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Continuity Properties of Concave Functions in Potential Theory*

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Given a bounded open set U in $\mathbb{R}^d$, the space of all continuous real functions on $\overline{U}$ which are harmonic on U is denoted by $H(U)$. Further, a lower bounded, Borel measurable numerical function s on $\overline{U}$ is said to be $H(U)$-concave if $\int s\,d\mu \leq s(x)$ for every $x \in \overline{U}$ and every measure μ on $\overline{U}$ satisfying $\int h\,d\mu = h(x)$ for all $h \in H(U)$. It is shown that every $H(U)$-concave function is continuous on U and, under additional assumptions on U, several characterizations of $H(U)$-concave functions are given.

For compact sets K in $\mathbb{R}^d$, continuity properties of $H_0(K)$-concave functions are studied, where $H_0(K)$ is the space of all functions on K which can be extended to be harmonic in some neighborhood of K (depending on the given function). We prove that these functions are finely upper semicontinuous on the fine interior of K, but not necessarily finely continuous there.

Most of the results are established in the context of harmonic spaces, covering solutions of elliptic and parabolic second order partial differential equations. For example, it is shown that $H(U)$-concave functions are always continuous on U if and only if the underlying harmonic space has the Brelot convergence property.

Keywords: Harmonic functions, Choquet theory, concave functions, balayage, fine topology, function spaces

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1. Introduction

Let X be a metrizable compact space, $\mathcal{C}(X)$ the space of continuous real functions on X, and H a *function space* on X. This means that H is a linear subspace of $\mathcal{C}(X)$ containing the constant functions and separating the points of X. We shall write H^+ for the set of positive functions in H, and $W(H)$ for the set of finite infima of functions from H.

Let $\mathcal{M}_+(X)$ denote the set of positive Radon measures on X. For $\mu \in \mathcal{M}_+(X)$ and $f \in \mathcal{C}(X)$, we often write $\mu(f)$ instead of $\int_X f\,d\mu$. With every $x \in X$ we associate the set of *H-representing measures* for x, defined by

$$\mathcal{M}_x(H) := \{\mu \in \mathcal{M}_+(X) \colon \mu(h) = h(x) \text{ for all } h \in H\}.$$

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A real function g on X is called *H-affine*, if g is bounded, Borel measurable, and $\mu(g) = g(x)$, whenever $x \in X$ and $\mu \in \mathcal{M}_x(H)$. A numerical function s on X, that is, a function $s\colon X \to [-\infty, \infty]$, is said to be *$H$-concave*, if s is lower bounded, Borel measurable, and if

$$\mu(s) \le s(x) \quad \text{for every } x \in X \text{ and every } \mu \in \mathcal{M}_x(H).$$

Obviously, a function g on X is *H-affine* if both g and $-g$ are H-concave. Moreover, the minimum of any finite set of H-concave functions is H-concave. In particular, finite minima of H-affine functions are H-concave. Further, the limes inferior of any lower bounded sequence of H-concave functions is H-concave.

The symbol $\mathrm{Ch}_H X$ stands for the *Choquet boundary* of X with respect to H, that is, the set of all $x \in X$ such that $\mathcal{M}_x(H)$ consists only of the point measure ε_x concentrated at x. It is known that the Choquet boundary is a G_δ-set. Choquet's theorem says that, for every $x \in X$, there exists $\mu \in \mathcal{M}_x(H)$ carried by $\mathrm{Ch}_H X$ in the sense that $\mu(X \setminus \mathrm{Ch}_H X) = 0$ (see [12, pp. 14,29]). A function space H is said to be *simplicial* if, for every $x \in X$, there exists a *unique* representing measure μ_x for x carried by $\mathrm{Ch}_H X$.

If H is simplicial, then g is H-affine if and only if there exists a bounded Borel measurable function f on $\mathrm{Ch}_H X$ such that

$$g(x) = \mu_x(f) \quad (x \in X) \tag{1}$$

(this follows immediately from [4, Proposition 2.8] and the trivial observation that $g(x) = \mu_x(g)$ for all H-affine functions g and $x \in X$).

This paper deals with continuity properties of H-concave functions. Of course, continuity should not be expected in general. If, for example, $H = \mathcal{C}(X)$, then every lower bounded Borel measurable function is trivially H-concave. Another simple example is related to one-dimensional potential theory, as follows. Let X be a closed interval $[a, b]$ and H the space of affine functions on X. Then every H-concave (that is, concave) function is continuous on (a, b), but it may be discontinuous at one or both endpoints a, b, as can be seen from the example $s = 1_{(a,b)}$.

We shall consider the more interesting situation of function spaces intensively studied in potential theory: For every bounded domain U in $\mathbb{R}^d$, $d \ge 2$, we denote by $\mathcal{H}(U)$ the space of harmonic functions on U and define

$$H(U) := \{h \in \mathcal{C}(\overline{U})\colon h|_U \in \mathcal{H}(U)\}. \tag{2}$$

It is known that $H(U)$ is simplicial, the Choquet boundary of $\overline{U}$ with respect to $H(U)$ is the set of all regular points of U, and the measures μ_x, $x \in \overline{U}$, are obtained by balayage $\varepsilon_x^{U^c}$ of ε_x on the complement U^c of U. In view of (1) this implies that already $H(U)$-*affine* functions need not be continuous on $\overline{U}$ (see Example 2.1), while they are obviously harmonic and hence continuous on U. We shall show that every $H(U)$-*concave* function is continuous on U. In several important cases, we fully characterize the set of all $H(U)$-concave functions.

There is another class of spaces studied in this paper. For every compact set $K \subset \mathbb{R}^d$, consider the space

$$H_0(K) := \bigcup\{H(U)|_K\colon U \text{ is a relatively compact open set, } K \subset U\}. \tag{3}$$

We shall prove that $H_0(K)$-concave functions are finely *upper* semicontinuous on the fine interior V of K, but not necessarily finely continuous on V (here we refer to the *fine topology* on $\mathbb{R}^d$). In particular, $H_0(K)$-affine functions are finely continuous on V. But they may be discontinuous on V. For details see Examples 3.1.

Most of our results will be established in the context of $\mathcal{P}$-harmonic spaces $(Y, \mathcal{H})$ (with countable base), covering potential theory for a wide class of second order linear elliptic and parabolic partial differential operators (see [7, 5]). For a relatively compact open set $U \subset Y$ and a compact set $K \subset Y$, the spaces $H(U)$ and $H_0(K)$ are defined as in (2) and (3), respectively. In considering these spaces, we shall tacitly assume that they contain the constant functions and separate points. It is known that spaces of these classes are simplicial, and a potential theoretic description of all representing measures is available (see [5, VII.9.6,6]).

2. H(U)-concave functions

The following example shows that already in the classical case $H(U)$-affine functions are not necessarily continuous on $\overline{U}$.

Example 2.1. a) Let f be a *discontinuous* bounded Borel measurable function on the boundary ∂U of the open unit ball in $\mathbb{R}^d$, $d \geq 2$, and $h\colon \overline{U} \to \mathbb{R}$ defined by $h(x) := \varepsilon_x^{U^c}(f)$. Then h is $H(U)$-affine (see (1)), but not continuous at the points $z \in \partial U$, where f is discontinuous.

b) If U is not regular, this may happen even for continuous boundary functions: Let U be the set obtained by removing a Lebesgue spine with cusp at 0 from the open unit ball in $\mathbb{R}^d$, $d \geq 3$. Let us define functions f and h by

$$f(x) := |x|, \quad h(x) := \varepsilon_x^{U^c}(f) \quad (x \in \overline{U}).$$

As before, h is $H(U)$-affine. Since all points in $\partial U \setminus \{0\}$ are regular, there exists a sequence (x_n) in U tending to the origin such that $\lim_{n\to\infty} h(x_n) = 0$. Since the probability measure $\varepsilon_0^{U^c}$ does not charge the origin, it is clear that $h(0) > 0$. So we see that already the restriction of h to $U \cup \{0\}$ is not continuous.

In the case of the harmonic space associated with the heat equation, $H(U)$-concave functions may have discontinuities even on U:

Example 2.2. Let $U := (0,1) \times (0,1)$ and

$$H(U) := \{h \in \mathcal{C}(\overline{U})\colon h|_U \text{ is caloric on } U\}.$$

For every $t \in (0,1)$, let $R_t := [0,1] \times (t,1]$. Let us fix $t \in (0,1)$ and let $s := 1_{\overline{R}_t}$, that is, let s be the characteristic function of the compact set $\overline{R}_t$ (which is upper semicontinuous, but not continuous on U). We claim that s is $H(U)$-concave. To show this, we fix $x \in \overline{U}$ and $\mu \in \mathcal{M}_x(H(U))$. The inequality $\mu(s) \leq s(x)$ is obvious if $x \in \overline{R}_t$. So suppose that $x \in \overline{U} \setminus \overline{R}_t$. Let $0 < \tilde{t} < t$ such that $x \notin R_{\tilde{t}}$. We choose $f \in \mathcal{C}(\partial U)$ such that $f = 0$ on $\partial U \setminus R_{\tilde{t}}$ and $f > 0$ on $\partial U \cap R_{\tilde{t}}$. Then the Perron-Wiener-Brelot solution $H_U f$ for f vanishes on $U \setminus R_{\tilde{t}}$ and is strictly positive on $U \cap R_{\tilde{t}}$. The function $H_U f$ extends continuously to a function $h \in H(U)$ and $h > 0$ on $\overline{U} \cap R_{\tilde{t}}$. Then $0 = h(x) = \mu(h)$, and hence the support of μ is contained in $\overline{U} \setminus R_{\tilde{t}}$. Consequently, $0 = \mu(s) = s(x)$.

We could show just as well that 1_{R_t} is $H(U)$-concave. But, of course, this also follows from $1_{R_t} = \sup_n 1_{\overline{R}_{t+1/n}}$. Let us note that 1_{R_t} is lower semicontinuous on $\overline{U}$ and finely continuous on U.

Lemma 2.3. *Let H be a function space on a metrizable compact space X, $x, y \in X$, $\mu \in \mathcal{M}_x(H)$, $\nu \in \mathcal{M}_y(H)$, and $c > 0$ such that $\mu \le c\nu$. Then $s(x) \le cs(y)$, whenever s is a positive H-concave function on X.*

Proof. Define

$$\sigma := \frac{1}{c}\varepsilon_x + (\nu - \frac{1}{c}\mu).$$

Then $\sigma \ge 0$ by assumption. Since $\varepsilon_x = \mu$ on H, we know that $\sigma \in \mathcal{M}_y(H)$. Taking an H-concave function $s \ge 0$ we thus have $\sigma(s) \le s(y)$, whence

$$\frac{1}{c}s(x) \le \sigma(s) \le s(y).$$

□

Proposition 2.4. *Let U be a relatively compact open subset of a $\mathcal{P}$-harmonic space $(Y, \mathcal{H})$, $c > 0$, and $x, y \in U$ such that $h(x) \le ch(y)$ for every $h \in H^+(U)$. Then, for every $H(U)$-concave function $s \ge 0$,*

$$s(x) \le cs(y).$$

Further, if $(Y, \mathcal{H})$ is a $\mathcal{P}$-harmonic Brelot *space and U is connected, then, for every compact subset K of U and every $x_0 \in U$, there exists a constant $c > 0$ such that, for every $H(U)$-concave function $s \ge 0$,*

$$s \le cs(x_0) \quad \text{on } K.$$

In particular, it follows that every $H(U)$-concave function which is finite at some point of U is locally bounded on U.

Proof. Let us recall that $H(U)$ is simplicial and that, for every $z \in \overline{U}$, there is a unique measure $\mu_z \in \mathcal{M}_z(H(U))$ carried by the Choquet boundary $\mathrm{Ch}_{H(U)}\overline{U}$ ([4, Corollary II.3.8]). We choose a continuous real potential p such that $p \ge 1$ on $\overline{U}$ and let $a := \sup p(\overline{U})$. Moreover, we fix $f \in \mathcal{C}(\partial U)$, $0 \le f \le 1$, $\eta > 0$, and a compact set $L \subset \mathrm{Ch}_{H(U)}\overline{U}$ such that $(\mu_x + \mu_y)(\overline{U} \setminus L) < \eta/a$. By [5, VII.5.10], there exists $h \in H(U)$, $0 \le h \le a$, such that $h = f$ on L. Then

$$\mu_x(f) \le \mu_x(h) + \eta = h(x) + \eta \le ch(y) + \eta = c\mu_y(h) + \eta \le c\mu_y(f) + (c+1)\eta,$$

and hence $\mu_x \le c\mu_y$. Now the first part of the result follows from Lemma 2.3.

The second part follows easily, since we have Harnack's inequalities for positive harmonic functions on U ([6, p. 15]). For the final part, we note that, for every $H(U)$-concave function s, the positive function $s - \inf s(\overline{U})$ is also $H(U)$-concave. □

Example 2.5. Let U be the open unit ball in $\mathbb{R}^d$, $d \geq 3$, and $y \in U \setminus \{0\}$. Choose $c \geq 0$ such that $h(0) \leq ch(y)$ for every harmonic function $h \geq 0$ on U. Define

$$s(x) := \min(t(x), (c+1)t(y)), \qquad \text{where } t(x) := |x|^{2-d} - 1 \quad (x \in \mathbb{R}^d).$$

Then s is a continuous bounded function on $\overline{U}$ which is superharmonic on U and vanishes on ∂U. But $s(0) = (c+1)t(y) = (c+1)s(y)$. By Proposition 2.4, we know that s is not $H(U)$-concave.

The relation between the notion of $H(U)$-concavity and the notion of superharmonicity has already been clarified in [11].

Furthermore, there obviously exists an open neighborhood V of 0 in U such that s is equal to the (harmonic) constant $C := (c+1)t(y)$ on $\overline{V}$. And, on any neighborhood of $\overline{U} \setminus V$ not containing the origin, s is the minimum of the two functions C and t, which are harmonic on $\mathbb{R}^d \setminus \{0\}$. So $H(U)$-concave functions are far from having a sheaf property.

Theorem 2.6. *Let U be a relatively compact domain in a $\mathcal{P}$-Brelot space $(Y, \mathcal{H})$. Then every locally lower bounded set $\mathcal{F}$ of $H(U)$-concave functions, which is upper bounded at some point of U, is locally bounded and locally equicontinuous on U.*

In particular, every $H(U)$-concave function which is finite at some point of U is continuous and real-valued on U.

Proof. Assume that the set $\mathcal{F}$ is upper bounded at $x_0 \in U$. Choose a compact set K in U containing x_0. Since Y is locally connected, there exists a domain V containing K such that $\overline{V} \subset U$ (see [8, p. 167]). We may assume without loss of generality that $s|_{\overline{V}} \geq 0$ for every $s \in \mathcal{F}$ (by adding a suitable constant). Then, by Proposition 2.4, $\mathcal{F}$ is bounded on K, say by a constant a. Let ρ be a metric on Y which is compatible with the topology of Y, and fix $\eta > 0$. By [6, pp. 20, 22], there exists $\delta > 0$ such that

$$\varepsilon_x^{V^c} \leq (1+\eta)\varepsilon_y^{V^c} \qquad \text{whenever } x, y \in K \text{ and } \rho(x,y) < \delta. \tag{4}$$

By Proposition 2.4, we see that, for every $s \in \mathcal{F}$ and for all $x, y \in K$ satisfying $\rho(x,y) < \delta$,

$$s(x) \leq (1+\eta)s(y) \leq s(y) + \eta a.$$

□

Corollary 2.7. *For every $\mathcal{P}$-harmonic space $(Y, \mathcal{H})$ the following properties are equivalent:*

(*i*) *$(Y, \mathcal{H})$ is a Brelot space.*
(*ii*) *For every relatively compact domain U in Y, each $H(U)$-concave function which is not identically infinite on U is continuous and real-valued on U.*

Remark 2.8. Let us already note that $H(U)$-concave functions will *always* be finely upper semicontinuous on U (see Corollary 3.8).

Proof of Corollary 2.7. $(i) \Rightarrow (ii)$: Theorem 2.6.

$(ii) \Rightarrow (i)$: Suppose that $(Y, \mathcal{H})$ is not a Brelot space. Then there exists a domain V in Y and an increasing sequence (h_n) of harmonic functions on V such that $h := \sup h_n \not\equiv \infty$

and h is not harmonic. Therefore h is not both continuous and real-valued. But, of course, for every relatively compact domain U with $\overline{U} \subset V$, the restriction $h|_{\overline{U}}$ is $H(U)$-concave.

If $h < \infty$ on V, we may choose a relatively compact domain U such that $\overline{U} \subset V$ and h is not continuous on U.

So suppose that the set A where $h = \infty$ is not empty. Since V is connected, there exists a point z in the boundary of A. If U is any relatively compact connected neighborhood of z such that $\overline{U} \subset V$, then $h(x_0) < \infty$ and $h(x_1) = \infty$ for some points $x_0, x_1 \in U$. $\square$

Corollary 2.9. *Let U be a relatively compact domain in a $\mathcal{P}$-Brelot space and let s be an $H(U)$-concave function which is finite at some point of U. Then the Perron-Wiener-Brelot solution $H_U s$ for the boundary function s satisfies $H_U s \le s$ on U and, for every compact subset K of U, $s|_K$ belongs to the uniform closure of $W(H(U))|_K$.*

Proof. For every $x \in U$, $\varepsilon_x^{U^c} \in \mathcal{M}_x(H(U))$ and therefore $H_U s(x) = \varepsilon_x^{U^c}(s) \le s(x)$.

The proof of the approximation is based on the pattern in [2, Satz 7]. Fix a compact set $K \subset U$, $x \in K$, and put $t := s|_K$. By Theorem 2.6, t is a continuous real function on K. For $f \in \mathcal{C}(K)$ define

$$Q(f) := \inf\{h(x)\colon h \in H(U), h|_K \ge f\}.$$

Then Q is a sublinear functional on $\mathcal{C}(K)$. Defining $\mu_0(\lambda t) := \lambda Q(t)$, $\lambda \in \mathbb{R}$, μ_0 is a linear functional on $\mathbb{R}\cdot t$ and it is easy to see that $\mu_0 \le Q$ on $\mathbb{R}\cdot t$. By the Hahn-Banach theorem, there exists a linear functional μ on $\mathcal{C}(K)$ such that $\mu(t) = Q(t)$ and $\mu \le Q$. If $f \in \mathcal{C}(K)$, $f \le 0$, then $\mu(f) \le Q(f) \le 0$, and hence μ is a Radon measure on K. Obviously, $Q(h|_K) = h(x)$ for every $h \in H(U)$, so $\mu \in \mathcal{M}_x(H(U))$ and $Q(t) = \mu(t) \le t(x)$. Since trivially $t(x) \le Q(t)$, we see that $t(x) = Q(t)$. So, for every $\delta > 0$, there exists a function $h \in H(U)$ such that $h \ge t$ on K and $h(x) \le t(x) + \delta$. A standard compactness argument now yields the existence of functions $h_1, \dots, h_n \in H(U)$ such that $t \le \min(h_1, \dots, h_n) \le t + \delta$ on K. $\square$

Proposition 2.10. *Let U be a relatively compact domain in a $\mathcal{P}$-Brelot space. Then, for every $H(U)$-concave function which is finite at some point of U, its restriction to U is equal to the infimum of its harmonic majorants there.*

In fact, for every lower bounded numerical function s on U, $s \not\equiv \infty$, the following statements are equivalent:

(*i*) *For every open set V with $\overline{V} \subset U$, the restriction $s|_{\overline{V}}$ belongs to the uniform closure of $W(H(V))$.*

(*ii*) *s is the infimum of its harmonic majorants.*

Proof. Once we have shown that (*i*) implies (*ii*), the first statement follows since, by Corollary 2.9, for every $H(U)$-concave function which is finite at some point of U, the restriction to U satisfies (*i*).

To prove the equivalence of (*i*) and (*ii*), we may assume without loss of generality that $s \ge 0$. Suppose first that (*i*) holds. Fix $x \in U$ and choose an increasing sequence (V_n) of open sets such that $x \in V_1$, each $\overline{V}_n$ is a subset of U and $U = \bigcup_{n=1}^\infty V_n$. For every $n \in \mathbb{N}$, there exists $h_n \in H(V_n)$ such that $h_n \ge s$ on V_n and $s(x) \le h_n(x) \le s(x) + 1/n$. There exists a subsequence (g_n) of (h_n) which converges locally uniformly on U to a harmonic function g on U. Clearly, $g \geq s$ and $g(x) = s(x)$ proving (ii).

Assume now conversely that $s = \inf\{g \in \mathcal{H}(U) \colon g \geq s\}$ and fix an open set V with $\overline{V} \subset U$. By Harnack inequalities, this implies that s is continuous on $\overline{V}$ (cf. (4)). Obviously, the restriction of any function $g \in \mathcal{H}(U)$ to $\overline{V}$ is contained in $H(V)$. A standard compactness argument finishes the proof. □

Theorem 2.11. *Let U be a relatively compact regular domain in a $\mathcal{P}$-Brelot space having the following contraction property:*

(C) *For every compact subset K of U and for every $\eta > 0$, there exist an open neighborhood V of K and a Borel measurable mapping $\varphi \colon \partial U \to \overline{V}$ such that $\overline{V} \subset U$ and, for every $y \in K$,*

$$(1-\eta)\varepsilon_y^{V^c} \leq \varphi(\varepsilon_y^{U^c}) \leq (1+\eta)\varepsilon_y^{V^c}. \tag{5}$$

Then, for every lower semicontinuous function $s \colon \overline{U} \to (-\infty, \infty]$, the following statements are equivalent:

(i) *s is $H(U)$-concave and finite at some point of U.*
(ii) *For every compact set K in U, the restriction $s|_K$ belongs to the uniform closure of $W(H(U))|_K$.*
(iii) *For every open V with $\overline{V} \subset U$, the restriction $s|_{\overline{V}}$ belongs to the uniform closure of $W(H(V))$.*
(iv) *For every open $V \neq \emptyset$ with $\overline{V} \subset U$, the restriction $s|_{\overline{V}}$ is $H(V)$-concave and finite at some point of V.*

If s is not lower semicontinuous, then these equivalences still hold if we assume instead that s is lower bounded and Borel measurable, and if the condition that $H_U s \leq s$ on U is added to the statements (ii), (iii), and (iv).

Proof. Let s be a Borel measurable function on $\overline{U}$ which is lower bounded. We may assume without loss of generality that $s \geq 0$. The implications $(ii)\Rightarrow(iii)\Rightarrow(iv)$ are trivial.

If any one of the properties (i) – (iv) holds, then s is superharmonic on U. If, moreover, s is lower semicontinuous on $\overline{U}$, then s is an upper function for $s|_{\partial U}$ and hence $H_U s \leq s$ on U.

By Corollary 2.9, (i) implies that $H_U s \leq s$ on U and that (ii) holds. Assuming finally that $H_U s \leq s$ holds on U, it remains to show that (iv) implies (i). To that end we may suppose without loss of generality that $s \leq 1$. Indeed, for every $n \in \mathbb{N}$, the function $s_n := \min(1, s/n)$ has property (iv) and satisfies $H_U s_n \leq s_n$ on U; having shown that each s_n is $H(U)$-concave, we see that $s = \lim_{n\to\infty} n s_n$ is $H(U)$-concave.

Fix $x \in \overline{U}$ and $\mu \in \mathcal{M}_x(H(U))$. We want to prove that $\mu(s) \leq s(x)$. If $x \in \partial U$, then $\mu = \varepsilon_x$, since U is regular, and $\mu(s) = s(x)$. So suppose that $x \in U$. Note that, for every $h \in H(U)$,

$$\mu^{U^c}(h) = \int \varepsilon_y^{U^c}(h)\, d\mu(y) = \int h(y)\, d\mu(y) = h(x) = \varepsilon_x^{U^c}(h).$$

Since $H(U)|_{\partial U} = \mathcal{C}(\partial U)$, this implies that $\mu^{U^c} = \varepsilon_x^{U^c}$. Moreover, $\tau^{U^c} = \tau$ for every measure τ on ∂U. Therefore,

$$\mu = \varepsilon_x^{U^c} + 1_U\mu - (1_U\mu)^{U^c}.$$

Fix $0 < \eta < 1/2$. There exists a compact subset K of U such that $x \in K$ and $\mu(U\setminus K) < \eta$. Then the measure $\nu := 1_K\mu$ satisfies

$$\mu(s) < (\varepsilon_x^{U^c} + \nu - \nu^{U^c})(s) + \eta, \qquad \nu^{U^c} \le \varepsilon_x^{U^c}. \tag{6}$$

Now choose an open neighborhood V of K and a Borel measurable mapping $\varphi\colon \partial U \to \overline{V}$ such that $\overline{V} \subset U$ and (5) holds for every $y \in K$.

Integrating (5) with respect to the measure ν on K we see that

$$(1-\eta)\nu^{V^c} \le \varphi(\nu^{U^c}) \le \varphi(\varepsilon_x^{U^c}) \le (1+\eta)\varepsilon_x^{V^c}.$$

Thus $\sigma := \varepsilon_x^{V^c} - (1-2\eta)\nu^{V^c} \ge \varepsilon_x^{V^c} - (1-\eta)(1+\eta)^{-1}\nu^{V^c} \ge 0$,

$$\rho := \sigma + (1-2\eta)\nu \in \mathcal{M}_x(H(V)), \qquad \rho(s) \le s(x). \tag{7}$$

Since $s \ge H_U s$ on U and $\varepsilon_y^{V^c}(H_U s) = H_U s(y) = \varepsilon_y^{U^c}(s)$ for every $y \in U$, we have

$$\sigma(s) \ge \sigma(H_U s) = \varepsilon_x^{U^c}(s) - (1-2\eta)\nu^{U^c}(s). \tag{8}$$

Combining (6), (7), and (8) we finally conclude that

$$\begin{aligned} \mu(s) &< \varepsilon_x^{U^c}(s) - (1-2\eta)\nu^{U^c}(s) + \nu(s) + \eta \\ &\le \sigma(s) + (1-2\eta)\nu(s) + 2\eta\nu(s) + \eta \\ &\le \rho(s) + 3\eta \le s(x) + 3\eta. \end{aligned}$$

□

Example 2.12. It may be instructive to consider the following example (not having the property (C)). Let B denote the open unit disk,

$$S := \{(t,0)\colon |t| \le 1/2\}, \qquad U := B \setminus S.$$

There exists a positive harmonic function h on U which vanishes at ∂B and satisfies

$$\lim_{t\downarrow 0} h(0,t) = 1, \qquad \lim_{t\uparrow 0} h(0,t) = 0.$$

We extend h by 0 to be a function on $\overline{U}$. Then h is lower semicontinuous on $\overline{U} = \overline{B}$. Obviously, $h|_{\overline{V}} \in H(V)$ for every open V with $\overline{V} \subset U$. So *(iii)* and *(iv)* in Theorem 2.11 hold. However, *(i)* does not hold, that is, h is not $H(U)$-concave. Indeed, by Harnack's inequalities, there exists a constant $K \ge 1$ such that, for all $y, \tilde{y} \in A := \partial B(0, 3/4)$,

$$\varepsilon_y^{U^c} \le K\varepsilon_{\tilde{y}}^{U^c} \quad \text{on } \partial B. \tag{9}$$

We choose $0 < t < 3/4$ such that $x := (0,t)$ and $\tilde{x} := (0,-t)$ satisfy $h(x) > K^2 h(\tilde{x})$. By symmetry, $\varepsilon_x^{U^c}|_S = \varepsilon_{\tilde{x}}^{U^c}|_S$ and $\varepsilon_x^{S\cup A}(A) = \varepsilon_{\tilde{x}}^{S\cup A}(A)$. By [5, VI.9.4],

$$\varepsilon_x^{U^c} = (\varepsilon_x^{S\cup A}|_A)^{U^c} \quad \text{on } \partial B, \qquad \varepsilon_{\tilde{x}}^{U^c} = (\varepsilon_{\tilde{x}}^{S\cup A}|_A)^{U^c} \quad \text{on } \partial B.$$

Fixing a point $y_0 \in A$, we hence see by (9) that

$$\begin{aligned} \varepsilon_x^{U^c} = \int_A \varepsilon_y^{U^c} d\varepsilon_x^{S\cup A}(y) &\leq K\varepsilon_x^{S\cup A}(A)\varepsilon_{y_0}^{U^c} \\ &= K\varepsilon_{\tilde{x}}^{S\cup A}(A)\varepsilon_{y_0}^{U^c} \leq K^2 \int_A \varepsilon_y^{U^c} d\varepsilon_{\tilde{x}}^{S\cup A}(y) = K^2 \varepsilon_{\tilde{x}}^{U^c} \end{aligned}$$

on ∂B and therefore $\varepsilon_x^{U^c} \leq K^2 \varepsilon_{\tilde{x}}^{U^c}$. Thus

$$\sigma := \varepsilon_{\tilde{x}}^{U^c} + K^{-2}(\varepsilon_x - \varepsilon_x^{U^c}) \in \mathcal{M}_{\tilde{x}}(H(U)).$$

In particular, for every $H(U)$-concave function s which vanishes on ∂U,

$$K^{-2}s(x) = \sigma(s) \leq s(\tilde{x}).$$

However, $h(x) > K^2 h(\tilde{x})$. So h is not $H(U)$-concave.

Remark 2.13. The condition $H_U s \leq s$ on U can be viewed as saying that s is majorized by boundary values of s on ∂U.

Let U be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, $x_0 \in U$, let M be the Martin kernel of U relative to x_0 and let κ be the measure on the (minimal) Martin boundary corresponding to the function $h = 1$ on U in the Martin representation; see [1, Chapter 8].

Assume that the Martin boundary of U can be identified with ∂U (as is, for example, the case when U is a Lipschitz domain). Then, by [1, Theorem 9.1.7],

$$\varepsilon_x^{U^c} = M(x,\cdot)\kappa \quad (x \in U).$$

By [1, Lemma 8.1.2], the measures $\varepsilon_x^{U^c}$ and κ are mutually absolutely continuous and a function f on ∂U is κ-integrable if and only if f is $\varepsilon_x^{U^c}$-integrable.

Let s be an $H(U)$-concave function which is finite at some point of U. By Corollary 2.9, s is κ-integrable and for minimal fine limits we have by [1, Theorem 9.4.5],

$$\text{mf-}\lim_{x\to z} H_U s(x) = s(z) \quad \text{ for } \kappa\text{-almost every } z \in \partial U.$$

Being lower bounded and superharmonic, s has minimal fine limits κ-almost everywhere on ∂U by [1, Theorem 9.4.6]. Therefore the inequality $H_U s \leq s$ implies that

$$\text{mf-}\lim_{x\to z} s(x) \geq s(z) \quad \text{ for } \kappa\text{-almost every } z \in \partial U. \tag{10}$$

Conversely, if (10) holds, then

$$\text{mf-}\lim_{x\to z} (s - H_U s)(x) \geq 0 \quad \text{ for } \kappa\text{-almost every } z \in \partial U,$$

whence $s - H_U s \geq 0$ on U by [1, Theorem 9.3.7].

Thus we see that the following conditions are equivalent:

(i) $H_U s \leq s$ on U.

(ii) $\text{mf-}\lim_{x\to z} s(x) \geq s(z)$ for all $z \in \partial U$, except for a set of harmonic measure zero.

Note that, for a ball U, minimal fine limits in (ii) can be replaced by non-tangential limits (see [1, Theorems 9.7.4 and 9.7.6]).

Proposition 2.14. *Let U be a bounded domain in $\mathbb{R}^d$, $d \ge 2$. Then U has the contraction property (C) of Theorem* 2.11 *if any of the following properties holds:*

- (i) *U is convex.*
- (ii) *U is strictly star-shaped, that is, there is a point $x_0 \in U$ such that the sets $x_0 + r(\overline{U - x_0})$, $0 < r < 1$, are contained in U.*
- (iii) *$d = 2$ and U is a Jordan domain.*
- (iv) *For every compact subset L of U and for every $\delta > 0$, there exists an open neighborhood V of L and a homeomorphism $\varphi : \overline{U} \to \overline{V}$ such that $\overline{V} \subset U$, $H(V) \circ \varphi = H(U)$, and $|x - \varphi^{-1}(x)| < \delta$ for every $x \in L$.*

Proof. $(i)\Rightarrow(ii)$: Trivial.

$(ii)\Rightarrow(iv)$: If $a \ge 2\delta$ such that $\overline{U} \subset B(x_0, a/2)$, then the contraction defined by $x \mapsto x_0 + (1 - \delta/a)(x - x_0)$ has the desired properties.

$(iii)\Rightarrow(iv)$: By [13, Theorem 14.19, Remark 14.20], there exists a homeomorphism ψ of $\overline{U}$ onto the closure $\overline{D}$ of the open unit disk D such that $\psi(\partial U) = \partial D$, and $\psi|_U$ is a conformal mapping from U on D. Therefore $H(D) \circ \psi = H(U)$. Knowing already that D has property (iv), we now obtain easily that U also has property (iv).

$(iv)\Rightarrow(C)$: Fix a compact subset $K \neq \emptyset$ of U and $\eta > 0$. Let L be a compact neighborhood of K in U. By [6, p. 20, 22], there exists $0 < \delta < \operatorname{dist}(K, L^c)$ such that

$$\varepsilon_{y_1}^{U^c} \le (1+\eta)\varepsilon_{y_2}^{U^c} \quad \text{whenever } y_1, y_2 \in L \text{ and } |y_1 - y_2| < \delta. \tag{11}$$

Let V be an open neighborhood of L and $\varphi\colon \overline{U} \to \overline{V}$ a homeomorphism such that $\overline{V} \subset U$, $H(V) \circ \varphi = H(U)$, and $|x - \varphi^{-1}(x)| < \delta$ for every $x \in L$. The identity $H(V) \circ \varphi = H(U)$ implies that

$$\varepsilon_y^{V^c} = \varphi\big(\varepsilon_{\varphi^{-1}(y)}^{U^c}\big) \quad \text{for every } y \in \overline{V}.$$

Indeed, it is easy to see that $\mathcal{M}_y(H(V)) = \varphi(\mathcal{M}_{\varphi^{-1}(y)}(H(U)))$ for every $y \in \overline{V}$ and $\mathrm{Ch}_{H(V)}\overline{V} = \varphi(\mathrm{Ch}_{H(U)}\overline{U})$. Since the measure $\varepsilon_{\varphi^{-1}(y)}^{U^c}$ is carried by $\mathrm{Ch}_{H(U)}\overline{U}$, the measure $\varphi(\varepsilon_{\varphi^{-1}(y)}^{U^c})$ is carried by $\mathrm{Ch}_{H(V)}\overline{V}$, and so it coincides with $\varepsilon_y^{V^c}$ by simpliciality of $H(V)$.

Given $y \in K$, let $\tilde{y} := \varphi^{-1}(y)$. Then $|y - \tilde{y}| < \delta$, so $y, \tilde{y} \in L$ and, using $1 - \eta < (1+\eta)^{-1}$,

$$(1-\eta)\varepsilon_{\tilde{y}}^{U^c} \le \varepsilon_y^{U^c}, \qquad \varepsilon_y^{U^c} \le (1+\eta)\varepsilon_{\tilde{y}}^{U^c},$$

$$(1-\eta)\varepsilon_y^{V^c} = (1-\eta)\varphi(\varepsilon_{\tilde{y}}^{U^c}) \le \varphi(\varepsilon_y^{U^c}) \le (1+\eta)\varphi(\varepsilon_{\tilde{y}}^{U^c}) = (1+\eta)\varepsilon_y^{V^c}.$$

□

We recall that, by Proposition 2.10, the restriction of every $H(U)$-concave $s \not\equiv \infty$ to U is the infimum of its harmonic majorants. If s is *continuous on* $\overline{U}$, then [2, Satz 7] implies that a much stronger result holds:

$$s = \inf\{h \in H(U)\colon h \ge f \text{ on } \overline{U}\}.$$

We stress that even for $H(U)$-affine functions the continuity assumption cannot be omitted, as the following example will show.

Example 2.15. Let U be the open unit ball in $\mathbb{R}^d$, $d \geq 2$, and let σ denote the normalized surface measure on ∂U. Fix a relatively open dense subset A of ∂U such that $\sigma(A) < 1$. We know that the function $s\colon x \mapsto \varepsilon_x^{U^c}(A)$, $x \in \overline{U}$, is $H(U)$-affine on $\overline{U}$ (see (1)). By [1, Theorem 4.6.6], s has non-tangential limit 1 at σ-almost every point of A, and hence on a dense subset of ∂U. Consequently, every $h \in H(U)$ majorizing s satisfies $h \geq 1$ on ∂U, and thus on $\overline{U}$ by minimum principle. So we see that

$$s(0) < 1 = \inf\{h(0)\colon h \in H(U), h \geq s\}.$$

Remark 2.16. Already in the situation of Example 2.15 we do not know whether, for every bounded $H(U)$-concave function s and every point $x \in U$,

$$s(x) = \inf\{t(x)\colon t \text{ is } H(U)\text{-affine, } t \geq s\}.$$

3. $\mathbf{H_0(K)}$-concave functions

In the following let $(Y, \mathcal{H})$ be an arbitrary $\mathcal{P}$-harmonic space. Given a finely open subset V of a compact set K in Y, let $H(K,V)$ denote the set of all continuous real functions h on K which are finely median on V, that is,

$$\varepsilon_x^{W^c}(h) = h(x)$$

whenever $x \in V$ and W is a Borel measurable finely open neighborhood of x having compact closure in V. It is known that $H(K,V)$ is always simplicial (see [5, VII,5.6]).

Let us note first that, for every relatively compact open set U in Y, $H(U) = H(\overline{U}, U)$ (see [5, VII.8.4]). Further, if K is a compact subset of Y and V denotes its fine interior, then $H(K,V)$ is the uniform closure of the space

$$H_0(K) = \bigcup\{H(U)|_K\colon U \text{ is a relatively compact open neighborhood of } K\}$$

(see [5, VII.9.2]) and therefore functions are $H_0(K)$-concave if and only if they are $H(K,V)$-concave. In this case the measures μ_x, $x \in K$, are the measures $\varepsilon_x^{K^c} = \varepsilon_x^{V^c}$ ([4, Corollary 3.14, Corollary 3.8], [5, VI.4.3, 4.4]).

Examples 3.1. 1. Already in the classical case, $H_0(K)$-affine functions need not be continuous on the fine interior V of K. To see this let U, h be as in Example 2.1 and $K := \overline{U}$. Then the fine interior V of K is $U \cup \{0\}$ and hence h is not continuous on V. Since $\mu_x = \varepsilon_x^{V^c} = \varepsilon_x^{U^c}$ for every $x \in K$, we know that h is $H_0(K)$-affine (see (1)).

Moreover, we claim that the function $s := 1_{\{0\}}$ which obviously is not finely continuous on V is $H_0(K)$-concave. Indeed, fix $y \in K$ and $\mu \in \mathcal{M}_y(H_0(K))$. Recall that $\mathrm{Ch}_{H_0(K)}K = \partial K \setminus \{0\}$. Since $H_0(K)$ is simplicial and the measures $\mu^{K^c}, \varepsilon_y^{K^c}$ from $\mathcal{M}_y(H_0(K))$ are carried by $\mathrm{Ch}_{H_0(K)}K$, they coincide. If $y = 0$, then certainly $\mu(s) \leq 1 = s(y)$. If $y \in \partial K \setminus \{0\}$, then $\mu = \varepsilon_y$ whence trivially $\mu(s) = s(y)$. So suppose that $y \in U$ and let $a := \mu(\{0\})$. Then

$$a\varepsilon_0^{U^c} \leq \mu^{U^c} = \mu^{K^c} = \varepsilon_y^{K^c} = \varepsilon_y^{U^c}$$

and therefore $a = 0$ by [10, Example 2.4]. Thus $\mu(s) = as(0) = 0 = s(y)$.

2. In Example 2.2 for the heat equation, the fine interior of $\overline{U}$ is U and hence $H(U) = H(\overline{U}, U) = \overline{H_0(\overline{U})}$. Therefore functions are $H(U)$-concave if and only if they are $H_0(\overline{U})$-concave. The $H(U)$-concave functions 1_{R_t} are not finely lower semicontinuous on the (fine) interior U of $\overline{U}$, the $H(U)$-concave functions $1_{\overline{R}_t}$ are not lower semicontinuous on U.

It follows from Theorem 2.6 that, for every compact K in a $\mathcal{P}$-*Brelot* space and for every finely open subset V of K, all $H(K,V)$-concave functions are *continuous on the interior of* V. We are going to show that, for *every* $\mathcal{P}$-harmonic space, these functions are *finely upper semicontinuous on* V. In fact, we shall prove that this is true for all lower bounded, Borel measurable functions s satisfying $\varepsilon_x^{W^c}(s) \le s(x)$ for all sufficiently small Borel measurable finely open neighborhoods of $x \in V$:

Theorem 3.2. *Let $x \in Y$, V be a fine neighborhood of x, and s be a lower bounded numerical function on $\overline{V}$ which is Borel measurable or finely lower semicontinuous. Assume that, for some fine neighborhood W of x in V and every compact subset L of $W \setminus \{x\}$,*

$$\left(\varepsilon_x^{(V\setminus L)^c}\right)_*(s) \le s(x).^1$$

Then s is finely upper semicontinuous at x.

For the proof we need some preparations. For the convenience of the reader we first write down the following statement which ought to be known (in the case of a finely open set it could also be obtained as a consequence of the quasi-Lindelöf property of the fine topology).

Lemma 3.3. *Let B be a subset of Y which is Borel measurable or finely open. Then there exists a K_σ-set A contained in B such that $\varepsilon_x^A = \varepsilon_x^B$ for every $x \in Y$.*

Proof. Fix a strict continuous real potential p on Y. The proof will be finished if we find a K_σ-set A contained in B such that $\hat{R}_p^A = \hat{R}_p^B$. Indeed, having trivially $\hat{R}_q^A \le \hat{R}_q^B$ for every potential q on Y, we then get that $\hat{R}_q^A = \hat{R}_q^B$ for every potential q on Y ([5, I.1.5 and p. 117]), that is, that $\varepsilon_x^A = \varepsilon_x^B$ for every $x \in Y$.

If B is a Borel set, then the existence of a K_σ-set A satisfying $\hat{R}_p^A = \hat{R}_p^A$ is guaranteed by [5, VI.1.9]. So assume that B is finely open. Let $\mathcal{W}$ denote the set of all functions $\hat{R}_p^K$, K compact in B. Obviously, $\mathcal{W}$ is an upper directed set of positive hyperharmonic functions on Y. Therefore $w := \sup \mathcal{W}$ is hyperharmonic and, of course, $w \le \hat{R}_p^B = R_p^B$ by [5, p. 243]. Given $y \in B$, we may choose a compact fine neighborhood K of y in B and then $\hat{R}_p^K(y) = p(y)$ again by [5, p. 243]. So $w = p$ on B, whence conversely $w \ge R_p^B$. By [5, I.1.7], there exists a sequence (K_n) of compact subsets of B such that

$$\sup_n \hat{R}_p^{K_n} = \sup \mathcal{W} = R_p^B = \hat{R}_p^B.$$

Taking $A := \bigcup_{n=1}^\infty K_n$ we clearly have $\hat{R}_p^A = \hat{R}_p^B$. □

Lemma 3.4. *Let $x \in Y$ and $A \subset Y \setminus \{x\}$. Assume that A is not thin at x and that A is Borel measurable or finely open. Then there exists a subset $\tilde{A}$ of A such that $\tilde{A} \cup \{x\}$ is compact and $\tilde{A}$ is not thin at x.*

Proof. By Lemma 3.3, there exists a sequence (K_n) of compact subsets of A such that $\bigcup_{n=1}^\infty K_n$ is not thin at x. We may assume without loss of generality that $K_n \subset K_{n+1}$ for every $n \in \mathbb{N}$. We fix a strict continuous real potential p on Y and a sequence (L_m)

[1]Here μ_* denotes the *lower* integral with respect to μ.

of compact neighborhoods of x such that $L_m \downarrow \{x\}$. Then, for every $m \in \mathbb{N}$, the union $\bigcup_{n=1}^{\infty}(K_n \cap L_m)$ is not thin at x, so there exists $n_m \in \mathbb{N}$ such that

$$\hat{R}_p^{K_{n_m} \cap L_m}(x) > p(x) - \frac{1}{m}.$$

Then

$$\tilde{A} := \bigcup_{m=1}^{\infty} K_{n_m} \cap L_m \subset A,$$

$\tilde{A} \cup \{x\}$ is compact, and $\hat{R}_p^{\tilde{A}}(x) = p(x)$, that is, $\tilde{A}$ is not thin at x. □

Proposition 3.5. *Let A, B be subsets of Y, $x \in Y \setminus (A \cup B)$. Assume that A is not thin at x, B is thin at x, and that A is Borel measurable or finely open. Then, for every $\delta > 0$, there exists a compact subset L of A such that*

$$\varepsilon_x^{L \cup B}(L) > 1 - \delta.$$

Proof. By [5, VI.4.1], we may assume without loss of generality that B is a Borel measurable finely closed set. Then ε_x^B is supported by the set B, which does not contain x (see [5, VI.4.6]). Therefore $\varepsilon_x^B(\{x\}) = 0$ and, given $\delta > 0$, there exists an open neighborhood U of x such that $\varepsilon_x^B(U) < \delta/2$. Choose $\tilde{A}$ according to Lemma 3.4. Then $\tilde{A} \cup B$ is not thin at x, $\varepsilon_x^{\tilde{A} \cup B} = \varepsilon_x$. Since $\varepsilon_x^{(\tilde{A} \setminus V) \cup B}$ converges weakly to $\varepsilon_x^{\tilde{A} \cup B} = \varepsilon_x$ as the neighborhoods V of x decrease to $\{x\}$, there exists an open neighborhood V of x such that the compact subset $L := \tilde{A} \setminus V$ of A satisfies

$$\varepsilon_x^{L \cup B}(U) > 1 - \frac{\delta}{2}. \tag{12}$$

We note that the measure $\varepsilon_x^{L \cup B}$ is supported by the Borel measurable finely closed set $L \cup B$. By [5, VI.9.4.], $\varepsilon_x^{L \cup B}|_B \leq \varepsilon_x^B$ and therefore

$$\varepsilon_x^{L \cup B}(U \cap B) \leq \varepsilon_x^B(U) < \frac{\delta}{2}.$$

Combining this estimate with (12) we finally conclude that $\varepsilon_x^{L \cup B}(U \cap L) > 1 - \delta$. □

Proof of Theorem 3.2. Since we can add constants, we may suppose that $s \geq 0$. Moreover, we may assume that W is finely open and Borel measurable (we can replace W by the fine interior of a smaller compact fine neighborhood).

Suppose that s is not finely upper semicontinuous at x. Then $s(x) < \infty$ and there exists $\eta > 0$ such that the set $A := \{y \in W \colon s(y) > s(x) + \eta\}$ is not thin at x. Note that, of course, $x \notin A$ and that A is Borel measurable or finely open. We may choose $\delta > 0$ such that

$$(1 - \delta)(s(x) + \eta) > s(x). \tag{13}$$

By Proposition 3.5, there exists a compact subset L of A such that

$$\varepsilon_x^{L \cup V^c}(L) > 1 - \delta.$$

Since $s \geq 0$ on $\overline{V}$ and $s > s(x) + \eta$ on L, we therefore obtain the estimate

$$\left(\varepsilon_x^{L \cup V^c}\right)_*(s) \geq (s(x)+\eta)\varepsilon_x^{L \cup V^c}(L) > (s(x)+\eta)(1-\delta) > s(x). \tag{14}$$

Having $L \cup V^c = (V \setminus L)^c$, inequality (14) contradicts our assumption. Thus s is finely upper semicontinuous at x. □

Let s be a numerical function on a finely open set V in Y. We will say that s is *finely supermedian on V*, if it is lower bounded, Borel measurable, and

$$\varepsilon_x^{W^c}(s) \leq s(x) \tag{15}$$

whenever $x \in V$ and W is a Borel measurable finely open neighborhood of x having compact closure in V.

Corollary 3.6. *For every finely open set V in Y, every finely supermedian function on V is finely upper semicontinuous on V.*

Remark 3.7. Having finished an earlier version of this paper, the authors learnt from L. Beznea that Corollary 3.6 could also be deduced from [3, Theorem 6.3].

Corollary 3.8. *Let V be a finely open subset of a compact set K in Y. Then every $H(K,V)$-concave function is finely upper semicontinuous on V.*

In particular, for every compact set K in Y, all $H_0(K)$-concave functions are finely upper semicontinuous on the fine interior of K and, for every relatively compact open set U in Y, all $H(U)$-concave functions are finely upper semicontinuous on U.

Furthermore, an easy consequence of Corollary 3.6 is the fine continuity of functions $x \mapsto \varepsilon_x^A(f)$ established in [9, Corollary 12.13] in a different way.

Corollary 3.9. *Let A be a subset of Y and f a Borel measurable function on Y such that $|f| \leq p$ for some continuous real potential p on Y. Then the function $x \mapsto \varepsilon_x^A(f)$ is finely continuous on the complement V of the fine closure of A.*

Proof. Fix $x \in V$ and let W be a Borel measurable finely open neighborhood of x such that $\overline{W}$ is a compact subset of V. By [5, VI.2.2], we may assume that A is a Borel measurable set not intersecting $\overline{W}$. Then $(\varepsilon_x^{W^c})^A = \varepsilon_x^A$ by [5, VI.9.4], and therefore the function $h : x \mapsto \varepsilon_x^A(f)$ satisfies

$$\varepsilon_x^{W^c}(h) = \int \varepsilon_y^A(f)\, d\varepsilon_x^{W^c}(y) = (\varepsilon_x^{W^c})^A(f) = \varepsilon_x^A(f) = h(x).$$

Applying Corollary 3.6 to h and $-h$ we obtain that h is finely continuous on V. □

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