathbb{R}^N$ has closed graph; moreover, $T^F_K(x) = (N^C_K(x))^0$;*

(*2*) *for all $x_i \in K$, $v_i \in N^P_K(x_i)$, $i = 1, 2$, it holds:*

$$\langle v_2 - v_1, x_2 - x_1\rangle \geq -(\varphi(x_1)|v_1| + \varphi(x_2)|v_2|)|x_2 - x_1|^2;$$

(*3*) *for $\mathcal{H}^{N-1}$-a.e. $x \in \operatorname{bdry} K$ there exists $v_x \in \mathbb{R}^N$, $|v_x| = 1$, such that*

$$N^P_K(x) \subseteq \mathbb{R}v_x$$

(here $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure);

(*4*) *$\operatorname{bdry} K$ is countably $\mathcal{H}^{N-1}$-rectifiable and K has locally finite perimeter in $\mathbb{R}^N$.*

Proof. The proof of (*1*) can be found, e.g., in [13, Propositions 6.2 and 4.2] and [15, Theorem 4.8 (12)]. Statement (*2*) is an immediate consequence of the definition of φ-convex sets, while (*3*) can be found in [15, §4] or [11, Corollary 4.1]. Finally, (*4*) is Theorem 4.2 in [11]. $\square$

As examples show (see Example 4.1 in [11]), it may occur that $N^P_K(x) = \mathbb{R}v$ for a subset of $\operatorname{bdry} K$ of positive $\mathcal{H}^{N-1}$-measure. In order to avoid such problems, we introduce the following definition (see [11, Definition 4.1]).

Definition 3.6. Let $K \subset \mathbb{R}^N$ be φ-convex. We say that K is nondegenerate if

$$\mathcal{H}^{N-1}\big\{x \in \operatorname{bdry} K \,:\, N_K^P(x) \text{ contains a line}\big\} = 0.$$

Remark 3.7. If K is φ-convex and nondegenerate, then for $\mathcal{H}^{N-1}$-a.e. $x \in \operatorname{bdry} K$ there exists v_x, $|v_x| = 1$ such that

$$N_K^P(x) = \mathbb{R}^+ v_x. \tag{9}$$

Observe also that, recalling Proposition 3.5(*3*), the nondegeneracy condition can be substituted by

$$\mathcal{H}^{N-1}\big\{x \in \operatorname{bdry} K \,:\, N_K^P(x) \text{ is a line}\big\} = 0.$$

Let now $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be continuous. In the following we will mostly consider functions f with the further assumption that

$$\operatorname{epi}(f) \text{ is } \varphi\text{-convex.}$$

Of course, if f is convex then epi(f) is φ-convex. An important class of nonsmooth and nonconvex functions with φ-convex epigraph is that of semiconvex functions.

Definition 3.8. Let Ω be an open and convex subset of $\mathbb{R}^N$. A function $f : \Omega \to \mathbb{R}$ is semiconvex in Ω if there exists a constant $C \geq 0$ such that for all $x_1, x_2 \in \Omega$, $\lambda \in [0, 1]$ it holds

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) + C\lambda(1-\lambda)|x_1 - x_2|^2. \tag{10}$$

It can be proved that every semiconvex function has φ_0-convex epigraph, see, e.g., [7, §3.6], where the symmetric class of semiconcave functions is thoroughly studied. Moreover, among locally Lipschitz functions semiconvexity is characterized exactly by the φ-convexity of the epigraph.

Functions with φ-convex epigraph were studied in [11], which in particular contains examples and some regularity properties, and in [4, 19]. Observe that the φ-convexity of epi(f) does not imply that f is locally Lipschitz. Actually, the epigraph of the functions f in (1) and T in (2) is φ_0-convex for a suitable φ_0, but neither f nor T are locally Lipschitz.

Concerning the regularity of functions with φ-convex epigraph, we observe first that $\partial_P f(x) = \partial f(x)$ for all $x \in \Omega$, both sets being possibly empty. The main result on the regularity of functions with φ-convex epigraph is (see [11, Theorem 5.1, Corollary 6.1])

Theorem 3.9. *Let $\Omega \subset \mathbb{R}^N$ be open, and let $f : \overline{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous, and such that* epi(f) *is φ-convex. Then there exists a sequence of sets $\Omega_h \subseteq \Omega$ such that Ω_h is compact in* dom(f) *and*

(*1*) *the union of Ω_h covers $\mathcal{L}^N$-almost all* dom(f), *i.e.,*

$$\mathcal{L}^N\left(\operatorname{dom}(f) \setminus \bigcup_h \Omega_h\right) = 0; \tag{11}$$

(*2*) *for all $x \in \bigcup_h \Omega_h$ there exist $\delta = \delta(x) > 0$, $L = L(x) > 0$ such that*

$$f \textit{ is Lipschitz on } B(x,\delta) \textit{ with ratio } L; \tag{12}$$

(*3*) *for all* $x \in \bigcup_h \Omega_h$

$$f \text{ is (strictly) Fréchet differentiable at } x; \tag{13}$$

in particular, for $\mathcal{H}^{N-1}$*-a.e.* $x \in \Omega$ *there exists* $v_x \in \mathbb{R}^{N+1}$, $|v_x| = 1$, *such that* $\langle v_x, e_{N+1}\rangle < 0$ *and*

$$N^P_{\mathrm{epi}(f)}(x, f(x)) = \{\lambda v_x : \lambda \geq 0\}; \tag{14}$$

(*4*) f *is a.e. twice differentiable in* $\bigcup_h \Omega_h$.

Remark 3.10. The differentiability result goes through showing that at every point x in $\bigcup_h \Omega_h$ the proximal subgradient $\partial f(x)$ is a singleton, and next that this fact implies that f is differentiable at x. The set Σ of nondifferentiability points can therefore be expressed as the union of the sets Σ_k where $\partial_P f$ has dimension $\geq k$, $k = 1, \ldots, N$, together with $\Sigma_\infty = \{x : \partial_P f(x) = \emptyset\}$. It can be shown (see [15, Remark 4.15]) that Σ_k is countably $N-k$-rectifiable, $k = 1, \ldots N$, while the set Σ_∞, although being of N-dimensional Lebesgue measure zero, does not admit lower dimensional estimates (see [11, Example 5.2]).

The next result of this section will be used in Sect. 5.

Proposition 3.11. *Let* $f : \Omega \to \mathbb{R}$ *be a continuous function with* φ*-convex epigraph. Let* $x \in \Omega$ *such that* $\partial^F f(x) \neq \emptyset$. *Then* f *is (strictly) differentiable at* x *and there exists a neighborhood of* x *where* f *is semiconvex.*

Proof. We prove the differentiability of f at x. The strict differentiability and the semiconvexity follow from [11, Claim 4 in the proof of Theorem 5.1 and Theorem 6.1]. It is sufficient, according to [7, Proposition 3.1.5], to show that $N^P_{\mathrm{epi}(f)}(x, f(x))$ contains a vector ζ such that $\langle \zeta, e_{n+1}\rangle < 0$, i.e., $\partial_P f(x) \neq \emptyset$. By contradiction, assume that $N^P_{\mathrm{epi}(f)}(x, f(x)) \subseteq \mathbb{R}^N \times \{0\}$, and let $z \in \mathbb{R}^N$, $|z| = 1$, be such that $(z, 0) \in N^P_{\mathrm{epi}(f)}(x, f(x))$. Let $v \in \partial^F f(x)$ and Φ be a C^1 function in a neighborhood of x such that $D\Phi(x) = v$ and Φ touches f from above at $(x, f(x))$ (cp. [7, Definition 3.1.6, Proposition 3.1.7]). In particular, for all y sufficiently near to x, we have $f(y) \leq \Phi(y)$. According to the properties of $(z, 0)$, there exists $r > 0$ such that

$$\mathrm{epi}(f) \cap B((x, f(x)) + r(z, 0), r) = \emptyset.$$

However, for $h > 0$ sufficiently small we have that

$$(x + hz, \Phi(x + hz)) \in \mathrm{epi}(f) \cap B((x, f(x)) + r(z, 0), r),$$

which leads to a contradiction. □

Finally, we show that the epigraph of f is nondegenerate, provided f is continuous.

Proposition 3.12. *Let* $\Omega \subseteq \mathbb{R}^N$ *be open and let* $f : \Omega \to \mathbb{R}$ *be continuous with* φ*-convex epigraph. Then epi*(f) *is nondegenerate.*

Proof. We want to show that the set

$$E := \{(x, f(x)) \in \mathrm{graph}(f) : N_{\mathrm{epi}(f)}(x, f(x)) \text{ contains a line}\}$$

has $\mathcal{H}^N$-measure zero. To this aim, observe first that it is easy to see that if $f:\Omega\to\mathbb{R}$ is continuous, then

$$\operatorname{graph}(f)=\operatorname{bdry} epi(f).$$

Recalling Proposition 3.5, part *(4)*, we obtain that $\operatorname{graph}(f):=G$ is countably $\mathcal{H}^N$-rectifiable and hence also E is countably $\mathcal{H}^N$-rectifiable. By Theorem 3.9, for $\mathcal{H}^N$-a.e. $x\in\Omega$ the classical tangent hyperplane to G at $(x,f(x))$, $\mathrm{Tan}_{\mathrm{graph}(f)}(x)$, exists and is not vertical. Moreover, for $\mathcal{H}^N$-a.e. $x\in\Omega$, $\mathrm{Tan}_{\mathrm{graph}(f)}(x)$ coincides with the approximate tangent space to G at $(x,f(x))$ (see [16, Theorem 3.2.19]). Consider now the Lipschitz map $\pi:\mathbb{R}^{N+1}\to\mathbb{R}^N$, $\pi(x,y)=x$. By Theorem [16, 3.2.19], for $\mathcal{H}^N$-a.e. $x\in\Omega$ the restriction of π to the N-dimensional affine space $x+\mathrm{Tan}_{\mathrm{graph}(f)}(x)$ is differentiable at x. Denote by $\nabla^G\pi(x)$ the corresponding matrix and observe that it has $\mathcal{H}^N$-a.e. rank N, since the tangent hyperplane $\mathrm{Tan}_{\mathrm{graph}(f)}(x)$ is not vertical. Therefore the N-dimensional area element

$$J_N^G\pi(x)=\sqrt{\det\nabla^G\pi(x)(\nabla^G\pi(x))^T}$$

is $\mathcal{H}^N$-a.e. not zero. By the area formula (see, e.g., [16, 3.2.22]), observing that the restriction of π to $\operatorname{graph}(f)$ is $1-1$, we have

$$\mathcal{H}^N(\pi(E))=\int_E J_N^G\pi(x)\,d\mathcal{H}^N(x).$$

By Theorem 3.9, the lefthand side of the above expression is zero. Since $J_N^G\pi(x)$ is $\mathcal{H}^N$-a.e. not zero, then $\mathcal{H}^N(E)=0$. The proof is concluded. $\square$

4. Representation of normal cone and subdifferential

We start our analysis of the set of the points on the boundary of a φ-convex set where the dimension of the normal cone N_K^P is strictly greater than 1. Our aim is giving a characterization in terms of the closure of the convex hull of limits of normals at nearby points where N_K^P has dimension 1. In all this section K will be a φ-convex subset of $\mathbb{R}^N$, mostly nondegenerate.

Proposition 3.5, part *(3)* allows to give the following definition, which is in the spirit of [7, Definition 3.1.10] (see also [9, Theorem 8.1]).

Definition 4.1. We say that $p\in\mathbb{R}^N$, $p\neq 0$, is a reachable normal of K at $x\in K$ if there exists a sequence $\{(x_n,v_n)\}_{n\in\mathbb{R}^N}\in K\times\operatorname{bdry}B(0,1)$ such that $x_n\to x$, $x_n\neq x$, $v_n\to p/|p|$ and $N_K^P(x_n)=\{\lambda v_n:\lambda\geq 0\}$. In this case, we will write $p\in N_K^R(x)$ and we will say that the sequence $\{(x_n,v_n):n\in\mathbb{N}\}$ reaches (x,p).

By Proposition 3.5, $p\in N_K^P(x)$. The following Proposition provides a finer result, in the spirit of [7, Proposition 3.3.4].

Proposition 4.2. *Let $x\in K$. Then $N_K^R(x)\subseteq\operatorname{bdry}N_K^P(x)$.*

Proof. Let $p\in N_K^R(x)$, $|p|=1$. Let $\{(x_n,v_n)\}_{n\in\mathbb{N}}$ be a sequence reaching (x,p). Up to passing to a subsequence, there exists $w\in\mathbb{R}^N$ with $|w|=1$ such that

$$\lim_{n\to\infty}\frac{x_n-x}{|x_n-x|}=w.$$

By definition, $w \in T_K^F(x)$. We claim that for all $t > 0$, $p + tw \notin N_K^P(x)$.
Indeed, we have:

$$\begin{aligned}
\langle p + tw, w \rangle &= \lim_{n\to\infty} \langle p + tw, \frac{x_n - x}{|x_n - x|} \rangle \\
&= \lim_{n\to\infty} \langle p - v_n, \frac{x_n - x}{|x_n - x|} \rangle + \langle v_n, \frac{x_n - x}{|x_n - x|} \rangle + t \langle w, \frac{x_n - x}{|x_n - x|} \rangle \\
&= \lim_{n\to\infty} -\langle v_n, \frac{x - x_n}{|x - x_n|} \rangle + t \langle w, \frac{x_n - x}{|x_n - x|} \rangle \\
&\geq \lim_{n\to\infty} -\varphi(x_n)|x - x_n| + t \langle w, \frac{x_n - x}{|x_n - x|} \rangle = t > 0.
\end{aligned}$$

Recalling (4) and (*1*) in Proposition 3.5, we obtain that $p + tw \notin N_K^F(x) = N_K^P(x)$ for all $t > 0$, which concludes the proof. □

The following result provides an alternative definition of $N_K^R(x)$.

Proposition 4.3. *Let $K \subset \mathbb{R}^N$ be φ-convex and nondegenerate, and let $x \in \operatorname{bdry} K$. Then:*

$$\begin{aligned}
N_K^R(x) = \Big\{ \lambda p : \lambda \geq 0, \, p = \lim p_i, |p_i| = 1, p_i \in N_K^P(x_i), \\
x_i \to x, \, \operatorname{diam}\big(N_K^P(x_i) \cap \operatorname{bdry} B(0,1)\big) \to 0 \Big\}
\end{aligned} \tag{15}$$

Proof. Denote by $N^\sharp$ the right hand side of (15). Clearly $N_K^R(x) \subseteq N^\sharp$. Conversely, let $p \in N^\sharp$, $p = \lim p_i$, $p_i \in N_K^P(x_i)$, $|p_i| = 1$, $x_i \to x$. Since $N_K^R(y) \neq \emptyset$ for all $y \in \operatorname{bdry} K$, choose $p_i^* \in N_K^R(x_i)$, $|p_i^*| = 1$, and let $y_i \in K$ and $q_i \in \mathbb{R}^N$ be such that

$$|y_i - x_i| < \frac{1}{i}, \qquad N_K^P(y_i) = \mathbb{R}^+ q_i, \qquad |q_i| = 1, \qquad |q_i - p_i^*| < \frac{1}{i}.$$

Then $y_i \to x$ and

$$\begin{aligned}
|q_i - p| &\leq |q_i - p_i^*| + |p_i^* - p_i| + |p_i - p| \\
&\leq \frac{1}{i} + \operatorname{diam}(N_K^P(x_i) \cap \operatorname{brdy} B(0,1)) + |p_i - p| \to 0.
\end{aligned}$$

This concludes the proof. □

In the sequel, we will need to represent the tangent cone by limits along special directions.

Lemma 4.4. *Let Q be a dense subset of K, and let $w \in T_K^F(x)$, $|w| = 1$. Then there exists a sequence $\{y_n\}_{n\in\mathbb{N}}$ in Q such that $y_n \neq x$, $y_n \to x$ and*

$$\lim_{n\to\infty} \frac{y_n - x}{|y_n - x|} = w.$$

Proof. By assumption, there exists a sequence $x_n \to x$ in $K \setminus \{x\}$ such that

$$\lim_{n\to\infty} \frac{x_n - x}{|x_n - x|} = w.$$

By the density of Q, for all $n \in \mathbb{N}$ there exists $y_n \in Q$ such that

$$|y_n - x_n| \leq |x_n - x|^2.$$

Obviously $y_n \to x$. Moreover,

$$\frac{|y_n - x|}{|x_n - x|} \to 1. \tag{16}$$

In fact,

$$\frac{|x_n - x| - |x_n - y_n|}{|x_n - x|} \leq \frac{|y_n - x|}{|x_n - x|} \leq \frac{|y_n - x_n| + |x - x_n|}{|x - x_n|}.$$

Moreover,

$$\begin{aligned} \left| \frac{y_n - x}{|y_n - x|} - w \right| &\leq \frac{|y_n - x_n|}{|y_n - x|} + \left| \frac{x_n - x}{|y_n - x|} - w \right| \\ &\leq \frac{|y_n - x_n|}{|y_n - x|} + \left| \frac{x_n - x}{|x_n - x|} \cdot \frac{|x_n - x|}{|y_n - x|} - w \cdot \frac{|x_n - x|}{|y_n - x|} - w\left(1 - \frac{|x_n - x|}{|y_n - x|}\right) \right| \\ &\leq \frac{|y_n - x_n|}{|y_n - x|} + \frac{|x_n - x|}{|y_n - x|} \left| \frac{x_n - x}{|x_n - x|} - w \right| + \left| 1 - \frac{|x_n - x|}{|y_n - x|} \right|. \end{aligned}$$

By (16) the proof is concluded, since all summands in the above expression tend to 0. $\square$

In order to state our representation formula for the normal cone, we need to introduce a nondegeneracy condition at a point x of K, namely that $N_K^P(x)$ contains no lines, i.e., x is not the tip of a cusp. This assumption has a first consequence on the Hausdorff measure of the boundary of K around x.

Proposition 4.5. *Let $K \subset \mathbb{R}^N$ be compact and φ-convex. Let $x \in \operatorname{bdry} K$ be such that $N_K^P(x)$ contains no lines. Then the $(N-1)$-density of x with respect to $\operatorname{bdry} K$ is positive, i.e.,*

$$\lim_{\varepsilon \to 0^+} \frac{\mathcal{H}^{N-1}(\operatorname{bdry} K \cap B(x,\varepsilon))}{\omega_{N-1}\varepsilon^{N-1}} > 0,$$

where ω_{N-1} is the $\mathcal{L}^{N-1}$-measure of the unit ball in $\mathbb{R}^{N-1}$. In particular, for every $\varepsilon > 0$, $\mathcal{H}^{N-1}(\partial K \cap B(x,\varepsilon)) > 0$.

Proof. By our assumption, the Hausdorff dimension of the convex set $T_K^F(x)$ is N. In fact, otherwise there exists a vector $v \neq 0$ such that $\langle v, w \rangle = 0$ for all $w \in T_K^F(x)$, which implies $\mathbb{R}v \subseteq N_K^F(x) = N_K^P(x)$. We now compute the N-dimensional density of x with respect to K. By [15, Remark 4.15 (ii)] we have:

$$\lim_{r \to 0^+} \frac{\mathcal{L}^N(K \cap B(x,r))}{\mathcal{L}^N(B(x,r))} = \frac{\mathcal{L}^N(K \cap B(x,r))}{\mathcal{L}^N(T_K^F(x) \cap B(x,r))} \frac{\mathcal{L}^N(T_K^F(x) \cap B(x,r))}{\mathcal{L}^N(B(x,r))} > 0.$$

By the isoperimetric inequality (see [14, Theorem 2, p. 190]), recalling that K and hence $K \cap B(x,\varepsilon)$ for a.e. $\varepsilon > 0$ have finite perimeter in $\mathbb{R}^N$ (see *(4)* in Proposition 3.5 above), and De Giorgi's structure theorem (see [14, Theorem 2, (iii), p. 205]), we have that

$$\mathcal{H}^{N-1}(\operatorname{bdry} K \cap B(x,\varepsilon)) \geq P(K, B(x,\varepsilon)) \geq \text{const} \cdot \left(\mathcal{L}^N(K \cap B(x,\varepsilon))\right)^{1-\frac{1}{N}}.$$

Hence

$$\lim_{\varepsilon\to 0}\frac{\mathcal{H}^{N-1}(\operatorname{bdry} K\cap B(x,\varepsilon))}{\omega_{N-1}\varepsilon^{N-1}}\geq \text{const}\lim_{\varepsilon\to 0}\left(\frac{\mathcal{L}^N(K\cap B(x,\varepsilon))}{\omega_N\varepsilon^N}\right)^{1-\frac{1}{N}}>0.$$

The proof is concluded. □

The following is our representation theorem.

Theorem 4.6. *Let K be φ-convex, let $x\in\operatorname{bdry} K$ and assume that $N_K^P(x)$ contains no lines. Then*

$$N_K^P(x)=\overline{co}(N_K^R(x)). \tag{17}$$

Proof. Since $N_K^P(x)$ is closed and convex, it is sufficient to prove that if $w\in\mathbb{R}^N$, $|w|=1$, generates an exposed ray of $N_K^P(x)$, then there exists a sequence (x_n,v_n) reaching (x,w). In fact, by [21, Corollary 18.7.1, p. 169], $N_K^P(x)$ is the closed convex hull of its exposed rays. We recall (see [21, p. 163]) that an exposed ray $\mathbb{R}^+v$ of a convex cone C is defined by the property that there exists a linear functional h which is zero on it and if $h(p)=0$ and $p\in C$ then $p\in\mathbb{R}^+v$.
Let π be a support hyperplane for $N_K^P(x)$ at w and let $\theta\in\mathbb{R}^N$, $|\theta|=1$, be such that $\pi=\{v:\langle\theta,v-w\rangle=\langle\theta,v\rangle=0\}$, $\langle\theta,z\rangle\leq 0$ for all $z\in N_K^P(x)$, and $\mathbb{R}^+w=\pi\cap N_K^P(x)$. Hence $\theta\in(N_K^P(x))^0=T_K^F(x)$ (recall Proposition 3.5, part (*1*)). By Proposition 4.5, Lemma 4.4 and Proposition 3.5, part (*3*), there exists a sequence $x_n\to x$, such that $N_K^P(x_n)\subseteq\mathbb{R}v_n$, $|v_n|=1$, such that v_n converges to some $p_0\in N_K^P(x)$ and

$$\frac{x_n-x}{|x_n-x|}\to\theta.$$

By φ-convexity, we have:

$$\langle p_0,-\theta\rangle=\lim_{n\to\infty}\langle v_n,\frac{x-x_n}{|x-x_n|}\rangle\leq\lim_{n\to\infty}\varphi(x_n)|x-x_n|=0.$$

Furthermore, since $p_0\in N_K^P(x)$, we have that $\langle p_0,\theta\rangle\leq 0$. Hence $\langle p_0,\theta\rangle=0$, and so $p_0\in\pi\cap N_K^P(x)$. Thus, according to the definition of exposed ray, we have $p_0=w$. □

In order to apply the previous results to functions, we focus our attention on normals to epigraphs. In the remainder of this section, $f:\Omega\to\mathbb{R}$ will denote a *continuous* function, with *φ-convex epigraph*, from an open subset Ω of $\mathbb{R}^N$.

Lemma 4.7. *Let $x\in\Omega$ such that $N^P_{\mathrm{epi}(f)}(x,f(x))=\mathbb{R}^+v$, $v\in\mathbb{R}^{N+1}$, $|v|=1$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in Ω such that $x_n\to x$, f is differentiable at x_n and*

$$\frac{(Df(x_n),-1)}{|(Df(x_n),-1)|}\to v.$$

Proof. Since f is differentiable almost everywhere, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in Ω such that $x_n\to x$ and f is differentiable at x. We may assume

$$\frac{(Df(x_n),-1)}{|(Df(x_n),-1)|}\to w\in\mathbb{R}^N,$$

with $|w| = 1$. By the closedness of the graph of the map $x \mapsto N^P_{\mathrm{epi}(f)}(x, f(x))$, we have that $w \in N^P_{\mathrm{epi}(f)}(x, f(x))$, hence $w = v$. □

Definition 4.8. We define the cone $N^G_{\mathrm{epi}(f)}(x, f(x))$ of normal vectors reachable by gradients of f at $x \in \Omega$ by setting

$$N^G_{\mathrm{epi}(f)}(x, f(x)) := \{\lambda v : v \in \mathbb{R}^{N+1}, |v| = 1, \lambda \in \mathbb{R}^+, \exists x_n \to x, \{x_n\}_{n\in\mathbb{N}} \subset \Omega \setminus \{x\} \text{ such that } \exists Df(x_n) \text{ and } \frac{(Df(x_n), -1)}{|(Df(x_n), -1)|} \to v\}.$$

Observe that, in particular, if $N^P_{\mathrm{epi}(f)}(x, f(x)) = \mathbb{R}^+ v$, then $N^P_{\mathrm{epi}(f)}(x, f(x)) = N^G_{\mathrm{epi}(f)}(x, f(x))$, see Lemma 4.7. Moreover, this is true for $\mathcal{L}^N$-a.e. $x \in \Omega$, by Theorem 3.9.

We are now ready to state the representation formula for the generalized gradient (we recall that under our assumptions the generalized gradient equals the proximal subgradient).

Theorem 4.9. *Let $x \in \Omega$, and assume that*

$$N^P_{\mathrm{epi}(f)}(x, f(x)) \textit{ contains no lines.} \tag{18}$$

Then

$$N^P_{\mathrm{epi}(f)}(x, f(x)) = \overline{co} N^G_{\mathrm{epi}(f)}(x, f(x)). \tag{19}$$

Moreover, let $v \in \partial_P f(x)$. Then there exist sequences $(x_i^h)_{h\in\mathbb{N}}$ and $(\gamma_i^h)_{h\in\mathbb{N}}$, $i = 1, \dots, N+1$, such that

(i) $x_i^h \in \Omega$ *for all* $h \in \mathbb{N}$ *and* $i = 1, \dots, N+1$, $\lim_{h\to\infty} x_i^h = x$ *for every* $i = 1, \dots, N+1$, *and f is differentiable at x_i^h for all $h \in \mathbb{N}$ and $i = 1, \dots, N+1$;*

(ii) $\gamma_i^h \in [0, 1]$ *for all* $h \in \mathbb{N}$ *and* $i = 1, \dots, N+1$ *and* $\sum_{i=1}^{N+1} \gamma_i^h = 1$ *for all* $h \in \mathbb{N}$;

(iii) $v = \lim_{h\to\infty} \sum_{i=1}^{N+1} \gamma_i^h Df(x_i^h)$ *and, for all* $i = 1, \dots, N+1$, *the sequence* $\{\gamma_i^h Df(x_i^h)\}_{h\in\mathbb{N}}$ *is bounded.*

Proof. In order to prove (19) it is sufficient to show that every exposed ray $\mathbb{R}^+ w$ of $N^P_{\mathrm{epi}(f)}(x, f(x))$ belongs to $N^G_{\mathrm{epi}(f)}(x, f(x))$. According to Theorem 4.6 and (14), there exists a sequence $y_j \to x$ of points with $N^P_{\mathrm{epi}(f)}(y_j, f(y_j)) = \mathbb{R}^+ v_j$, $|v_j| = 1$ such that $v_j \to w$. According to Lemma 4.7, there exists a sequence $\{x_{j,k}\}_{k\in\mathbb{R}^N}$ of differentiability points for f such that:

$$x_{j,k} \to y_j, \qquad \frac{(Df(x_{j,k}), -1)}{|(Df(x_{j,k}), -1)|} \to v_j \text{ as } k \to \infty.$$

We set now $x_n = x_{n,n}$: f is differentiable at x_n and

$$x_n \to x, \qquad \frac{(Df(x_n), -1)}{|(Df(x_n), -1)|} \to w.$$

To show (i), (ii) and (iii), let $d = \dim N^P_{\mathrm{epi}(f)}(x, f(x))$ and fix $v \in \partial_P f(x)$. By (19), there exist $\lambda > 0$ and sequences $(x_i^h)_{h\in\mathbb{N}}$, $(\lambda_i^h)_{h\in\mathbb{N}}$, $\lambda_i^h \in [0, 1]$, $i = 1, \dots, d+1$, $\sum_{i=1}^{d+1} \lambda_i^h = 1$ for all $h \in \mathbb{N}$ such that:

- f is differentiable at x_i^h and $\lim_{h\to\infty} x_i^h = x$ for every $i = 1, \dots, d+1$;
- it holds

$$\frac{(v,-1)}{|(v,-1)|} = \lambda \lim_{h\to\infty} \sum_{i=1}^{d+1} \lambda_i^h \frac{(Df(x_i^h),-1)}{|(Df(x_i^h),-1)|}. \tag{20}$$

We claim now that there necessarily exist $k \in \{1, \dots, d+1\}$ and $M > 0$ such that, up to rearranging indexes,

- $\sup_h |Df(x_i^h)| \leq M$ for all $i = 1, ..., k$;
- $\sup_h |Df(x_i^h)| = +\infty$ for all $i \in \{k+1, ..., d+1\}$, where we mean that the last set is empty if $k = d+1$.

In fact, if for all $i = 1, \dots, d+1$ it holds that $\sup_h |Df(x_i^h)| = +\infty$, then the $N+1$-component of the limit in (20) vanishes.
According to the definition of the index k, we then have

$$\lim_{h\to\infty} \frac{\lambda_i^h}{|(Df(x_i^h),-1)|} = 0, \text{ for } i = k+1, ..., d+1,$$

$$\lim_{h\to\infty} \sum_{i=1}^{k} \frac{\lambda \lambda_i^h |(v,-1)|}{|(Df(x_i^h),-1)|} = 1, \text{ for } i = 1, ..., k.$$

We set, for $i = 1, \dots, d+1$,

$$\alpha_i^h = \frac{\lambda \lambda_i^h |(v,-1)|}{|(Df(x_i^h),-1)|}$$

and

$$\gamma_i^h = \begin{cases} \alpha_i^h \dfrac{1 - \displaystyle\sum_{j=k+1}^{d+1} \alpha_j^h}{\displaystyle\sum_{j=1}^{k} \alpha_j^h} & \text{for } i = 1, ..., k \\ \alpha_i^h & \text{for } i = k+1, ..., d+1. \end{cases}$$

We have, for every $i, h \in \mathbb{N}$ that $\gamma_i^h \geq 0$, $\gamma_i^h - \alpha_i^h \to 0$ as $h \to \infty$ for $i = 1, ..., k$ and

$$\sum_{i=1}^{d+1} \gamma_i^h = 1.$$

Now we observe that

$$(v,-1) = \lim_{h\to\infty} \sum_{i=1}^{d+1} \gamma_i^h (Df(x_i^h),-1)$$

and, by construction, the sequences $\{\gamma_i^h Df(x_i^h) : h \in \mathbb{N}\}$, $i = 1, \dots, d+1$ are bounded. The proof is concluded. $\square$

Remark 4.10. Recalling that a set K is epi-Lipschitz at x if and only if the normal cone $N_K^C(x)$ contains no lines (see Theorem 3 in [22]), Theorem 4.9 is not surprising. Moreover, let K be the epigraph of $f(x) = \sqrt{|x|}$, $x \in \mathbb{R}$. Then $N^P_{\mathrm{epi}(f)}(0,0) = \mathbb{R} \times (-\infty, 0]$ contains the line $\mathbb{R} \times \{0\}$, and the representation formulas (17) and (19) do not hold. However, parts (i), (ii), and (iii) for $\partial_P f(0) = \mathbb{R}$ in the statement of Theorem 4.9 do hold. In fact, let $\alpha \geq 0$. By setting for $h \in \mathbb{N}$, $x_1^h = -1/h$, $x_2^h = 1/h$, $\gamma_1^h = (1 - 2\alpha/\sqrt{h})/2$, and $\gamma_2^h = 1 - \gamma_1^h$, properties (i) to (iii) are satisfied for $v = \alpha \in \partial_P f(0)$. The case $\alpha < 0$ is symmetric. Actually, we conjecture that the restriction (18) in Theorem 4.9 is not necessary for (i)–(iii).

5. Viscosity solutions

For the theory of viscosity solutions and its application to Optimal Control we refer to [3].

In all this section, Ω will be an open subset of $\mathbb{R}^N$ and

$$H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$$

will be a continuous function. We are interested in the following general nonlinear first order equation:

$$H(x, u, Du) = 0, \quad x \in \Omega \tag{21}$$

in the unknown $u : \Omega \to \mathbb{R}$.

Definition 5.1. A continuous function $u : \Omega \to \mathbb{R}$ is called a *viscosity subsolution* of equation (21) if for any $x \in \Omega$ it satisfies:

$$H(x, u(x), p) \leq 0, \quad \forall p \in \partial^F u(x).$$

In the same way, a continuous function $u : \Omega \to \mathbb{R}$ is called a *viscosity supersolution* of equation (21) if for any $x \in \Omega$ it satisfies:

$$H(x, u(x), q) \geq 0, \quad \forall q \in \partial_F u(x).$$

A function u which is both sub- and supersolution is called a *viscosity solution* of equation (21).

In this section we switch to statements of "concave type", in order to simplify the comparison with the existing literature. We focus our attention on noncoercive Hamiltonians, i.e., we do not require that $\lim_{|p|\to+\infty} H(x, u, p) = +\infty$. Consequently we allow nonLipschitz solutions. The following result permits to estimate the Hamiltonian at limit points of sequences of gradients.

Lemma 5.2. *Let $\varphi : \overline{\Omega} \to [0, +\infty[$ be continuous, and let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, with φ-convex hypograph. Let $\alpha : \Omega \to \mathbb{R}$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be continuous and assume that $H(x, u, \cdot)$ is convex for all $(x, u) \in \Omega \times \mathbb{R}$ and*

$$H(y, u(y), Du(y)) \leq \alpha(y)$$

for all $y \in \mathrm{dom}(Du)$. Let $\bar{x} \in \mathrm{dom}(\partial^P u)$ and $v \in \partial^P u(\bar{x})$ be such that there exist sequences $(x_i^h)_{h\in\mathbb{N}}$, $x_i^h \in \Omega$, and $(\gamma_i^h)_{h\in\mathbb{N}}$ such that (i)–(iii) of the statement of Theorem 4.9 hold (with f replaced by u).
Then, if either

(a) $(x,u) \mapsto H(x,u,p)$ *is continuous uniformly with respect to* $p \in \{Du(x) : x \in \mathrm{dom}(Du)\}$,

or

(b) $H(x,u,\lambda p) = \lambda H(x,u,p)$ *for all* $(x,u,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$ *and all* $\lambda \geq 0$,

then

$$H(\bar{x}, u(\bar{x}), v) \leq \alpha(\bar{x}).$$

Proof. Let $(x_i^h)_{h\in\mathbb{N}}$, $x_i^h \in \Omega$, and $(\gamma_i^h)_{h\in\mathbb{N}}$ satisfy the assumptions. Then

$$\begin{aligned} H(\bar{x}, u(\bar{x}), v) &= \lim_{h\to\infty} H\Big(\bar{x}, u(\bar{x}), \sum_{i=1}^{d+1} \gamma_i^h Du(x_i^h)\Big) \\ &\leq \limsup_{h\to\infty} \sum_{i=1}^{d+1} \gamma_i^h H(\bar{x}, u(\bar{x}), Du(x_i^h)) \\ &= \limsup_{h\to\infty} \left(\sum_{i=1}^{d+1} \gamma_i^h H(x_i^h, u(x_i^h), Du(x_i^h)) + \right. \\ &\qquad \left. + \sum_{i=1}^{d+1} \gamma_i^h \Big(H(\bar{x}, u(\bar{x}), Du(x_i^h)) - H(x_i^h, u(x_i^h), Du(x_i^h)) \Big) \right). \end{aligned} \tag{22}$$

The first summand in the above expression is $\leq \alpha(\bar{x})$ by assumption.
The second summand tends obviously to zero, as $h \to \infty$, if (a) is valid.
Let now (b) hold. Then the second summand equals

$$\sum_{i=1}^{d+1} \Big(H(\bar{x}, u(\bar{x}), \gamma_i^h Du(x_i^h)) - H(x_i^h, u(x_i^h), \gamma_i^h Du(x_i^h)) \Big).$$

Since the sequences $\{\gamma_i^h Du(x_i^h)\}$ are bounded, the above expression tends to zero as well, for $h \to \infty$. The proof is concluded. □

Theorem 5.3. *Let* Ω *be an open subset of* $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}$ *be a continuous function with* φ*-convex hypograph, and assume that, for all* $x \in \Omega$, u *satisfies the statements* (i), (ii), (iii) *of Theorem* 4.9. *Let* $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *and* $\alpha : \Omega \to \mathbb{R}$ *be continuous. Assume furthermore that* $H(x,u,\cdot)$ *is convex for all* $(x,u) \in \Omega \times \mathbb{R}$ *and either* (a) *or* (b) *in the statement of Lemma* 5.2 *hold. Suppose finally that*

$$H(x, u(x), Du(x)) = \alpha(x) \quad \text{for a.e. } x \in \Omega.$$

Then $u(\cdot)$ *is a viscosity solution of the equation* $H(x,u,Du) = \alpha$.

Proof. The proof is divided into three claims, and follows the lines of the proof of [7, Proposition 5.3.1].

Claim 1: $u(\cdot)$ satisfies the equation at all points of differentiability.
Indeed, let $x \in \Omega$ be such that there exists $Du(x)$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence such that

$x_k \to x$, $Du(x_k)$ exists and $H(x_k, u(x_k), Du(x_k)) = \alpha(x_k)$. We show that the sequence of gradients $(Du(x_k))_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$. By asssumption,

$$\zeta_k := \frac{(-Du(x_k), 1)}{|(-Du(x_k), 1)|} \in N^P_{\mathrm{hypo}(u)}(x_k, u(x_k));$$

and if there exists a subsequence $(x_{k_m})_{m\in\mathbb{N}}$ such that $|Du(x_{k_m})| \to +\infty$, then (we can assume $\zeta_{k_m} \to \zeta \in \mathbb{R}^N$) we have $\zeta = (v, 0) \in N^P_{\mathrm{hypo}(u)}(x, u(x))$ according to part (*1*) of Proposition 3.5. But u is differentiable at x, hence $\partial^P u(x) \subseteq \{Du(x)\}$. However, $(-Du(x), 1) \in N^F_{\mathrm{hypo}(u)}(x, u(x)) = N^P_{\mathrm{hypo}(u)}(x, u(x)) = \mathbb{R}^+(-Du(x), 1)$. By convexity of $N^P_{\mathrm{hypo}(u)}(x, u(x))$, it follows that $\partial^P u(x)$ contains infinitely many elements, which leads to a contradiction. Hence there exists a subsequence (still denoted by $(x_k)_{k\in\mathbb{N}}$) such that $Du(x_k)$ converges, say $Du(x_k) \to v$. By regularity, we have $(-v, 1) \in N^P_{\mathrm{hypo}(u)}(x, u(x)) = \mathbb{R}^+(-Du(x), 1)$, hence $v = Du(x)$. By continuity of H and α, it follows $H(x, u(x), Du(x)) = \alpha(x)$. This concludes the proof of Claim 1.

Claim 2: Let $x \in \Omega$, $v \in \partial^F u(x)$. Then $H(x, u(x), v) \le \alpha(x)$.
To show this claim, apply Lemma 5.2.

Claim 3: Let $x \in \Omega$, $v \in \partial_F u(x)$. Then $H(x, u(x), v) \ge \alpha(x)$.
By Proposition 3.11, $u(\cdot)$ is differentiable at x. From Claim 1 it follows that actually

$$H(x, u(x), Du(x)) = \alpha(x).$$

The proof is concluded. □

Corollary 5.4. *Let $\Omega \subseteq \mathbb{R}^N$ be open and $u : \Omega \to \mathbb{R}$ be continuous with φ-convex hypograph and assume that, for all $x \in \Omega$, $N_{\mathrm{hypo}(u)}(x, u(x))$ contains no lines. Let $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\alpha : \Omega \to \mathbb{R}$ be continuous. Assume furthermore that $H(x, u, \cdot)$ is convex for all $(x, u) \in \Omega \times \mathbb{R}$ and either (a) or (b) in the statement of Lemma 5.2 hold. Suppose finally that*

$$H(x, u(x), Du(x)) = \alpha(x) \quad \textit{for a.e. } x \in \Omega.$$

Then $u(\cdot)$ is a viscosity solution of the equation $H(x, u, Du) = \alpha$.

Proof. By Theorem 4.9, Theorem 5.3 can be applied. □

Corollary 5.5. *Let $\Omega \subset \mathbb{R}^N$ be bounded and open and let $H : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and $\alpha : \Omega \to \mathbb{R}$ be satisfying the assumptions of Theorem 5.3, together with the following:
there exists $\omega : [0, +\infty) \to \mathbb{R}$ continuous, such that $\omega(0) = 0$ and*

$$|H(x, p) - H(y, p)| \le \omega\big(|x - y|(1 + |p|)\big)$$

for all $x, y \in \Omega$ and $p \in \mathbb{R}^N$. Let $u_1, u_2 : \bar{\Omega} \to \mathbb{R}$ be continuous, with φ-convex hypograph, and satisfying the statements (i)–(iii) of Theorem 4.9. Assume furthermore that

(*a*) $u_1 = u_2$ *on* $\mathrm{bdry}\,\Omega$;
(*b*) $H(x, Du_1(x)) = H(x, Du_2(x)) = \alpha(x)$ *a.e. on* Ω;
(*c*) *there exists $\psi : \bar{\Omega} \to \mathbb{R}$ continuous, continuously differentiable on Ω such that $\psi \le u_1$ on $\bar{\Omega}$ and*

$$\sup_{x\in\Omega'} \big\{H(x, D\psi(x)) - \alpha(x)\big\} < 0 \quad \forall \Omega' \subset\subset \Omega.$$

Then $u_1 = u_2$ on Ω.

Proof. Since u_1 and u_2 are viscosity solutions of the equation $H(x, Du(x)) = \alpha(x)$, the result is a consequence of Theorem 5.9, p. 82, in [3]. □

The last result of the section concerns viscosity solutions of equations with a convex Hamiltonian which are continuous and have φ-convex epigraph. We recall (see [12]) that such solutions may arise in some minimal time problem. By Theorem 3.9, a function with φ-convex epigraph is a.e. differentiable. By adding the information that it is a viscosity solution of a Hamilton-Jacobi equation, one obtains that it is of class $\mathcal{C}^1$ on an open set with full measure.

Proposition 5.6. *Let $u : \Omega \to \mathbb{R}$ be a continuous function with φ-convex epigraph such that u is a viscosity solution of $H(x, u, Du) = 0$. Assume that H is convex in the third variables and is such that for every $(x, u) \in \Omega \times \mathbb{R}$ the zero level of $H(x, u, \cdot)$ does not contain line segments. Then there exists an open subset $\Omega' \subset \Omega$ such that $\mathcal{L}^N(\Omega \setminus \Omega') = 0$, $u \in C^1(\Omega')$ and $H(x, u(x), Du(x)) = 0$ for all $x \in \Omega'$.*

Proof. There exists an open set Ω' with full measure in Ω where $u(\cdot)$ is locally semiconvex by [11, Theorem 5.1, Theorem 6.1]. Then apply [7, Proposition 5.3.4]. □

Example 5.7. Define

$$H(x, u, p) = e^{\frac{1}{1+|u|} - p} - e^{g(x)}, \qquad g(x) = \frac{\sqrt{|x|} - 1}{2(|x| + \sqrt{|x|})}.$$

The function

$$u(x) = \operatorname{sign}(x)\sqrt{|x|}.$$

is a viscosity solution of $H(x, u, Du) = 0$ by Theorem 5.3.

Example 5.8. The Hamiltonians

$$H_1(x, p) = \begin{cases} 0 & x < 0,\ p \in \mathbb{R} \\ x|p| & x \geq 0,\ p \in \mathbb{R}. \end{cases}$$

and

$$H_2((x_1, x_2), (p_1, p_2)) = -\left\langle (p_1, p_2), (|x_1|, \frac{\operatorname{sign}(x_1)\sqrt{|x_1|}}{2}) \right\rangle$$

satisfy (b) of Lemma 5.2, together with the other assumptions of continuity and convexity. The functions

$$u_1(x) = \begin{cases} -\sqrt{|x|} & x < 0 \\ 0 & x \geq 0 \end{cases} \qquad \text{and} \qquad u_2(x_1, x_2) = x_2 - \sqrt{|x_1|}$$

satisfy the assumptions of Theorem 5.3, and therefore are viscosity solutions of the equations

$$H_1(x, Du(x)) = 0 \qquad \text{and} \qquad H_2((x_1, x_2), Du(x_1, x_2)) = 0$$

respectively.

Finally, we reconsider the function defined in (2).

Example 5.9. The function defined in (2) is a viscosity solution of the Hamilton-Jacobi equation

$$H((x,y),(T_x,T_y)) = 1,$$

where $H((x,y),(p,q)) = |p|$, by Proposition 2.3, p. 240, in [3]. The same result could be recovered by our Theorem 5.3 above.

6. Propagation of singularities

This section is devoted to the study of the singular set of a continuous function f with φ-convex epigraph. The first subsection is concerned with sufficient conditions in order the subdifferential of f be not a singleton on a Lipschitz arc. Our argument is based on a study of the normal cone to the epigraph, and so the first result of the subsection is devoted to the propagation of singularities at boundary point of φ-convex sets. Then we apply this result to functions. The main source for this subsection is Chapter 4 in [7] (see also [1]). The second subsection is concerned with sufficient conditions in order both the subdifferential and the superdifferential of f be empty on a Lipschitz arc.

6.1. Propagation of nonsmoothness

Proposition 6.1. *Let K be φ-convex and nondegenerate, and let $x_0 \in K$ and $p_0 \in \mathbb{R}^N$ be such that*

$$p_0 \in \operatorname{bdry} N_K^P(x_0) \setminus N_K^R(x_0).$$

Then there exist $\rho > 0$, $\delta > 0$ and a Lipschitz arc $x : [0,\rho] \to \operatorname{bdry} K$ such that:

$$x(0) = x_0, \tag{23}$$

$$\lim_{s\to 0^+} \frac{x(s) - x_0}{s} \neq 0, \tag{24}$$

$$\operatorname{diam}\{v \in N_K^P(x(s) \ : \ |v| = 1\} \geq \delta \textit{ for all } s \in [0,\rho]. \tag{25}$$

Proof. The proof is divided into two claims, and it is inspired by [7, §4.2].

Claim 1: Let $p_0 \in \operatorname{bdry} N_K^P(x_0)$, $q \in \mathbb{R}^N \setminus \{0\}$ be such that:

$$\langle q, p - p_0\rangle \leq 0, \quad \forall p \in N_K^P(x_0). \tag{26}$$

Then there exist $\rho > 0$ and a Lipschitz arc $x : [0,\rho] \to \operatorname{bdry} K$ satisfying (23), (24) and

$$\lim_{s\to 0^+} \frac{x(s) - x_0}{s} = q, \tag{27}$$

$$p(s) := p_0 + q - \frac{x(s) - x_0}{s} \in N_K^P(x(s)) \text{ for all } s \in [0,\rho]. \tag{28}$$

Moreover,

$$\operatorname{Lip}(x(\cdot)) \leq 2|p_0 + q|. \tag{29}$$

Proof of Claim 1. Let $r > 0$ be so small that $4r\varphi(y) \leq 1$ for all $y \in B(x_0, r)$. Set $\rho = r/(2|p_0 + q|)$. Then, recalling Proposition 3.2, for all $s \in [0,\rho]$ the projection $x(s) :=$

$\pi_K(x_0+s(p_0+q))$ is unique and is Lipschitz with respect to s with ratio $2|p_0+q|$. Observe that $p_0+q \notin N_K^P(x_0)$, because $q \neq 0$ is normal to $N_K^P(x_0)$ at p_0. As a consequence, $x(s) \neq x_0$ for all $s \in [0,\rho]$.
By construction, $x_0+s(p_0+q)-x(s) \in N_K^P(x(s))$, whence $p(s) \in N_K^P(x(s))$. Moreover $|p(s)| \leq 2|p_0+q|$ for all $s \in [0,\rho]$. By definition of $p(s)$ and Proposition 3.5, we have

$$\begin{aligned} \langle p(s)-p_0, p(s)-(p_0+q)\rangle &= \langle p(s)-p_0, \frac{x_0-x(s)}{s}\rangle \\ &\leq \frac{1}{s}(\varphi(x_0)|p_0|+\varphi(x(s))|p(s)|)|x(s)-x_0|^2. \end{aligned} \tag{30}$$

Let $s_k \to 0$ be such that $p(s_k) \to \bar{p} \in N_K^P(x_0)$. Then by passing to the limit in (30) and recalling that $x(\cdot)$ is Lipschitz, we obtain:

$$|\bar{p}-p_0|^2 - \langle q, \bar{p}-p_0\rangle \leq 0.$$

By (26) we obtain that $\bar{p}=p_0$. As a consequence we obtain (27). □

Claim 2: Let $p_0 \in \operatorname{bdry} N_K^P(x_0) \setminus N_K^R(x_0)$, and let $q \neq 0$ be such that $\langle q, p-p_0\rangle \leq 0$ for all $p \in N_K^P(x_0)$. Let $x:[0,\rho] \to K$ be the arc constructed in Claim 1, corresponding to p_0, q. Then that there exists $\delta > 0$ such that $\operatorname{diam}\{v \in N_K^P(x(s) : |v|=1\} \geq \delta$ for all $s \in [0,\rho]$.

Proof of Claim 2. If there exists a sequence $s_k \to 0$ such that $\operatorname{diam}\{v \in N_K^P(x(s_k) : |v| = 1\} \to 0$, then by Proposition 4.3

$$p_0 = \lim_{k\to\infty} p(s_k) \in N_K^R(x_0),$$

a contradiction. □

Corollary 6.2. *Let $\Omega \subset \mathbb{R}^N$ be open and let $f:\Omega \to \mathbb{R}$ be continuous with φ-convex epigraph. Let $x_0 \in \Omega$ be such that:*

$$\operatorname{bdry} N^P_{\operatorname{epi}(f)}(x_0, f(x_0)) \setminus N^R_{\operatorname{epi}(f)}(x_0, f(x_0)) \neq \emptyset. \tag{31}$$

Then there exists $\rho > 0$, $\delta > 0$ and a Lipschitz arc $y:[0,\rho] \to \Omega$ such that:

$$\lim_{s\to 0^+} \tfrac{y(s)-x_0}{s} \neq 0,$$
$$f \textit{ is not differentiable at } y(s) \textit{ for all } s \in [0,\rho],$$
$$f(y(s)) \textit{ is Lipschitz continuous,}$$

and $y(s) \neq x_0$ for all $s \in (0,\delta]$.

Proof. By Proposition 3.12, epi(f) is nondgenerate. By Proposition 6.1, there exists a Lipschitz arc $x(s):[0,\rho] \to \operatorname{bdry}\operatorname{epi}(u)$ satisfying (23), (24), (25). Then $x(s) = (y(s), f(y(s)))$ and both $y(\cdot)$ and $f(y(\cdot))$ are Lipschitz. The proof is concluded. □

Remark 6.3. If $\partial_P f(x_0)$ is nonempty and bounded, then (31) reads as:

$$\operatorname{bdry}\partial_P f(x_0) \setminus \{v \in \mathbb{R}^N : v = \lim_{h\to\infty} Df(x_h), x_h \to x, \exists Df(x_h)\} \neq \emptyset.$$

Recalling Proposition 3.5, part (*1*), there exists a neighborhood $\mathcal{U}$ of x_0 where $\partial_P f$ is nonempty and bounded. Hence $f_{|\mathcal{U}}$ is semiconvex, and one can apply [7, Theorem 4.2.2], which yields numbers $\rho,\delta > 0$ and a Lipschitz arc $x(\cdot)$ such that $x(0) = x_0$, $\operatorname{diam}(\partial_P f(x(s))) \geq \delta$ for all $s \in [0,\rho]$ and $x(s) \neq x_0$ for all $s \in [0,\rho]$.

Example 6.4. Consider the function

$$f(x,y) = (x^2+y^2)^{1/4}.$$

Then f has φ_0-convex epigraph, with $\varphi_0 = 1/2$, and the only singular point is $(0,0)$. Actually, bdry $N^P_{\mathrm{epi}(f)}(0,0) = \mathbb{R}^2 \times \{0\}$, and condition (31) is not satisfied.

6.2. Propagation of non-sub/superdifferentiability

In this subsection we are first concerned with continuous functions with smooth graph, and study the propagation of "vertical tangents" to the graph. Then we apply this result to a suitable neighborhood of a φ-convex epigraph, obtaining propagation of vertical tangents for general continuous functions with φ-convex epigraph.

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be continuous and such that its graph is a $\mathcal{C}^{1,1}$-manifold, i.e., for all $x \in \Omega$ there exists a neighborhood $U_x \subset \mathbb{R}^{N+1}$ of $(x, f(x))$ and a $\mathcal{C}^{1,1}$-function $g : U_x \to \mathbb{R}$ such that

$$\mathrm{graph}(f) \cap U_x = \big\{(x,\xi) \in \mathbb{R}^N \times \mathbb{R} \,:\, g(x,\xi) = 0\big\}, \qquad Dg(x,\xi) \neq 0 \ \forall (x,\xi) \in U_x.$$

Let $x_0 \in \Omega$ be such that the normal cone to epi(f) at $(x, f(x))$ is horizontal, i.e.,

$$\frac{\partial g}{\partial \xi}(x_0, f(x_0)) = 0.$$

We seek for sufficient conditions in order that there exists (at least) a Lipschitz curve $\gamma : [-a,a] \to \Omega$ such that

$$\gamma(0) = x_0,\ g(\gamma(t), f(\gamma(t))) = 0, \qquad \text{and} \qquad \frac{\partial g}{\partial \xi}(\gamma(t), f(\gamma(t))) = 0 \ \text{ for all } \ t \in (-a,a).$$

This amounts to saying that the system of two equations in $N+1$ variables

$$\begin{cases} g(x,\xi) = 0 \\ \frac{\partial g}{\partial \xi}(x,\xi) = 0 \end{cases} \tag{32}$$

can be solved in a neighborhood of $(x_0, f(x_0))$. To this aim, we observe first that the Lipschitz version of the classical implicit function theorem (see, e.g., [8, Sect. 7.1]) cannot be used. More precisely, recall that the generalized Jacobian matrix ∂G of the $\mathbb{R}^2$-valued Lipschitz map $G := (g, \partial g/\partial \xi)$ is defined by (see [9, p. 133])

$$\partial G(x,\xi) = \mathrm{co}\big\{\lim_{i\to\infty} J_G(x_i,\xi_i) \,:\, J_G(x_i,\xi_i) \text{ exists and } x_i \to x, \xi_i \to \xi\big\},$$

where J_G denotes the classical Jacobian matrix. We have:

Proposition 6.5. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \to \mathbb{R}$ be continuous and such that* $\mathrm{graph}(f) = \{(x,\xi) \in \mathbb{R}^{N+1} \,:\, g(x,\xi) = 0\}$*, where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^{1,1}$. Let the map $G : \Omega \times \mathbb{R} \to \mathbb{R}^2$ be defined by $G(x,\xi) = (g(x,\xi), \partial g(x,\xi)/\partial \xi)$ and let $x_0 \in \Omega$ be such that $G(x_0, f(x_0)) = (0,0)$. Then there exists a matrix $M \in \partial G(x_0, f(x_0))$ such that*

$$\mathrm{rk}(M) < 2.$$

Proof. Set $\Gamma = \mathrm{graph}(f)$. Without loss of generality we can assume that $Dg(x_0, f(x_0)) \neq 0$ (otherwise the proof is concluded) and that $g(x,\xi)$ is positive on $\{(x,\xi) : \xi > f(x)\}$ and negative on $\{(x,\xi) : \xi < f(x)\}$. Hence $\partial g/\partial\xi(x, f(x)) \geq 0$ for all $x \in \Omega$, which implies that $(x_0, f(x_0))$ is a minimum point for $\partial g/\partial\xi(x,\xi)$ under the constraint $(x,\xi) \in \Gamma$. By the nonsmooth version of the Lagrange multipliers rule, (see, e.g., [9, Theorem 3.1.3]) there exists a matrix $M \in \partial G(x_0, f(x_0))$ such that $\mathrm{rk}(M) < 2$. □

However, a slightly more general version of the classical implicit function theorem can be used in order to solve (32), at least in some examples. We state this theorem, due to Hildebrandt and Graves [17], in a particular version, directly applicable to (32).

Theorem 6.6. *Let $D \subseteq \mathbb{R}^{N-1} \times \mathbb{R}^2$ be open and let $G : D \to \mathbb{R}^2$ be Lipschitz. Let $(y_0, z_0) \in D$ be such that $G(y_0, z_0) = 0$ and let $\rho_0, \sigma_0 > 0$ be such that $\bar{B}(y_0, \rho_0) \times \bar{B}(z_0, \sigma_0) \subset D$. Assume furthermore that there exists a linear invertible operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $0 < \rho \leq \rho_0$, $\alpha_0 \geq 0$ such that*

(*i*) $|G(y, z_2) - G(y, z_1) - A(z_2 - z_1)| \leq \alpha_0 |z_2 - z_1| < \|A^{-1}\|^{-1} |z_2 - z_1|$ *for all $y \in \bar{B}(y_0, \rho_0)$, for all $z_1, z_2 \in \bar{B}(z_0, \sigma_0)$;*

(*ii*) $|G(y, z_0)| \leq (\|A^{-1}\|^{-1} - \alpha_0)\sigma_0$ *for all $y \in \bar{B}(y_0, \rho)$.*

Then for each $y \in \bar{B}(y_0, \rho)$ the system $G(y, \cdot) = 0$ has a solution $z(y)$ which is unique in that ball. Moreover, the function $y \mapsto z(y)$ is Lipschitz, the Lipschitz constant being dependent only on the Lipschitz constant of G.

Proof. The above statement is a particular case of Theorem 3 in [17]. The Lipschitz continuity of $z(\cdot)$ is not stated there; it is however easy to obtain by the successive approximation method (see, e.g., [5, Theorem 2.1]). □

Here a simple case is presented, where the propagation occurs and Theorem 6.6 can be applied.

Example 6.7. Let $f(x,y) = \mathrm{sign}(x)\sqrt{|x|}$. The graph of f is a $\mathcal{C}^{1,1}$-manifold, globally represented by the $\mathcal{C}^{1,1}$-constraint $g(x,y,\xi) = 0$, $g(x,y,\xi) = x - \mathrm{sign}(\xi)\xi^2$. Of course, the system (32) can be readily solved by setting $x(y) = \xi(y) \equiv 0$. However, we wish to show that Theorem 6.6 can be applied. So, we are going to use Theorem 6.6 for the system $G(x,y,\xi) := (x - \xi^2, -2\xi) - (0, -2\eta) = (0,0)$ in a neighborhood of the point $(\eta^2, 0, \eta)$, for $0 < \eta < 1/6$. To this aim, let ρ_0 be arbitrary and $\sigma_0 = \eta/2$. Let $N = 2$, $D = \mathbb{R}^2 \times (0, +\infty)$ and set

$$A = \frac{\partial G}{\partial(x,\xi)}(\eta^2, 0, \eta) = \begin{pmatrix} 1 & -2\eta \\ 0 & -2 \end{pmatrix}$$

$z_0 = (\eta^2, \eta)$, $y_0 = 0$ (the space $\mathbb{R}^2$ in Theorem 6.6 is meant to be generated by the (x,ξ)-variables, while $\mathbb{R}$ is meant to be generated by the y-variable). For $\xi, \xi' \in \bar{B}(\eta, \eta/2)$, $x, x' \in \mathbb{R}$, $y \in \mathbb{R}$ one has

$$\begin{aligned}\left| G(x',y,\xi') - G(x,y,\xi) - A\begin{pmatrix} x'-x \\ \xi'-\xi \end{pmatrix} \right| &= |((\xi'-\xi)(2\eta - (\xi+\xi')), 0)| \\ &\leq 3\eta\,|(x'-x, \xi'-\xi)| < \frac{1}{2}\,|(x'-x, \xi'-\xi)|\,.\end{aligned}$$

Moreover, $\|A^{-1}\|^{-1} = 1$, so that condition (i) of Theorem 6.6 is satisfied. Condition (ii) holds trivially. Thus, Theorem 6.6 yields, for each η, Lipschitz functions $x_\eta(y), \xi_\eta(y)$ such that

$$G(x_\eta(y), y, \xi_\eta(y)) = (0, -2\eta) \quad \text{for all } y$$

(of course $x_\eta(y), \xi_\eta(y)$ are readily computable). Since the Lipschitz constants are independent of η, by letting $\eta \downarrow 0$ one obtains Lipschitz functions $x(\cdot), \xi(\cdot)$ such that

$$G(x(y), y, \xi(y)) = (0, 0) \quad \text{for all } y.$$

More in general, Theorem 6.6 may be used according to a similar pattern. In fact, let $N^P_{\mathrm{epi}(f)}(x, f(x)) = \mathbb{R}^+(v, 0)$. Recalling Theorem 4.9, there exists a sequence $x_n \to x_0$ such that $N^P_{\mathrm{epi}(f)}(x_n, f(x_n)) = \mathbb{R}^+ v_n$, $v_n \to (v, 0)$, $v_n \neq (v, 0)$. This fact can be rewritten as $\eta_n := \partial g(x_n, f(x_n))/\partial \xi \to 0$; if the system

$$\begin{cases} g(x, \xi) = 0 \\ \frac{\partial g}{\partial \xi}(x, \xi) = \eta_n \end{cases}$$

can be solved around $(x_n, f(x_n))$ by Lipschitz functions, with domains and Lipschitz constants independent of n, then by passing to the limit a Lipschitz solution of (32) is obtained.

In the following example we provide a function where no propagation occurs.

Example 6.8. Let

$$f(x, y) = \begin{cases} \sqrt[4]{x + y^8} - y^2 & \text{for } x \geq 0 \\ y^2 - \sqrt[4]{y^8 - x} & \text{for } x < 0. \end{cases}$$

The graph of f is a $\mathcal{C}^{1,1}$-manifold. The only point with horizontal normal cone is $(0, 0, 0)$.

Let now f be a function with φ-convex epigraph. We recall (see Theorem 3.2) that every sufficiently small ε-neighborhood of $\mathrm{epi}(f)$ is a $\mathcal{C}^{1,1}$-manifold. The point of the last result of this section is showing that this neighborhood is actually the graph of a function f_ε, to which one may try to apply Theorem 6.6. Then a Lipschitz curve in *graph*(f) with horizontal normals can be obtained by projecting a Lipschitz singular curve in *graph*(f_ε).

Proposition 6.9. *Let $\Omega \subset \mathbb{R}^N$ be open, and let $f : \bar{\Omega} \to \mathbb{R} \cup \{+\infty\}$ be continuous in $\bar{\Omega}$. Let $\varphi : \mathrm{epi}(f) \to [0, +\infty[$ be continuous and assume that $\mathrm{epi}(f)$ is φ-convex. Then, for all open set $\Omega' \subset \Omega$, with $\bar{\Omega}'$ compact and contained in Ω, and for all $\varepsilon > 0$ there exists a continuous function $f_{\varepsilon,\Omega'} : \Omega' \to \mathbb{R}$ with the properties:*

a) *there exists $0 < \eta < \varepsilon$ such that $\mathrm{graph}(f_{\varepsilon,\Omega'}) = \mathrm{bdry}\, B(\mathrm{epi}(f), \eta) \cap (\Omega' \times \mathbb{R})$, so that $\mathrm{graph}(f_{\varepsilon,\Omega'})$ is a $C^{1,1}$-manifold;*

b) *for all $x \in \Omega'$, $f_{\varepsilon,\Omega'}(x) \leq f(x)$;*

c) *for every $x \in \Omega'$,*

$$N^P_{epi(f_{\varepsilon,\Omega'})}(x, f_{\varepsilon,\Omega'}(x)) \subseteq N^P_{epi(f)}(\pi_{epi(f)}(x, f_{\varepsilon,\Omega'}(x))).$$

In particular, if $\gamma : [a, b] \to \mathrm{graph}(f_{\varepsilon,\Omega'})$ *is a Lipschitz curve such that for all* $s \in [a, b]$ *one has*

$$\langle v, e_{N+1}\rangle = 0 \;\; \text{for all} \;\; v \in N_{\mathrm{epi}(f_{\varepsilon,\Omega'})}(\gamma(s))$$

then, for all $s \in [a, b]$ *the normal cone* $N^P_{epi(f)}(\pi_{epi(f)}(\gamma(s)))$ *contains a vector* $w(s)$ *such that* $\langle w(s), e_{N+1}\rangle = 0$*. Moreover,* $w(\cdot)$ *is Lipschitz.*

Proof. Let $\Omega' \subset \Omega$ be such that $\bar{\Omega}' \subset \Omega$ is compact, and fix $\varepsilon > 0$. Let $0 < \bar{\eta} < \varepsilon$ be so small that $\bar{B}(\Omega', \bar{\eta}) \subset \Omega$, and set

$$\varphi_{\bar{\eta}} := \max\{(\varphi(x, f(x)) : x \in \bar{B}(\Omega', \bar{\eta})\} < +\infty.$$

Let $0 < \eta < \bar{\eta}$ be so small that $4\eta\varphi_{\bar{\eta}} < 1$. Fix $x \in \Omega'$ and define

$$f_{\varepsilon,\Omega'}(x) := \inf\{y \in \mathbb{R} : (x, y) \in \mathrm{epi}_\eta(f)\},$$

where $\mathrm{epi}_\eta(f)$ denotes the closed η-neighborhood of $\mathrm{epi}(f)$. Observe that $f_{\varepsilon,\Omega'}(x)$ is finite, and is actually a minimum. Indeed, if a sequence $y_n \to -\infty$ exists, together with a sequence $\{x_n\} \subset \bar{B}(x, \eta)$ such that $|(x_n, f(x_n)) - (x, y_n)| \le \eta$, then, up to a subsequence, $x_n \to \bar{x}$, $f(x_n) \to f(\bar{x}) = -\infty$, a contradiction. Choose $(\hat{x}, f(\hat{x})) \in \pi_{\mathrm{epi}(f)}(x, f_{\varepsilon,\Omega'}(x))$, and observe that $(x, f_{\varepsilon,\Omega'}(x)) \in \mathrm{Unp}(\mathrm{epi}(f))$. Moreover, by construction, $f_{\varepsilon,\Omega'}(x) \le f(x)$ and $\mathrm{graph}(f_{\varepsilon,\Omega'}) \subseteq \mathrm{bdry}\,\mathrm{epi}_\eta(f) \cap (\Omega' \times \mathbb{R})$. We wish to show that actually $\mathrm{graph}(f_{\varepsilon,\Omega'}) = (\mathrm{bdry}\,\mathrm{epi}_\eta(f)) \cap (\Omega' \times \mathbb{R})$. To this aim, assume by contradiction that there exist $\bar{x} \in \Omega'$ and $\bar{y} \in \mathbb{R}$ such that $(\bar{x}, \bar{y}) \in \mathrm{bdry}\,\mathrm{epi}_\eta(f) \cap (\Omega' \times \mathbb{R})$, but $\bar{y} > f_{\varepsilon,\Omega'}(\bar{x})$. Set $(\tilde{x}, f(\tilde{x})) = \pi_{\mathrm{epi}(f)}(\bar{x}, f_{\varepsilon,\Omega'}(\bar{x}))$, $\bar{v} = (\bar{x}, f_{\varepsilon,\Omega'}(\bar{x})) - (\tilde{x}, f(\tilde{x}))$, and $(\tilde{x}, \tilde{y}) = (\bar{x}, \bar{y}) - \bar{v}$. Observe that, by construction, $\tilde{y} > f(\tilde{x})$. However, since $|\bar{v}| = \eta$, by the uniqueness of the projection it follows that $(\tilde{x}, \tilde{y}) \in \mathrm{bdry}\,\mathrm{epi}(f)$. Thus $\tilde{y} = f(\tilde{x})$, a contradiction. Hence property *a*) follows. Property *b*) holds by construction, while *c*) follows from Corollary 3.4 above. Let now $\gamma(\cdot)$ be a Lipschitz curve in $\mathrm{graph}(f_{\varepsilon,\Omega'})$ such that $N_{\mathrm{epi}(f_{\varepsilon,\Omega'})}(\gamma(s))$ is horizontal. Recalling *a*), the unit external normal to $\mathrm{epi}(f_{\varepsilon,\Omega'})$ at $\gamma(s)$ can be computed as $v(s) := Dd_{\mathrm{epi}(f)}(\gamma(s))$ which is Lipschitz continuous w.r.t. s. Recalling Corollary 3.4, we have that $w(s) = v(s)$. The proof is concluded. □

Remark 6.10. The Moreau regularization of f, i.e., for a given $\lambda > 0$ the function

$$e_\lambda(x) := \min_{x' \in \mathbb{R}^N} \left\{ f(x') + \frac{1}{\lambda}|x' - x|^2 \right\},$$

does not satisfy the property *c*) in the above statement, as it is always locally Lipschitz. The same happens with a regularization by convolution.

As a conclusion, we go back to the examples presented in the Introduction, and show that the structure of their singular sets is covered by our results.

Example 6.11. Let f be the function defined in (1). All points of nonsmoothness, i.e., the points of the set $A := \{xy = 0\}$, satisfy the assumptions of Corollary 6.2. All points not in A where both sub and superdifferential of f are empty can be treated as in Example 6.7. Now consider the points $(0, \pm 1)$ and $(\pm 1, 0)$: the same method can be applied to the function $f_{\varepsilon,\Omega'}$ constructed in Proposition 6.9, for $\varepsilon > 0$ sufficiently small, in a neighborhood of points in the boundary of $\mathrm{epi}(f)$ which project onto $(0, \pm 1)$ or $(\pm 1, 0)$.

Example 6.12. The minimal time function T described in (2) satisfies the assumptions of Proposition 6.7 at all points $(x,0)$, $x>0$.

Finally, we consider another minimum time function, arising in a classical linear problem: the rocket railroad car (see, e.g., [3, p. 242] and [11]).

Example 6.13. Set $R_-=\{(x_1,x_2):\,x_1>-\frac{1}{2}x_2|x_2|\}$, $R_+=\{(x_1,x_2):\,x_1<-\frac{1}{2}x_2|x_2|\}$ and

$$f(x_1,x_2)=\begin{cases} x_2+2\sqrt{\frac{x_2^2}{2}+x_1}, & (x_1,x_2)\in\overline{R}_-\\ -x_2+2\sqrt{\frac{x_2^2}{2}-x_1}, & (x_1,x_2)\in\overline{R}_+\end{cases}$$

The function f is actually the minimum time to reach the origin for the control system:

$$\begin{cases}\dot{\xi}_1=\xi_2, & \xi_1(0)=x_1\\ \dot{\xi}_2\in[-1,1], & \xi_2(0)=x_2\end{cases}$$

Assumptions of Corollary 6.2 are satisfied at all points of $\{(x_1,x_2)\;:\;x_1=-\frac{1}{2}x_2|x_2|\}$, which are exactly the nondifferentiability points for f.

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